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Analytical Solution of the Symmetric Circulant Tridiagonal Linear System

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Abstract
A circulant tridiagonal system is a special type of Toeplitz system that appears in a variety of problems in scientific computation. In this paper we give a formula for the inverse of a symmetric circulant tridiagonal matrix as a product of a circulant matrix and its transpose, and discuss the utility of this approach for solving the associated system.

1 Introduction
A real $N \times N$ matrix $C$ is said to be Toeplitz if $c_{i,j} = c_{i+1,j+1}$ (the matrix is constant along diagonals). A Toeplitz matrix is circulant if $c_{i,j} = c_{i+1,j+1}$ where the matrix is constant along diagonals, with row-wise wrap-around. We write $C = \text{circ}(c_0, \ldots, c_{N-1})$ to indicate the circulant matrix with first row $c_1, \ldots, n$.

Circulant matrices appear in many applications in scientific computing, including computational fluid dynamics [1], numerical solution of integral equations [2], [3], preconditioning Toeplitz matrices [3], and smoothing data [4]. Linear systems involving circulant matrices may be solved efficiently in $O(n \log n)$ operations using three applications of the Fast Fourier Transform (FFT) [3].

Circulant matrices may be banded. The $N \times N$ circulant tridiagonal matrix is the matrix $C = \text{circ}(c_0, c_1, 0, \ldots, 0, c_{N-1})$. If in addition $c_1 = 0$, we say that it is circulant lower bidiagonal; if instead $c_{N-1} = 0$, we say that it is circulant upper bidiagonal. The eigenvalues of the circulant tridiagonal matrix $\text{circ}(c_0, c_1, 0, \ldots, 0, c_1)$ are known to be

$$\lambda_i = c_0 + 2c_1 \cos \left( \frac{2\pi i}{N} \right), \quad i = 0, \ldots, N-1$$

Keywords and phrases: circulant matrix, circulant tridiagonal matrix, LU decomposition
In this paper we will focus on the symmetric circulant tridiagonal matrix in a normalized form that appears in a number of applications, including computational fluid dynamics [1]:

\[
\Gamma = \begin{pmatrix}
1 & a & 0 & 0 & a \\
0 & 1 & 0 & 0 & a \\
a & 0 & 1 & a & 0 \\
0 & a & 0 & 1 & a \\
a & 0 & 0 & a & 1
\end{pmatrix}
\] (2)

(shown for \(N = 5\)). In our case,

\[
\lambda_i = 1 + 2a \cos \left(\frac{2\pi i}{N}\right), \ i = 0, ..., N - 1
\] (3)

so that \(\Gamma\) is singular if \(a = -1/2\) \((i = 0)\) or if \(a = 1/2\) and \(N\) is even \((i = N/2)\). Note that for \(-1/2 < a < 1/2\), \(\Gamma\) is strictly diagonally dominant and, from (3), positive definite. Hence we expect it to be well-behaved numerically; in fact, we can easily generate its eigenvalues and use \(|\lambda_{\text{max}}| / |\lambda_{\text{min}}|\) as a check on its conditioning [5].

The inverse of a (symmetric) positive definite Toeplitz matrix such as \(\Gamma\) may be computed in \(O(n^2)\) operations [6]. Although the general circulant linear system \(Cx = b\) may be solved in \(O(n \log n)\) operations, Chen [5] develops a special LU decomposition for the strictly diagonally dominant symmetric circulant tridiagonal matrix \(c_0 \Gamma\), in the form \(c_0 \Gamma = \alpha \hat{L} \hat{U}\) where \(\hat{L}\) is lower bidiagonal and \(\hat{U}\) is upper bidiagonal, then solves \(c_0 \Gamma x = b\) as \(\alpha \hat{L} \hat{U} x = b\) with the aid of two applications of the Sherman-Morrison formula. The resulting algorithm is \(O(n)\) (about \(5n\) operations versus about \(12n \log_2 n\) for the general FFT-based approach).

We will use a convolution algebra and a \(z\)-transform [8] idea to develop a formula of the form \(\Gamma^{-1} = \gamma M M^T\), with \(M\) a circulant matrix that is dependent upon a single parameter. Once \(M\) and \(\gamma\) are known, \(\Gamma x = b\) may be solved as \(x = \gamma M (M^T b)\).

## 2 The Convolution Algebra

Consider \(\mathbb{Z}_N\), the cyclic group of integers mod \(N\), and take the convolution algebra \(\mathbb{C}(\mathbb{Z}_N)\) to be the complex vector space of all functions defined on \(\mathbb{Z}_N\), with convolution product \(*\) defined by

\[
f * g(r) = \sum_{k=0}^{N-1} f(k) g(r - k) \mod N
\]

giving an associative and commutative \(\mathbb{C}\)-algebra with multiplicative identity.

We use the time sample basis

\[
\delta_0, ..., \delta_{N-1}
\] (4)
for $\mathbb{C}(\mathbb{Z}_N)$, where $\delta_i(j) = \delta_{i,j}$ (the Kronecker delta function). Given any $f \in \mathbb{C}(\mathbb{Z}_N)$,

$$f = c_0\delta_0 + \ldots + c_{N-1}\delta_{N-1}$$

where $c_j = f(j)$, and so we may identify $f$ with the column vector $[c_0, c_1, \ldots, c_{N-1}]^T$. Also noting that $\delta_i \ast \delta_j = \delta_{i+j}$ (indices mod $N$) convolution products are easily calculated using basis expansion above and we see that $\delta_0$ serves as the multiplicative identity $1 \in \mathbb{C}(\mathbb{Z}_N)$.

To relate $\mathbb{C}(\mathbb{Z}_N)$ to circulant matrices, fix an $f \in \mathbb{C}(\mathbb{Z}_N)$ and use it to define a linear transformation

$$L_f : \mathbb{C}(\mathbb{Z}_N) \to \mathbb{C}(\mathbb{Z}_N)$$

by $L_f(g) = f \ast g$. The matrix of this linear transformation with respect to the basis (4) is

$$C = \text{circ}(c_0, c_{N-1}, c_{N-2}, \ldots, c_1)$$

(and so by proper choice of $f$ we may arrange for $C$ to be any desired circulant matrix). By associativity,

$$L_{f \ast g}(h) = (f \ast g) \ast h = f \ast (g \ast h) = L_f(L_g(h))$$

and hence $f \to L_f$ is an algebra isomorphism onto the subalgebra of circulant matrices. Hence we can find the inverse of the matrix $C$ by finding the inverse of $f$ in the convolution algebra.

### 3 The Symmetric Circulant Tridiagonal Case

We want to invert (2), $\Gamma = \text{circ}(1, a, 0, \ldots, 0, a)$, when it is nonsingular. The representer polynomial [4] for $\Gamma$ would be $p_\Gamma(z) = 1 + az + az^{N-1}$ (so that $p_\Gamma(1/z)$ is the corresponding $z$-transform), and similarly, the element of $\mathbb{C}(\mathbb{Z}_N)$ corresponding to $\Gamma$ is

$$f = 1\delta_0 + a\delta_1 + a\delta_{N-1} = 1 + a\delta_1 + a\delta_{N-1}$$

which we seek to factor as

$$f = c(1-r\delta_1)(1-r\delta_{N-1})$$

i.e. as $f = cf_1f_{-1}$, where $f_1 = 1-r\delta_1$, $f_{-1} = 1-r\delta_{N-1}$ (cf. the factorization into a product of circulant bidiagonals in [5]; in particular, $L_{f_1}$ is circulant lower bidiagonal and $L_{f_{-1}}$ is circulant upper bidiagonal). If we can find these factors, then we will have $L_f^{-1} = \gamma L_{f_1}^{-1}L_{f_{-1}}^{-1}$, where $\gamma = 1/c$. Comparing (5) and (6), we see that
\[
\begin{align*}
c(1 + r^2) &= 1 \\
cr &= -a
\end{align*}
\]
is required. If \(a = 0\) then \(\Gamma = I_N\); otherwise,

\[
\begin{align*}
 r_{1,2} &= \frac{-1 \pm \sqrt{1 - 4a^2}}{2a} \\
c_{1,2} &= \frac{1 \pm \sqrt{1 - 4a^2}}{2}
\end{align*}
\]
(which are complex when \(|a|\) exceeds \(1/2\); \(c_1\) is Chen’s \(\alpha\) in \(\Gamma = \alpha \hat{L} \hat{U}\)). Choose \((r, c) = (r_i, c_i)\) for \(i = 1\) or \(i = 2\). Since

\[
(1 - r_1 \delta_1)(1 + r_1 \delta_1 + r_2 \delta_2 + ... + r^{N-1} \delta_{N-1}) = 1 - r^N
\]
we have

\[
(1 - r_1 \delta_1)^{-1} = \frac{1}{1 - r^N} \delta_0 + \frac{r}{1 - r^N} \delta_1 + \frac{r^2}{1 - r^N} \delta_2 + ... + \frac{r^{N-1}}{1 - r^N} \delta_{N-1}
\]
and so \(L_{f_1}^{-1}\) has the matrix representation

\[
M = \frac{1}{1 - r^N} \text{circ}(1, r^{N-1}, r^{N-2}, ..., r)
\]
and similarly, the matrix representation of \(L_{f_i}^{-1}\) is found to be \(M^T\). From (6), then,

\[
\Gamma^{-1} = \gamma MM^T
\]
where \(\gamma = 1/c\), and \(c\) is nonzero when \(|a| < 1/2\). Because of the factor \(1/(1 - r^N)\), the value of \(r_{1,2}\) furthest from unity should usually be chosen (unless the corresponding \(c\) value is extremely small).

Solving \(Cx = b\) for the general symmetric circulant tridiagonal case \(C = \text{circ}(c_0, c_1, 0, ..., 0, c_1)\) is easily handled. We have

\[
C = c_0 \text{circ}(1, c_1/c_0, 0, ..., 0, c_1/c_0) = c_0 \Gamma
\]
if \(c_0\) is nonzero, and from (1) we see that \(C\) must have at least one null eigenvalue if \(c_0 = 0\).
4 Discussion

The method discussed here advances previous work by giving explicit formulas for the inverses of the two circulant bidiagonal factors. In addition, for $N$ odd our formula is valid for the weakly diagonally dominant case $a = 1/2$. But because $M$ is dense, solution of $\Gamma x = b$ by the use of (7) in the form

$$x = \gamma M(M^T b)$$

(8)

requires two circulant-matrix-by-vector multiplications, each of which requires three FFTs [3]. Hence the method is $O(n \log n)$ once the first row of $M$ is computed. Although we could simplify this somewhat after diagonalizing $M$ by the Fourier matrix [4], it will typically be less efficient than using the the $LU$ decomposition $\Gamma = \alpha \hat{L} \hat{U}$ in conjunction with the Sherman-Morrison formula, which requires approximately $5n$ operations, or when $\Gamma$ is not strictly diagonally dominant, directly solving $\Gamma x = b$ as a general circulant system using three FFTs.

Significantly, however, our formula applies whenever $\Gamma$ is nonsingular. It is apparent from (3) that for any fixed $N$ there are up to $N$ values of $a$ that may make $\Gamma$ singular, viz.

$$a = \frac{-1}{2 \cos \left( \frac{2\pi i}{N} \right)}$$

for $i = 0, \ldots, N - 1$; in fact, there are $1 + \lfloor N/2 \rfloor$ such distinct values of $a$. If we are willing to use complex arithmetic in (8) then we may solve $\Gamma x = b$ by this formula whenever $\Gamma$ admits an inverse. (Note that (8) and (7) remain correct as written; the transpose does not become the Hermitian transpose when $|a| > 1/2$.) Thus, the choices $(r, c) = (r_i, c_i)$ for $i = 1, 2$ give two distinct (if $a \neq 1/2$) decompositions of $\Gamma^{-1}$ whenever it exists.

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References

[1] Lui, Calvin, private communication

