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An Alternating Sum of Alternating Sign Matrices

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Abstract

An alternating-sign matrix (ASM) is a square matrix with entries from $\{-1, 0, 1\}$, row and column sums of 1, and in which the nonzero entries in each row and column alternate in sign. ASMs have many non-trivial parameters and symmetries that reveal their significant combinatorial structure. In this note, we will prove an identity that relates one parameter and one symmetry.

An alternating-sign matrix (ASM) is a square matrix with entries from $\{-1, 0, 1\}$, row and column sums of 1, and in which the nonzero entries in each row and column alternate in sign. In the early 1980s, D. Robbins and H. Rumsey discovered ASMs, which can be thought of as generalizations of permutation matrices, through their work on λ -determinants [2]. With W. Mills, they advanced several conjectures related to determining the number A_n of $n \times n$ ASMs—but their relatively simple formulas turned out to be surprisingly difficult to prove [9]. In fact, it wasn't until 1995 that D. Zeilberger published a proof of the ASM Conjecture, confirming their formula for A_n [3, 10]. Amusingly, Zeilberger organized his paper into lemmas, sublemmas, subsublemmas, and so on to (sub)⁶lemmas, and recruited 88 people (as well as his computer, Shalosh B. Ekhad) as referees. Each referee was assigned a single (sub)^{*i*}lemma with the charge to make sure that it followed from any (sub)^{*i*+1}lemmas [10].

The sequence A_n begins 1, 2, 7, 42, 429, 7436, . . . (Sloane's A005130), and the general formula is given below.

Theorem 1 (The ASM Conjecture) *For $n \geq 1$,*

$$A_n = \prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!}.$$

It would be difficult to guess this elegant formula for A_n based only on the first few terms of the sequence, but the ASM Conjecture was systematically developed once Robbins introduced a useful parameter to better attack the problem. Figure 1 lists the seven 3×3 ASMs, of which six are the 3×3 permutation matrices. The reader may notice that they each have exactly one 1 in their top row.

Theorem 2 (The Refined ASM Conjecture) For $n \geq 1$ and $1 \leq k \leq n$,

$$A_{n,k} = \binom{n+k-2}{k-1} \frac{(2n-k-1)!}{(n-k)!} \prod_{j=0}^{n-2} \frac{(3j+1)!}{(n+j)!}.$$

In 1996, Zeilberger published a proof of the Refined ASM conjecture [11] drawing from techniques G. Kuperberg introduced in his proof of the ASM conjecture [4]. For more information about ASMs, see [2], [3], and [7].

We will also use vertically symmetric ASMs (VSASMs) in our identity. Robbins again conjectured a formula for the number of VSASMs [8], which Kuperberg then proved [5]. We will use an equivalent formula that was later proved by A. Razumov and Y. Stroganov [6]. By considering the position of the 1 in the top row, it is clear that there are no $2n \times 2n$ VSASMs. The 5×5 VSASMs are given in Figure 2.

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 1 & -1 & 1 & -1 & 1 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

Figure 2. The three 5×5 VSASMs

Theorem 3 (Kuperberg) The number of $2n+1 \times 2n+1$ VSASMs, with $n \geq 0$, is

$$V(2n+1) = \frac{1}{2^n} \prod_{k=1}^n \frac{(6k-2)!(2k-1)!}{(4k-1)!(4k-2)!}.$$

It is well-known that the sum of the entries in the n th row of Pascal's triangle is 2^n , while the alternating sum is 0 if $n \geq 1$. Since we know the sum of the entries in the n th row of the ASM triangle is A_n , it is therefore natural to ask for the corresponding alternating sum. In fact, the original form of the Refined ASM Conjecture says that the ratio of entries in the ASM triangle looks like the ratio of a sum of entries from Pascal's triangle—so our analogy seems appropriate [2].

Theorem 4 For $n \geq 1$, let $S(n) = \sum_{k=1}^n (-1)^{k+1} A_{n,k}$. Then $S(n) = V(n)^2$.

Proof. First, we note that the alternating sum $S(2n)$ is zero, by symmetry of the $A_{n,k}$. Therefore we have only to consider $S(2n+1)$.

Starting with the Refined ASM Conjecture we can write

$$S(2n+1) = \sum_{k=1}^{2n+1} (-1)^{k+1} A_{2n+1,k}$$

$$\begin{aligned}
&= \sum_{k=1}^{2n+1} \left((-1)^{k+1} \binom{2n+k-1}{k-1} \frac{(4n-k+1)!}{(2n-k+1)!} \prod_{j=0}^{2n-1} \frac{(3j+1)!}{(2n+1+j)!} \right) \\
&= \left(\sum_{k=1}^{2n+1} (-1)^{k+1} \binom{2n+k-1}{k-1} \binom{4n-k+1}{2n-k+1} \right) \left((2n)! \prod_{j=0}^{2n-1} \frac{(3j+1)!}{(2n+1+j)!} \right)
\end{aligned}$$

Reindexing to have the sum begin at zero, and writing the product $(2n)! \prod_{j=0}^{2n-1} \frac{(3j+1)!}{(2n+1+j)!}$ as Π , we have

$$\frac{S(2n+1)}{\Pi} = \sum_{k=0}^{2n} (-1)^k \binom{2n+k}{k} \binom{4n-k}{2n-k}.$$

If we apply the identity $\binom{b+i-1}{i} = (-1)^i \binom{-b}{i}$ to both binomial coefficients in the sum, we get

$$\frac{S(2n+1)}{\Pi} = \sum_{k=0}^{2n} (-1)^k \binom{-(2n+1)}{k} \binom{-(2n+1)}{2n-k}.$$

Now, the coefficient of x^{2n} in the product $(1-x)^{-(2n+1)}(1+x)^{-(2n+1)}$ is exactly this sum. Since $(1-x)^{-(2n+1)}(1+x)^{-(2n+1)} = (1-x^2)^{-(2n+1)}$, we can equate coefficients of x^{2n} to obtain the identity

$$\sum_{k=0}^{2n} (-1)^k \binom{-(2n+1)}{k} \binom{-(2n+1)}{2n-k} = \binom{3n}{n}.$$

This is actually a special case of Kummer's ${}_2F_1$ identity—the proof we show above is from [1, Remark 3.4.1]. Therefore,

$$S(2n+1) = \binom{3n}{n} \Pi = \binom{3n}{n} (2n)! \prod_{j=0}^{2n-1} \frac{(3j+1)!}{(2n+1+j)!} = \frac{(3n)!}{n!} \prod_{j=0}^{2n-1} \frac{(3j+1)!}{(2n+1+j)!}.$$

It is not immediately clear that this number is actually a square. Regardless, we must now only check that this formula really is the same as that for $V(2n+1)^2$.

We proceed by induction. For both sides, evaluation at $n = 1$ gives one. Now, by our above simplification, we have

$$\frac{S(2n-1)}{S(2n+1)} = \left(\frac{n}{(3n)(3n-1)(3n-2)} \right) \left(\frac{(4n)!(4n-1)!(4n-2)!(4n-3)!}{(6n-2)!(6n-5)!(2n)!(2n-1)!} \right).$$

On the other hand, we also have

$$\frac{V(2n-1)^2}{V(2n+1)^2} = \left(\frac{2(4n-1)!(4n-2)!}{(6n-2)!(2n-1)!} \right)^2.$$

We leave it to the reader to check that these two expressions are equal, completing the proof. \square

We conclude by providing a combinatorial interpretation of this identity. Because the central column of a VSASM has no zeros, we can represent an ordered pair of VSASMs as a single ASM with a non-zero middle column by choosing the left half from the first VSASM and the right half from the second. Thus, $V(2n + 1)^2$ as the number of ASMs that have a central column with no zeros. It would therefore be interesting to see a direct bijection from $n \times n$ ASMs with their top-most one in an odd numbered column to the disjoint union of $n \times n$ ASMs with top-most one in an even numbered column and $n \times n$ ASMs whose middle column contains no zeros.

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References

- [1] G. Andrews, R. Askey, and R. Roy. *Special Functions*. Encyclopedia of Mathematics and its Applications, 71. Cambridge University Press, 1999.
- [2] D. Bressoud. *Proofs and Confirmations: The story of the alternating sign matrix conjecture*. Cambridge University Press, 1999.
- [3] D. Bressoud and J. Propp. How the alternating sign matrix conjecture was solved. *Notices Amer. Math. Soc.*, 46(6):637–646, 1999.
- [4] G. Kuperberg. Another proof of the alternating-sign matrix conjecture. *Internat. Math. Res. Notices*, 3:139–150, 1996.
- [5] G. Kuperberg. Symmetry classes of alternating-sign matrices under one roof. *Ann. of Math.*, 156(3):835–866, 2002.
- [6] A. Razumov and Y. Stroganov. On refined enumerations of some symmetry classes of ASMs. *Theor. Math. Phys.*, 141:1609-1630, 2004.
- [7] D. Robbins. The story of 1, 2, 7, 42, 429, 7436, \dots . *Math. Intelligencer*, 13:12–19, 1991.
- [8] D. Robbins. Symmetry classes of alternating sign matrices. arXiv:math/0008045v1, 2000.
- [9] D. Robbins and H. Rumsey. Determinants and alternating sign matrices. *Adv. in Math.*, 62(2):169–184, 1986.
- [10] D. Zeilberger. Proof of the alternating sign matrix conjecture. *Electron. J. Comb.*, 3 (2) Research Paper 13, 1996.

- [11] D. Zeilberger. Proof of the refined alternating sign matrix conjecture. *New York J. Math.*, 2:59–68, 1996.