Rose-Hulman Undergraduate Mathematics Journal

Volume 18
Issue 1

Article 10

Existence and Uniqueness of Solutions of an Einstein-Maxwell PDE System

Toby Aldape
Lewis and Clark College

Rose-Hulman Undergraduate Mathematics Journal

EXISTENCE AND UNIQUENESS OF SOLUTIONS OF AN EINSTEIN-MAXWELL PDE SYSTEM

Toby Aldape[a]

Volume 18, No. 1, Spring 2017

Sponsored by

Rose-Hulman Institute of Technology
Department of Mathematics
Terre Haute, IN 47803
mathjournal@rose-hulman.edu
scholar.rose-hulman.edu/rhumj

[a]Lewis and Clark College

Rose-Hulman Undergraduate Mathematics Journal
Volume 18, No. 1, Spring 2017

EXISTENCE AND UNIQUENESS OF SOLUTIONS OF AN EINSTEIN-MAXWELL PDE SYSTEM

Abstract. We consider a nonlinear coupled system of partial differential equations with asymptotic boundary conditions which is relevant in the field of general relativity. Specifically, the PDE system relates the factors of a conformally flat spatial metric obeying the laws of gravity and electromagnetism to its charge and mass distributions. The solution to the system is shown to be existent, smooth, and unique. While the discussion of the PDE assumes knowledge of physics and differential geometry, the proof uses only the PDE theory of flat space.

Acknowledgements: Thanks to Lewis and Clark College for allowing me to take their classes, and thanks especially to Iva Stavrov for teaching me, for suggesting the problem, and for her help with editing.

1 Introduction

In the Newtonian framework of gravity the gravitational potential, θ, is described by the equation

$$\Delta\theta = -4\pi\rho_m \tag{1}$$

where ρ_m is the mass density. In general relativity, gravitational "force" is seen to be caused by curved spatial geometry which evolves in time. This geometry is governed by a metric, measuring distances across a spatial slice. An equation similar to (1) arises from an examination of the conformally flat spatial metric

$$g = \theta^4\delta,$$

which arises naturally as a limit of spaces in which many, many Swarzschild solutions are superimposed (see Benko, Stavrov [1]). The mass density function is defined according to the metric as

$$R(g) = 16\pi\rho_m$$

where $R(g)$ signifies the scalar curvature of the metric g. We define the form

$$\omega = \omega(x^1, x^2, x^3)dx^1dx^2dx^3 = \rho_m dvol_g.$$

This form captures a metric independent notion of mass density. In the chosen coordinate system, $\omega(x^1, x^2, x^3) = \rho_m\theta^6$. Substituting this expression and evaluating the scalar curvature gives

$$-8\Delta\theta/\theta^5 = 16\pi\omega(x^1, x^2, x^3)/\theta^6$$

and so

$$\Delta\theta \cdot \theta = -2\pi\omega(x^1, x^2, x^3).$$

We move the 2π factor into the θ functions. Constant factors in the metric are irrelevant since they can be eliminated by a coordinate change. We are left with

$$\Delta\theta \cdot \theta = -\omega(x^1, x^2, x^3). \tag{2}$$

When θ is required to approach 1 along the boundary (infinity), solutions to this equation are known to be existent and unique. Furthermore, it is not hard to see that this equation parallels (1) with the conformal factor θ standing in for the gravitational potential. Equation (1) also has analogies within the Newtonian model of gravity and electromagnetism and within a relativistic model incorporating both gravity and electromagnetism, as we will see.

In the presence of charge, the Newtonian gravitational and electric potentials may be described by two Poisson-type equations:

$$\Delta\theta_m = -\rho_m,$$

$$\Delta\theta_e = -\rho_e.$$

When we let $\chi = \theta_m + \theta_e$, $\psi = \theta_m - \theta_e$, $\omega_1 = -\rho_m - \rho_e$, and $\omega_2 = -\rho_m + \rho_e$, we get

$$\begin{aligned} \Delta\chi &= \omega_1, \\ \Delta\psi &= \omega_2. \end{aligned} \tag{3}$$

It is required that $\rho_m \geq |\rho_e|$, therefore $\omega_1 \leq 0$ and $\omega_2 \leq 0$. Assuming suitable asymptotic decay of ω_1 and ω_2 and boundary conditions which require χ and ψ to approach some constant at infinity, solutions to this equation are once again existent and unique.

Now, in general relativity, a similar equation relating the metric to mass and charge distributions may be derived by guessing the metric

$$g = (\chi\psi)^2\delta$$

under the constraints

$$\begin{aligned} R(g) &= 16\pi\rho_m + 2\|\overrightarrow{E}\|^2 \\ \operatorname{div}\overrightarrow{E} &= 4\pi\rho_e, \end{aligned} \tag{4}$$

and a third equation relating ρ_m to $\Delta\chi$ and $\Delta\psi$. These constraints, and a definition of ω_1 and ω_2 as metric independent forms which are related to $-\rho_m - \rho_e$ and $-\rho_m + \rho_e$, respectively, lead to the equations

$$\begin{aligned} \Delta\chi \cdot \psi &= \omega_1 \\ \Delta\psi \cdot \chi &= \omega_2 \end{aligned}$$

where $\omega_1 \leq 0$ and $\omega_2 \leq 0$. These equations are analogous to both (2) and (3) for relativity in the absence of electromagnetism, and electromagnetism in the the absence of relativity, respectively. In analogy/extension of similar results for (1), (2) and (3) this paper proves the existence, uniqueness, and smoothness of solutions to the following problem:

$$\begin{aligned} &\text{Given smooth, compactly supported, non-positive functions } \omega_1 \text{ and } \omega_2 \text{ with} \\ &\text{their domain in } \mathbb{R}^3, \text{ find } \chi \text{ and } \psi \text{ for which} \\ &\chi, \psi > 0 \\ &\chi, \psi \to 1 \text{ as } \|x\| \to \infty \\ &\Delta\chi \cdot \psi = \omega_1, \\ &\Delta\psi \cdot \chi = \omega_2. \end{aligned} \tag{5}$$

The techniques used in the existence proof were inspired by a proof of the method of sub and super solutions from the lecture notes of Professor Iva Stavrov. As these have not been published, we have included a reference to a similar proof by Kazdan and Warner [2]. The proof of existence constitutes the bulk of the paper, but we believe the uniqueness proof is more original and of more value to the reader. To find it, we imagined two sets of solutions (χ_1, ψ_1) and (χ_2, ψ_2) satisfying equations (6) and (7), then simplified the problem by letting

$\omega_1 = \omega_2$, $\psi_2 = \chi_1$ and $\psi_1 = \chi_2$. These equalities reduce the two equations to one. We changed the form of this equation, and found a new, linear PDE solved by both χ_1 and χ_2 by fixing some variables and letting others vary. Since the equation is linear, linearly combining the the distinct solutions χ_1 and χ_2 produces a one parameter family of functions satisfying the equation and the boundary conditions imposed on χ_1 and χ_2. Some of these functions will have positive minimums and will therefore be unable to solve the equation under consideration. This contradiction disproves the assumption of two distinct solutions to the original problem. The actual proof is, of course, different, since in general it is not possible to assume $\omega_1 = \omega_2$ or $\psi_2 = \chi_1$ and $\psi_1 = \chi_2$, but this method inspired the proof in full generality.

In the first part of the existence section, Section 2, we build two sequences (χ_n and ψ_n) whose limits (χ_* and ψ_*) seem to satisfy our PDE system and which have properties allowing us to prove that their limits actually exist (Theorem 2.1). With this task accomplished, we prove that not only do the sequences converge pointwise, they also converge within the Sobolev spaces $H^k(B(r))$. We break the proof of this result into the base case Theorem 2.2 and an induction argument that follows in the proof of Theorem 2.3. From this result it follows that the sequence limits are smooth and satisfy the PDE system. In the last step of the existence proof, we demonstrate that the limits χ_* and ψ_* satisfy the boundary conditions of the problem.

Section 3 presents the uniqueness proof as a single theorem. We use proof by contradiction and a reformulation of the PDE system to show there is only one solution set solving our problem, not two.

Lastly, the appendix contains three technical lemmas concerning Cauchy sequences (used in Theorem 2.2), regularity (in Theorem 2.3), and the continuity of the infimum of a certain group of continuous functions (in Theorem 3.1).

2 Existence

To prove existence of solutions, we construct sequences χ_n and ψ_n which solve PDEs approaching our PDE as $n \to \infty$. Then we show that χ_n and ψ_n are Cauchy in the Sobolev spaces $H^k(B[r])$ for all positive integers k and all $r > 0$. Sequences χ_n and ψ_n will therefore converge to smooth solutions to the problem, χ and ψ.

Theorem 2.1. *Given smooth, non-positive, compactly supported functions ω_1 and ω_2, there exist sequences of functions χ_n and ψ_n and functions ψ_+ and χ_+ for which*

(i) $\psi_0 = 1$,

(ii) $\Delta\chi_n \cdot \psi_n = \omega_1$ *for each n,*

(iii) $\Delta\psi_{n+1} \cdot \chi_n = \omega_1$ *for each n,*

(iv) ψ_n *and* χ_n *are smooth functions for each n,*

(v) *the sequence* ψ_n *is non-decreasing.*

(vi) *the sequence* χ_n *is non-increasing.*

(vii) ψ_+ *and* χ_+ *approach 1 at infinity,*

(viii) $1 \leq \psi_n \leq \psi_+$ *and* $1 \leq \chi_n \leq \chi_+$*,*

(ix) ψ_+ *and* χ_+ *are bounded above.*

Proof. As ω_1 and ω_2 are continuous and compactly supported, they are also bounded. This allows us to define $\chi_+ = 1 - \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{\omega_1(y)}{\|y-x\|}dy$ and $\psi_+ = 1 - \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{\omega_2(y)}{\|y-x\|}dy$. The integrands are compactly supported and approach 0 for large x, thus χ_+ and ψ_+ approach 1 near infinity. The functions χ_+ and ψ_+ are differentiable by the Leibniz rule, therefore they are continuous. Their continuity and their boundary values imply that χ_+ and ψ_+ are bounded above. Also, $\psi_+ \geq 1$ and $\chi_+ \geq 1$, since ω_1 and ω_2 are non-positive. Note that $\Delta\chi_+ = \omega_1$ and $\Delta\psi_+ = \omega_2$. We show that for each k there are partial sequences ψ_0,ψ_1,...,ψ_{k+1} and χ_0,χ_1,...,χ_k satisfying (i)-(ix). This will demonstrate the existence of non-terminating sequences ψ_n and χ_n satisfying equations (i)-(ix).

Base case: Let $\psi_0 = 1$. Let $\chi_0 = \chi_+$. Clearly $1 \leq \chi_0 \leq \chi_+$, χ_0 approaches 1 at infinity, and χ_0 is smooth. Since $\Delta\chi_0 = \omega_1$, (i) is satisfied for $n = 0$. Let ψ_1 be the smooth function approaching 1 at infinity for which $\Delta\psi_1 = \omega_2/\chi_0$. Such a function exists because ω_2/χ_0 is bounded and compactly supported. This satisfies (ii) for $n = 0$. It is also smooth, and $1 \leq \psi_1 \leq \psi_+$. Since

$$\Delta\psi_1 = \omega_2/\chi_0 \leq 0 = \Delta\psi_0$$

and

$$\Delta\psi_1 \geq \omega_2 = \Delta\psi_+$$

it follows from the weak maximum principle that

$$\psi_0 \leq \psi_1 \leq \psi_+.$$

Inductive step: Assume that there exist finite sequences of smooth functions ψ_0, ψ_1,...,ψ_k and χ_0, χ_1,...,χ_{k-1} satisfying equations (i)-(ix). Because ω_1/ψ_{k+1} is bounded and compactly supported, we may define χ_k as the smooth function approaching 1 at infinity and satisfying $\Delta\chi_k = \omega_1/\psi_k$. This function satisfies (ii) for $n = k$. Furthermore

$$\Delta\chi_k = \omega_1/\psi_k \geq \omega_1/\psi_{k-1} = \Delta\chi_{k-1}$$

and

$$\Delta\chi_k = \omega_1/\psi_k \leq 0.$$

By the weak maximum principle

$$1 \leq \chi_k \leq \chi_{k-1} \leq \chi_+.$$

The function ω_2/χ_k is bounded and compactly supported. This allows us to define ψ_{k+1} as the smooth function approaching 1 at infinity and satisfying $\Delta\psi_{k+1} = \omega_2/\chi_k$. This satisfies (iii) for $n = k$. Furthermore

$$\Delta\psi_{k+1} = \omega_1/\chi_k \leq \omega_2/\chi_{k-1} = \Delta\psi_k$$

and

$$\Delta\psi_{k+1} = \omega_1/\chi_k \geq \omega_1 = \Delta\psi_+.$$

By the weak maximum principle

$$1 \leq \psi_k \leq \psi_{k+1} \leq \psi_+.$$

This completes our inductive proof. □

Since ψ_n and χ_n are both bounded and monotonic sequences, they converge pointwise. In order to prove that their pointwise limits satisfy the PDE, we need convergence in $C^2(B[r])$ (with $B[r]$ the closed ball of radius r centered about the origin) for any choice of r to move a limit through the Laplacians in equations (ii) and (iii). By proving even stronger regularity results for ψ_n and χ_n we will show that their limits, the functions solving the PDE, are in fact smooth. We start by proving the base case of the induction proof that ψ_n and χ_n are Cauchy sequences in all H^k and C^k spaces over balls.

Theorem 2.2. *Let $G_1(x, y) = \omega_1(x)/y$. Let $G_2(x, y) = \omega_2(x)/y$. For all positive r, the sequences ψ_n and χ_n are Cauchy in $H^2(B[r])$ and $C^0(B[r])$, and the sequences $G_1(x, \psi_n(x))$ and $G_2(x, \chi_n)$ are Cauchy in $L^2(B[r])$.*

Proof. Let $r > 0$. Recall that ω_1 and ω_2 are bounded. Since χ_n and ψ_n are bounded below by 1, $\Delta\chi_n = \omega_1/\psi_n$ and $\Delta\psi_n = \omega_2/\chi_{n-1}$ are uniformly pointwise bounded, and therefore uniformly bounded in $L^2(B[r + 2])$. By the Elliptic Regularity Theorem there exists a constant C such that

$$\|\psi_n\|_{H^2(B[r+1])} \leq C(\|\Delta\psi_n\|_{L^2(B[r+2])} + \|\psi_n\|_{L^2(B[r+2])}).$$

It follows that the uniform boundedness of ψ_n and $\Delta\psi_n$ in $L^2(B[r + 2])$ implies the uniform boundedness of the sequence ψ_n in $H^2(B[r+1])$. The Rellich Lemma asserts the existence of a subsequence ψ_{n_k} which converges in $H^1(B[r+1])$ and, in particular, is Cauchy in $L^2(B[r+1])$. Since the sequence ψ_n is monotonic and has a Cauchy subsequence in $L^2(B[r + 1])$, ψ_n itself is Cauchy in $L^2(B[r + 1])$. In case this fact is not obvious, it is proven in Lemma 4.1.

Let ψ_{++} be an upper bound on ψ_+ (and let χ_{++} be an upper bound on χ_+). The partial derivatives of G_1 in y are continuous. Because the set $B[r] \times [1, \psi_{++}]$ is compact we may define $C' = \max_{B[r]\times[1,\psi_{++}]} |\frac{\partial}{\partial y} G_1(x, y)|$. The Mean Value Theorem then shows

$$|G_1(x, \psi_m(x)) - G_1(x, \psi_l(x))| \leq C'|\psi_m(x) - \psi_l(x)|.$$

Changing this into a statement about norm,

$$\|G_1(x, \psi_m(x)) - G_1(x, \psi_l(x)\|_{L^2(B[r+1])} \leq C'\|\psi(x)_m - \psi(x)_l\|_{L^2(B[r+1])}.$$

Because ψ_n is Cauchy in $L^2(B[r+1])$, $G_1(x, \psi_n(x)) = \Delta\chi_n$ is Cauchy in $L^2(B[r+1])$. A similar argument shows $G_2(x, \chi_{n-1}) = \Delta\psi_n$ is Cauchy in $L^2(B[r+1])$. By the Elliptic Regularity Theorem, there is a constant C for which

$$\|\psi_m - \psi_l\|_{H^2(B[r])} \leq C''(\|\Delta\psi_m - \Delta\psi_l\|_{L^2(B[r+1])} + \|\psi_m - \psi_l\|_{L^2(B[r+1])}).$$

Because $\Delta\psi_n$ and ψ_n are Cauchy in $L^2(B[r+1])$, ψ_n is Cauchy in $H^2(B[r])$. A similar argument shows χ_n is Cauchy in $H^2(B[r])$. The Sobolev Embedding Theorem then implies that ψ_n and χ_n are Cauchy under $C^0(B[r])$ norm. □

Now comes the induction proof.

Theorem 2.3. *For all integers $k \geq 0$ and all $r > 0$, ψ_n and χ_n are Cauchy in $H^k(B[r])$.*

Proof. We apply a bootstrap argument to show that ψ_n and χ_n are Cauchy in $C^k(B[r])$ and $H^k(B[r])$ for all non-negative integers k and all $r > 0$. Let $r > 0$.

Base Case: By Theorem 2.2, ψ_n and χ_n are Cauchy in $C^0(B[r])$ and $H^2(B[r])$, $G_1(x, \psi_n)$ and $G_2(x, \chi_n)$ are Cauchy in $H^0(B[r])$ for all positive r.

Inductive Step: Assume ψ_n and χ_n are Cauchy in $C^k(B[r+1])$ and $H^{k+2}(B[r+1])$ for all $r > 0$. The sequence ψ_n is bounded between the constants 1 and ψ_{++}, and the sequence χ_n is bounded between the constants 1 and χ_{++}. The sequences are clearly also Cauchy in $H^{k+1}(B[r+1])$. From Lemma 4.2 in the appendix it follows that $G_1(x, \psi_n)$ and $G_2(x, \chi_n)$ are Cauchy in $H^{k+1}(B[r+1])$. Therefore $\Delta\chi_n = G_1(x, \psi_n)$ and $\Delta\psi_n = G_2(x, \chi_n)$ must be Cauchy in $H^{k+2}(B[r+1])$. By the Elliptic Regularity Theorem, there exists a C''' for which

$$\|\psi_m - \psi_l\|_{H^{k+3}(B[r])} \leq C'''(\|\Delta\psi_m - \Delta\psi_l\|_{H^{k+1}(B[r+1])} + \|\psi_m - \psi_l\|_{L^2(B[r+1])})$$

It follows that ψ_n must be Cauchy in $H^{k+3}(B[r])$. The Sobolev Embedding Theorem implies ψ_n and χ_n are Cauchy in $C^{k+1}(B[r])$. The principle of induction gives us, in particular, that ψ_n and χ_n are Cauchy in $C^{k+2}(B[r])$ for all non-negative integers k. □

Theorem 2.4. *There exist smooth solutions to* (5)*.*

Proof. The following remarks hold for any positive r and any non-negative integers k. By Theorem 2.3, ψ_n and χ_n are Cauchy in $H^{k+2}(B[r])$. As the Sobolev spaces are complete, ψ_n and χ_n converge in $H^{k+2}(B[r])$ for any r and k. By the Sobolev Embedding Theorem, the limits ψ_* and χ_* are elements of $C^k(B[r])$ and the sequences converge to these functions in $C^k(B[r])$. Because this applies for all positive integers k, ψ_* and χ_* must be smooth. Taking the pointwise limit of equations(ii) and (iii) gives

$$\Delta\chi_* \cdot \psi_* = \omega_1$$

$$\Delta\psi_* \cdot \chi_* = \omega_2$$

Moving the limit through the product is possible since $\psi_n, \chi_n \in C^0(B[r])$ for all r. Moving the limit through the laplacians is possible because convergence of χ_n and ψ_n occurs in $C^2(B[r])$. Also, these equations work for all x because each x is a member of some closed ball centered around the origin. It is only left show that ψ_* and χ_* satisfy the boundary condition. Since, for all n, $1 \leq \psi_n \leq \psi_+$ and $1 \leq \chi_n \leq \chi_+$ it follows that $1 \leq \psi_* \leq \psi_+$ and $1 \leq \chi_* \leq \chi_+$. The functions 1, ψ_+, and χ_+ all approach 1 at infinity, thus ψ_* and χ_* must also approach 1 at infinity. □

3 Uniqueness

We assume the existence of two distinct solutions to the problem, then modify them to produce a family of solutions to a related system of equations. We show that there must be members of this family with positive minimums, which yields a contradiction.

Theorem 3.1. *Solutions to* (5) *are unique.*

Proof. Assume that two distinct solutions to the problem exist. More explicitly, assume

$$\Delta\chi_1 \cdot \psi_1 = \omega_1 = \Delta\chi_2 \cdot \psi_2 \tag{6}$$

and

$$\Delta\psi_1 \cdot \chi_1 = \omega_2 = \Delta\psi_2 \cdot \chi_2 \tag{7}$$

where χ_1, ψ_1 and χ_2, ψ_2 satisfy the problem conditions and $\chi_1 \not\equiv \chi_2$ or $\psi_1 \not\equiv \psi_2$. It is easily seen that $\chi_1 \equiv \chi_2$ implies $\psi_1 \equiv \psi_2$ and vice-versa. Therefore $\chi_1 \not\equiv \chi_2$ if and only if $\psi_1 \not\equiv \psi_2$. From the assumptions that χ_1 and ψ_2 are positive while ω_1 and ω_2 are non-positive, it follows that $\Delta\psi_1 \leq 0$ and $\Delta\chi_2 \leq 0$, which implies $\psi_1 \geq 1$ and $\chi_2 \geq 1$.

Subtracting the far right side of the above equations from the far left gives

$$\Delta(\chi_1 - \chi_2)\psi_1 + \Delta\chi_2(\psi_1 - \psi_2) = 0$$

$$\Delta\psi_1(\chi_1 - \chi_2) + \Delta(\psi_1 - \psi_2)\chi_2 = 0$$

We set $v = \chi_1 - \chi_2$ and $w = \psi_1 - \psi_2$ getting

$$\Delta v \cdot \psi_1 + \Delta\chi_2 \cdot w = 0 \tag{8}$$

$$\Delta w \cdot \chi_2 + \Delta\psi_1 \cdot v = 0 \tag{9}$$

It must also be true that

$$\Delta v(\psi_1 + kw) + \Delta(\chi_2 - kv)w = 0 \tag{10}$$

$$\Delta w(\chi_2 - kv) + \Delta(\psi_1 + kw)v = 0 \tag{11}$$

for any real k. The functions $\inf_{x\in\mathbb{R}^3}(\chi_2 - kv)$ and $\inf_{x\in\mathbb{R}^3}(\psi_1 + kw)$ are continuous in k, by Lemma 4.3. Therefore, $\min(\inf_{x\in\mathbb{R}^3}(\chi_2 - kv), \inf_{x\in\mathbb{R}^3}(\psi_1 + kw))$ is also continuous in k. Evaluated at $k = 0$, $\min(\inf_{x\in\mathbb{R}^3}(\chi_2 - kv), \inf_{x\in\mathbb{R}^3}(\psi_1 + kw)) > 1/2$. Since $v \not\equiv 0$ there exists a $k \in \mathbb{R}$ at which $\min(\inf_{x\in\mathbb{R}^3}(\chi_2 - kv), \inf_{x\in\mathbb{R}^3}(\psi_1 + kw)) < 1/2$. It follows from the Intermediate Value Theorem that there exists a k_* at which

$$\min(\inf_{x\in\mathbb{R}^3}(\chi_2 - k_*v), \inf_{x\in\mathbb{R}^3}(\psi_1 + k_*w)) = 1/2.$$

In particular, both $\chi_2 - k_*v > 0$ and $\chi_2 + k_*w > 0$, and there exists a point p at which $\chi_2(p) - k_*v(p) < 1$ or $\psi_1(p) + k_*w(p) < 1$.

We present the argument in the case where the first inequality holds. The other case may be treated similarly. At some point $q \in \mathbb{R}^3$, $\Delta(\chi_2 - k_*v)(q) > 0$, because otherwise the weak maximum principle implies that $\chi_2 - k_*v \geq 1$ everywhere. Because $\Delta(\chi_2 - k_*v)(q) > 0$ yet $\Delta\chi_2 \leq 0$, it follows that $k_*\Delta v(q) < 0$. Multiplying (10) by k_* gives

$$(k_*\Delta v)(\psi_1 + k_*w) + \Delta(\chi_2 - k_*v)(k_*w) = 0$$

Since $k_*\Delta v(q) < 0$, $(\psi_1 + k_*w)(q) > 0$, and $\Delta(\chi_2 - k_*v)(q) > 0$, it follows that $k_*w(q) > 0$. Multiplying (8) by k_* gives

$$k_*\Delta v \cdot \psi_1 + \Delta\chi_2 \cdot k_*w = 0$$

But plugging $k_*\Delta v(q) < 0$, $\psi_1 > 0$, $\Delta\chi_2 \leq 0$ and $k_*w(q) > 0$ into (9) shows $0 < 0$, a contradiction.

The assumed statement, that $\chi_1 \not\equiv \chi_2$ or $\psi_1 \not\equiv \psi_2$, is false. Solutions to PDE problem (5) are unique. □

4 Appendix

This section contains proofs of three technical lemmas referenced in the paper.

Lemma 4.1. *If the sequence of functions f_n is monotonic (weakly), and has a subsequence f_{n_k} Cauchy in $L^2(B[r])$, then f_n is Cauchy in $L^2(B[r])$.*

Proof. Assume for simplicity that f_n is non-decreasing. Let $\epsilon > 0$. Let K_1 be a number such that for any $j, k \geq K_1$, $\|f_{n_j} - f_{n_k}\|_{L^2(B[r+1])} < \epsilon$. Then for $l, m > n_{K_1}$ there is a K_2 for which $n_{K_1} < l, m < n_{K_2}$. It follows from the monotonicity of f_n that $f_{n_{K_1}} \leq f_l, f_m \leq f_{n_{K_2}}$. Therefore

$$\|f_m - f_l\|_{L^2(B[r+1])} < \|f_{m_{K_2}} - f_{m_{K_1}}\|_{L^2(B[r+1])} < \epsilon.$$

□

Lemma 4.2. *If the sequence of functions f_n is bounded between constants a and b, is Cauchy in $C^k(B[r])$ and $H^{k+1}(B[r])$ for some $k \geq 0$, and $G(x, y)$ is a smooth function whose arguments are in $\mathbb{R}^3$ and $\mathbb{R}$, respectively, then $G(x, f_n(x))$ is Cauchy in $H^{k+1}(B[r])$.*

Proof. Let α be a multiset representing derivatives with respect to x for which $|\alpha| \leq k+1$. It suffices to show that $D^\alpha G(x, f_n(x))$ is Cauchy in $L^2(B[r])$. By the product rule, the function $D^\alpha G(x, f_n(x))$ is a sum of terms of the form

$$D^X D^Y G|_{(x,f_n(x))} \cdot D^\beta f_n$$

where X and Y represent derivatives of the first and second components of G respectively, β is a multi-index for which $\beta = (\beta_1, \beta_2, ..., \beta_{|Y|})$ and $D^\beta \psi_n = D^{\beta_1}\psi_n \cdot D^{\beta_2}\psi_n \cdot ... \cdot D^{\beta_{|Y|}}\psi_n$, $|X| + |Y| \leq k+1$ and $\sum_{i=1}^{|Y|} |\beta_i| \leq k+1$. Therefore it is enough to show that such terms are Cauchy in $L^2(B[r])$.

The term is of the form $D^X D^Y G|_{(x,f_n(x))} \cdot D^\beta f_n$. The smoothness of G allows us to define $C''' = \max_{B[r]\times[a,b]} \frac{\partial}{\partial y} D^X D^Y G$. By the Mean Value Theorem

$$D^X D^Y G|_{(x,f_m(x))} - D^X D^Y G|_{(x,f_l(x))} \leq C'''(f_m(x) - f_l(x)).$$

Now $D^X D^Y G|_{(x,f_n(x))}$ is Cauchy in $C^0(B[r])$ since f_n is Cauchy in $C^0(B[r])$.

Case 1: $|Y| \neq 1$. The term has either no derivatives of f_n or multiple derivatives of f_n in its product. Crucially, for each β_i, $|\beta_i| \leq k$ and $D^{\beta_i} f_n \in C^0(B[r])$ since $f_n \in C^k(B[r])$. For reasons mentioned above $D^X D^Y G|_{(x,f_n)}$ is also Cauchy in $C^0(B[r])$. It follows that $D^X D^Y G|_{(x,f_n(x))} \cdot D^{\beta_1} f_n \cdot ... \cdot D^{\beta_{|Y|}} f_n$ is Cauchy in $C^0(B[r])$, therefore Cauchy in $L^2(B[r])$.

Case 2: $|Y| = 1$. In this case it is more difficult to prove that the term is Cauchy, since it could include a partial derivative of f_n of order $k+1$. Our previous argument would fail because f_n is not necessarily in $C^{k+1}(B[r])$.

$D^\beta f_n$ is Cauchy in $L^2(B[r])$ because $|\beta| \leq k+1$ and f_n is Cauchy in $H^{k+1}(B[r])$. Now, since

$$\begin{aligned} D^X D^Y G|_{(x,f_m(x))} \cdot D^\beta|f_m - D^X D^Y G|_{(x,f_l(x))}) \cdot D^\beta f_l & \\ = (D^X D^Y G|_{(x,f_m(x))} - D^X D^Y G|_{(x,f_l)}) \cdot D^\beta f_m & \\ + D^X D^Y G|_{(x,f_l(x))} \cdot (D^\beta f_m - D^\beta f_l), & \end{aligned}$$

the fact that $D^X D^Y G|_{(x,f_n(x))}$ is Cauchy (and thus bounded) in $C^0(B[r])$ and $D^\beta f_n$ is Cauchy (and thus bounded) in $L^2(B[r])$ implies $D^X D^Y G|_{(x,f_l(x))} \cdot D^\beta f_n$ is Cauchy in $L^2(B[r])$. □

Lemma 4.3. *Let f be some function which is bounded below. Let g be some bounded function. Then $\inf(f - kg)$ is a continuous function of k.*

Proof.

$$\inf((k_2 - k_1)g) \leq \inf(f - k_1 g) - \inf(f - k_2 g) \leq \sup((k_2 - k_1)g)$$

As g is bounded, the function $(k_2 - k_1)g$ can be made arbitrarily close to zero by choosing a sufficiently small $(k_2 - k_1)$. Thus $\inf(f - kg)$ is a continuous function of k. □

References

[1] Benko, T., Stavrov Allen, I., *Intrinsic flat limit and discretizations of charged dust*, in preparation.

[2] Kazdan, J.L., Warner, F.W., *Curvature functions for compact* 2*-manifolds,* Ann. Math. (2), **99** (1974).