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GENERALIZATION OF PASCAL'S RULE AND LEIBNIZ'S RULE FOR DIFFERENTIATION

Rajeshwari Majumdar

Abstract. We generalize the combinatorial identity for binomial coefficients underlying the construction of Pascal's Triangle to multinomial coefficients underlying the construction of Pascal's Simplex. Using this identity, we present a new proof of the formula for calculating the n^{th} derivative of the product of k functions, a generalization of Leibniz's Rule for differentiation.

Acknowledgements: I took a freshman sequence in Advanced Calculus at the University of Connecticut using Volume 1 of Calculus by Apostol (1967) as the textbook in 2013-14 with Professor William Abikoff. Without Professor Abikoff's continuous encouragement to push the envelope and untiring patience in answering each and every question, I would not have developed the fortitude to undertake this investigation. While I started the work in the summer of 2014, the upper divisional Honors Probability course I took with the late Professor Evarist Giné in Fall 2014 was extremely influential in helping me formulate the extension of Pascal's Rule, a key step in my proof of the generalization of Leibniz's Rule. I would like to dedicate this paper to my fond memories of Professor Giné.

1 Introduction

For a real-valued function y defined on an open interval $I \subseteq \mathfrak{R}$, and any nonnegative integer k , let $y^{(k)}$ denote the k^{th} derivative of y with the convention that $y^{(0)} = y$. Let f, g be two real-valued functions defined on an open interval $I \subseteq \mathfrak{R}$ such that $f^{(n)}$ and $g^{(n)}$ exist for some nonnegative integer n . For $h(x) = f(x)g(x)$, Leibniz's Rule asserts

$$h^{(n)}(x) = \sum_{k=0}^n \binom{n}{k} f^{(k)}(x) g^{(n-k)}(x). \quad (1)$$

See, for example, Exercise 5.11.4 of Apostol's *Calculus, Volume 1* [1]. For $n = 1$, (1) reduces to the product rule for differentiation. The formula in (1) is proved by induction on n using Pascal's Rule

$$\binom{q}{j-1} + \binom{q}{j} = \binom{q+1}{j}, \quad (2)$$

where q and j are non-negative integers, $0 \leq j \leq q$. Note that

$$\binom{q}{-1} = 0$$

under the convention that $(r!)^{-1} = 0$ if r is a negative integer.

A formula for calculating the derivative of the product of k differentiable functions is outlined in Exercise 4.6.24 of Apostol's *Calculus, Volume 1* [1]. Let $\{f_1, \dots, f_k\}$ be k real-valued differentiable functions on an open interval $I \subseteq \mathfrak{R}$ and let

$$g = \prod_{i=1}^k f_i; \quad (3)$$

then

$$g^{(1)} = \sum_{i=1}^k f_i^{(1)} \prod_{\{j \neq i\}} f_j. \quad (4)$$

The equality in (4) is vacuously true for $k = 1$, is the product rule for $k = 2$, and follows easily by induction on k using the product rule.

Theorem 1 of this paper presents a generalization of the formula in (4) for higher order derivatives. We need to define a multi-index to state the theorem.

Definition 1. A k -dimensional multi-index

$$\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_k)$$

is a k -tuple of non-negative integers, with

$$|\boldsymbol{\alpha}| = \sum_{i=1}^k \alpha_i \text{ and } \boldsymbol{\alpha}! = \prod_{i=1}^k \alpha_i!.$$

Theorem 1. For $n \geq 1$, let $\{f_1, \dots, f_k\}$ be real-valued functions on an open interval $I \subseteq \mathfrak{R}$ such that $f_i^{(n)}$ exists for all $1 \leq i \leq k$. Let g be as in (3); then

$$g^{(n)} = \sum_{\{\alpha: |\alpha|=n\}} \frac{n!}{\alpha!} \prod_{i=1}^k f_i^{(\alpha_i)}, \quad (5)$$

where α is a k -dimensional multi-index.

Remark 1. It should be noted here that the coefficient of the product in the right-hand side of (5),

$$\frac{n!}{\alpha!} = \frac{n!}{\alpha_1! \dots \alpha_k!} = \binom{n}{\alpha_1 \dots \alpha_k},$$

is nothing but the multinomial coefficient.

Remark 2. The generalized Leibniz's Rule presented in (5) is not a new result. As pointed out by Thaheem and Laradji [2], this generalization is overlooked by most calculus textbooks, and those that mention it typically do so without a proof. Thaheem and Laradji [2] presented this generalization in their Theorem 2; they proved the formula in (5) by fixing the order of the derivative, which is n in our notation, and using induction on the number of factors, which is k in our notation. They assumed Leibniz's Rule stated in (1). Also see Mazkewitsch [3]. In the next section, we establish the formula in (5) by fixing k and using induction on n . The key ingredients of our proof are the equalities in (4) and (9) of Lemma 1 below. Note that (9) is a generalization of Pascal's Rule stated in (2).

2 Proof of Theorem 1

To prove Theorem 1, we first need to state and prove Lemma 1. The formulation of Lemma 1 requires the following definition.

Definition 2. Given a k -dimensional multi-index α and $i \in \{1, 2, \dots, k\}$, let ${}^+ \alpha^{(i)}$ be the k -dimensional multi-index with the j^{th} component given by

$${}^+ \alpha_j^{(i)} = \begin{cases} \alpha_j & \text{if } j \neq i \\ \alpha_i + 1 & \text{if } j = i; \end{cases} \quad (6)$$

if $\alpha_i > 0$, let ${}_- \alpha^{(i)}$ be the k -dimensional multi-index with the j^{th} component given by

$${}_- \alpha_j^{(i)} = \begin{cases} \alpha_j & \text{if } j \neq i \\ \alpha_i - 1 & \text{if } j = i. \end{cases} \quad (7)$$

Given a k -dimensional multi-index α , for $1 \leq i \leq k$, let

$$\gamma_i(\alpha) = \begin{cases} 0 & \text{if } \alpha_i = 0 \\ \frac{(|\alpha|-1)!}{{}_- \alpha^{(i)}!} & \text{if } \alpha_i > 0. \end{cases} \quad (8)$$

Remark 3. Note that the definition of γ_i is an extension of the convention that $(r!)^{-1} = 0$ if r is a negative integer. Before proceeding further, let us explicitly compute for a couple of multi-indices the quantities defined in (6), (7), and (8). Let $k = 3$ and $\alpha = (1, 4, 2)$. Then

$$\begin{aligned} +\alpha^{(1)} &= (2, 4, 2), \quad +\alpha^{(2)} = (1, 5, 2), \quad +\alpha^{(3)} = (1, 4, 3) \\ -\alpha^{(1)} &= (0, 4, 2), \quad -\alpha^{(2)} = (1, 3, 2), \quad -\alpha^{(3)} = (1, 4, 1) \\ \gamma_1(\alpha) &= 15, \quad \gamma_2(\alpha) = 60, \quad \gamma_3(\alpha) = 30. \end{aligned}$$

Let $k = 3$ and $\alpha = (3, 0, 5)$. Then

$$\begin{aligned} +\alpha^{(1)} &= (4, 0, 5), \quad +\alpha^{(2)} = (3, 1, 5), \quad +\alpha^{(3)} = (3, 0, 6) \\ -\alpha^{(1)} &= (2, 0, 5), \quad -\alpha^{(2)} \text{ is undefined, } -\alpha^{(3)} = (3, 0, 4) \\ \gamma_1(\alpha) &= 21, \quad \gamma_2(\alpha) = 0, \quad \gamma_3(\alpha) = 35. \end{aligned}$$

In both examples

$$\gamma_1(\alpha) + \gamma_2(\alpha) + \gamma_3(\alpha) = \frac{|\alpha|!}{\alpha!},$$

a fact that is true in general and is the assertion of Lemma 1.

Lemma 1. *Given a k -dimensional multi-index β ,*

$$\frac{|\beta|!}{\beta!} = \sum_{i=1}^k \gamma_i(\beta). \quad (9)$$

Proof. Since by definition

$$\gamma_i(\beta) = \frac{|\beta|!}{\beta!} \times \frac{\beta_i}{|\beta|} \text{ and } \sum_{i=1}^k \beta_i = |\beta|,$$

the algebraic proof of (9) follows. □

Remark 4. The identity in (9) can (and should) be interpreted in terms of a selection problem. To get there, let us recall the following interpretation of (2). Clearly, the right-hand side of (2) is the number of samples of size j that can be chosen from a population of size $q + 1$. Let us mark an element of the population as E . The collection of samples of size j can be partitioned into two subcollections, where one subcollection consists of all the samples of size j that include E and the other subcollection consists of all the samples of size j that exclude E . This partition is mutually exclusive and exhaustive. Since the first term in the left-hand side of (2) is equal to the number of samples of size j that include E and the second term is equal to the number of samples of size j that exclude E , the assertion of (2) is immediate.

Now consider a population of size $q + 1$, elements of which are to be partitioned into k mutually exclusive and exhaustive subsets of the population, say $\{G_1, \dots, G_k\}$, where the size of G_i is β_i , $1 \leq i \leq k$; that is, $\boldsymbol{\beta} = (\beta_1, \dots, \beta_k)$ is a k -dimensional multi-index with

$$|\boldsymbol{\beta}| = \sum_{i=1}^k \beta_i = q + 1.$$

Let \mathcal{P} be the collection of all such partitions. Note that the left-hand side of (9) is the cardinality of \mathcal{P} . Let us once again mark an element of the population as E . For $i \in \{1, \dots, k\}$, let \mathcal{P}_i be the subcollection of all partitions that place E in G_i . The fact that the subsets G_i are mutually exclusive and exhaustive implies that $\{\mathcal{P}_1, \dots, \mathcal{P}_k\}$ is a mutually exclusive and exhaustive partition of \mathcal{P} , so that

$$\text{left-hand side of (9)} = \sum_{i=1}^k \text{cardinality of } \mathcal{P}_i.$$

Note that

$$\text{cardinality of } \mathcal{P}_i = \begin{cases} 0 & \text{if } \beta_i = 0 \\ \frac{q!}{(\prod_{j \neq i} \beta_j!)^{(\beta_i - 1)!}} & \text{if } \beta_i > 0; \end{cases} \quad (10)$$

since right-hand side of (10) = $\gamma_i(\boldsymbol{\beta})$, (9) follows.

Proof of Theorem 1. We first observe that for $n = 1$, the formula in (5) reduces to the formula in (4); that is, the formula in (5) holds for $n = 1$. The number of k -dimensional multi-indices $\boldsymbol{\alpha}$ such that $|\boldsymbol{\alpha}| = 1$ is k ; they can be enumerated as $\{\mathbf{e}_i : 1 \leq i \leq k\}$, where \mathbf{e}_i has 1 in the i^{th} coordinate and 0 elsewhere. Since $\mathbf{e}_i! = 1$ and by convention $f_j^{(0)} = f_j$, the formula in (5) reduces to

$$g^{(1)} = \sum_{i=1}^k f_i^{(1)} \prod_{\{j \neq i\}} f_j,$$

which is precisely the formula in (4).

To prove the formula in (5) by induction on n , let us assume that (5) holds for $n = m$, that is,

$$g^{(m)} = \sum_{\{\boldsymbol{\alpha}: |\boldsymbol{\alpha}|=m\}} \frac{m!}{\boldsymbol{\alpha}!} \left(\prod_{i=1}^k f_i^{(\alpha_i)} \right). \quad (11)$$

From (11), by linearity of the operation of differentiation,

$$g^{(m+1)} = \sum_{\{\boldsymbol{\alpha}: |\boldsymbol{\alpha}|=m\}} \frac{m!}{\boldsymbol{\alpha}!} D \left(\prod_{i=1}^k f_i^{(\alpha_i)} \right), \quad (12)$$

where D denotes the differential operator. Using the formula for calculating the first derivative of the product of k functions, that is, the equality in (4), we obtain via (6)

$$D \left(\prod_{i=1}^k f_i^{(\alpha_i)} \right) = \sum_{i=1}^k f_i^{(\alpha_i+1)} \prod_{\{j \neq i\}} f_j^{(\alpha_j)} = \sum_{i=1}^k \left(\prod_{j=1}^k f_j^{(+\alpha_j^{(i)})} \right). \tag{13}$$

Substituting (13) in (12),

$$g^{(m+1)} = \sum_{\{\alpha:|\alpha|=m\}} \frac{m!}{\alpha!} \sum_{i=1}^k \left(\prod_{j=1}^k f_j^{(+\alpha_j^{(i)})} \right). \tag{14}$$

Interchanging the orders of (finite) summation over i and α in (14),

$$g^{(m+1)} = \sum_{i=1}^k \sum_{\{\alpha:|\alpha|=m\}} \frac{m!}{\alpha!} \left(\prod_{j=1}^k f_j^{(+\alpha_j^{(i)})} \right) = \sum_{i=1}^k T_i, \tag{15}$$

where

$$T_i = \sum_{\{\alpha:|\alpha|=m\}} \frac{m!}{\alpha!} \left(\prod_{j=1}^k f_j^{(+\alpha_j^{(i)})} \right). \tag{16}$$

Now let us fix $i \in \{1, 2, \dots, k\}$. Note that for every k -dimensional multi-index α such that $|\alpha| = m$, $+\alpha^{(i)}$ is a k -dimensional multi-index such that $|+\alpha^{(i)}| = m + 1$. Conversely, for every k -dimensional multi-index β such that $|\beta| = m + 1$ and $\beta_i > 0$, there exists a k -dimensional multi-index $\alpha (= -\beta^{(i)})$ such that $|\alpha| = m$ and $\beta = +\alpha^{(i)}$. Therefore, by a change of variable in the summation in the right-hand side of (16),

$$\begin{aligned} T_i &= \sum_{\{\beta:|\beta|=m+1, \beta_i>0\}} \frac{m!}{-\beta^{(i)}!} \left(\prod_{j=1}^k f_j^{(\beta_j)} \right) \\ &= \sum_{\{\beta:|\beta|=m+1\}} \gamma_i(\beta) \left(\prod_{j=1}^k f_j^{(\beta_j)} \right), \end{aligned} \tag{17}$$

where the second equality follows from the definition of γ_i in (8). Substituting in (15) the expression for T_i obtained in (17),

$$g^{(m+1)} = \sum_{i=1}^k \sum_{\{\beta:|\beta|=m+1\}} \gamma_i(\beta) \left(\prod_{j=1}^k f_j^{(\beta_j)} \right). \tag{18}$$

Interchanging the orders of (finite) summation over β and i in (18),

$$g^{(m+1)} = \sum_{\{\beta:|\beta|=m+1\}} \sum_{i=1}^k \gamma_i(\beta) \left(\prod_{j=1}^k f_j^{(\beta_j)} \right). \tag{19}$$

Since $\prod_{j=1}^k f_j^{(\beta_j)}$ does not depend on i , it follows from (19) and (9) that

$$g^{(m+1)} = \sum_{\{\beta:|\beta|=m+1\}} \left(\prod_{j=1}^k f_j^{(\beta_j)} \right) \sum_{i=1}^k \gamma_i(\beta) = \sum_{\{\beta:|\beta|=m+1\}} \frac{(m+1)!}{\beta!} \left(\prod_{j=1}^k f_j^{(\beta_j)} \right),$$

thereby completing the proof of the theorem. □

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