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Superelliptic surfaces as p-gonal surfaces

S. Allen Broughton

Dedicated to Emilio Bujalance for his sixtieth anniversary

Abstract. In this brief, expository paper, we discuss superelliptic surfaces and p-gonal surfaces, which generalize hyperelliptic surfaces. A superelliptic surface, or more generally, a p-gonal surface, has a conformal automorphism w of prime order such that $S/\langle w\rangle$ has genus zero. Alternatively, the surface has an equation of the form $y^p = f(x)$ for some rational function $f(x)$. We discuss normal forms, automorphism groups, and families of p-gonal surfaces.

1. Introduction

There is a strong interest in superelliptic and p-gonal surfaces. Historically, the surfaces, especially hyperelliptic surfaces, were linked to the study of certain integrals. Most recently, they are of interest in cryptography and fields of moduli. Among all surfaces, p-gonal surfaces are the surfaces with the simplest and most tractable equations. In this brief article we talk about normal forms of p-gonal surfaces, their automorphisms, and very briefly about families of p-gonal surfaces. Space does not allow discussion of applications to cryptography or fields of definition of superelliptic surfaces. See [**19**] and [**20**] for references on these topics.

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2. The equation point of view of p-gonal surfaces

2.1. Hyperelliptic surfaces. Hyperelliptic surfaces (curves) were introduced in studying the surface of the function $\sqrt{f(x)}$ and related integrals $\int \sqrt{f(x)}dx$ where $f(x)$ is a rational function, or alternatively the Riemann surface determined

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by $y^2 = f(x)$. By using elementary algebra we may assume that $f(x)$ is a square free polynomial. For, we may write $f(x) = u^2(x)v(x)$ where $v(x)$ is a square free polynomial and $u(x)$ is a rational function. Setting $y' = \frac{y}{u(x)}, x' = x$ we see that

$$(y')^2 = \frac{y^2}{u^2(x)} = \frac{u^2(x)v(x)}{u^2(x)} = v(x').$$

The transformation $\phi(x, y) = (x', y') = (\frac{y}{u(x)}, x)$ is a birational transformation of $\mathbb{P}^2$ carrying the surface $y^2 = f(x)$ to $(y')^2 = v(x')$. Since we are only interested in surfaces up to birational equivalence, let us assume that $f(x)$ is a square free polynomial and that S is the surface (projective plane curve) defined by

$$y^2 = f(x). \tag{2.1}$$

All of the finite points of S are smooth. At infinity the equation of S has the form $y^2 - x^t h(1/x) = \left(y - x^{t/2}\sqrt{h(1/x)}\right)\left(y + x^{t/2}\sqrt{h(1/x)}\right)$ for some polynomial h with nonzero constant term. If $t = deg(f)$ is even, then the projective completion of S has two branches at infinity otherwise it has a cusp. The normalization $S^\nu \to S$ is the smooth compactification of $S - \{\text{singular points}\}$. Therefore, the normalization $S^\nu \to S$ is 1-1 over the finite points of S and has one or two points lying over the infinite points of S, depending on the parity of t.

The map $\iota : (x, y) \to (x, -y)$ is an involution of S that fixes only the points $(a_i, 0)$ in the finite part of S. The involution ι lifts to the normalization $S^\nu \to S$ and we also denote this map by ι. The action of the involution ι on S^ν fixes all the lifts of the $(a_i, 0)$. In the case S^ν has a single point at infinity lying over the infinite point of S, the lift of ι fixes the infinite point, in case there are two points, the lift interchanges them. The quotient map $\pi : S^\nu \to S^\nu / \langle\iota\rangle$ is given by $(x, y) \to x$ at the finite points and hence $S^\nu / \langle\iota\rangle \simeq \mathbb{P}^1$. So we arrive at another characterization of a hyperelliptic surface namely a smooth surface S with an involutory automorphism ι such that $S / \langle\iota\rangle \simeq \mathbb{P}^1$. Indeed, starting out with such a surface a plane model given by equation 2.1 can be found.

2.2. n-gonal surfaces and equations. Now generalizing, we consider surfaces of the form $y^n = f(x)$ motivated by the study of the function $\sqrt[n]{f(x)}$. These surfaces are called *cyclic n-gonal surfaces*. Using calculations as above, we can show that S has a plane model of the form

$$y^n = f(x) = \prod_{i=1}^{s} (x - a_i)^{t_i} \tag{2.2}$$

where the a_i and t_i satisfy

(1) the a_i are distinct,
(2) $0 < t_i < n$, and
(3) $gcd(n, t_1, \ldots, t_s) = 1$.

The third condition comes from the assumed irreducibility of the surface, for otherwise $y^n - f(x)$ factors. In addition we may wish to impose the following

(4) n divides $t = t_1 + \cdots + t_s = deg(f)$.

The finite singular points of S are the points $(a_i, 0)$ where $t_i > 1$. By writing $y^n - f(x)$ as $y^n - (x - a_i)^{t_i}\phi_i(x)$ with $\phi_i(a_i) \neq 0$ we see that S has $d_i = gcd(t_i, n)$ local branches at $(a_i, 0)$. So the surface has a cusp or single branch at $(a_i, 0)$ if $\gcd(n, t_i) = 1$ and is even smooth if $t_i = 1$. It also follows that the normalization

$S^\nu \to S$ has d_i points lying over $(a_i, 0)$. If $t = deg(f) = n$ then S has n smooth points at infinity. Otherwise S has a single point at infinity. Writing, as before, $y^n - f(x) = y^n - x^t h(1/x)$ we see that in every case S has $gcd(t, n)$ local branches at infinity and the normalization $S^\nu \to S$ has $gcd(t, n)$ points lying over the infinite point(s) of S. We call S^ν the *smooth model* and S the *plane model* though we frequently loosely identify the two surfaces.

The projection $\pi : S^\nu \to \mathbb{P}^1$ induced by $(x, y) \to x$ is ramified over all of the points a_i with ramification degree $m_i = \frac{n}{d_i}$ and ramified over ∞ with degree $\frac{n}{\gcd(t,n)}$. If condition 4 holds $\pi : S^\nu \to \mathbb{P}^1$ has d_i points lying over a_i and n points lying over all other points, including the point ∞. By a simple ramified covering argument, the genus σ of S^ν is given by

$$\sigma = \frac{1}{2}\left(2 + (s-2)n - \sum_{i=1}^{s} d_i\right). \tag{2.3}$$

Sometimes it is convenient to shift the branch points away from infinity, i.e., to ensure that t is divisible by n. To this end pick r so that $t + r$ is divisible by n, $0 < r < n$ and pick any linear fractional transformation, L, of the coordinates

$$x' = L(x) = \frac{\alpha x + \beta}{\gamma x + \delta}, x = L^{-1}(x') = \frac{\delta x' - \beta}{-\gamma x' + \alpha}, \text{ and } \alpha\delta - \beta\gamma = 1.$$

It follows that

$$-\gamma x' + \alpha = \frac{1}{\gamma x + \delta}. \tag{2.4}$$

Further, impose the restrictions that

$$\forall i \ L(a_i) \neq \infty, \text{ and } \gamma \neq 0. \tag{2.5}$$

Next set $K = \prod_{i=1}^{s} (\gamma a_i + \delta)^{t_i}$, which is nonzero because of equation 2.5. Then, using $x = \frac{\delta x' - \beta}{-\gamma x' + \alpha}$, we obtain

$$\begin{aligned}
y^n &= f\left(\frac{\delta x' - \beta}{-\gamma x' + \alpha}\right) = \prod_{i=1}^{s} \left(\frac{\delta x' - \beta}{-\gamma x' + \alpha} - a_i\right)^{t_i} \\
&= K \frac{(-\gamma x' + \alpha)^r}{(-\gamma x' + \alpha)^{t+r}} \prod_{i=1}^{s} \left(x' - \frac{\alpha a_i + \beta}{\gamma a_i + \delta}\right)^{t_i} \\
&= K(-\gamma)^r (\gamma x + \delta)^{t+r} \left(x' - \frac{\alpha}{\gamma}\right)^r \prod_{i=1}^{s} (x' - L(a_i))^{t_i}
\end{aligned}$$

and so

$$\left(\frac{1}{\sqrt[n]{K(-\gamma)^r}} \frac{y}{(\gamma x + \delta)^{\frac{t+r}{n}}}\right)^n = \left(x' - \frac{\alpha}{\gamma}\right)^r \prod_{i=1}^{s} (x' - L(a_i))^{t_i}.$$

Now set

$$\begin{aligned}
y' &= \frac{1}{\sqrt[n]{K(-\gamma)^r}} \frac{y}{(\gamma x + \delta)^{\frac{t+r}{n}}}, \\
a_{s+1} &= \infty, \\
b_i &= L(a_i), \ b_{s+1} = \frac{\alpha}{\gamma} = L(\infty), \ t_{s+1} = r,
\end{aligned}$$

and we get

$$(y')^n = \prod_{i=1}^{s+1} (x' - b_i)^{t_i}. \tag{2.6}$$

Because $a_1, \ldots, a_{s+1}$ are distinct, so are $b_1, \ldots, b_{s+1}$ and we arrive at a birationally equivalent surface where $t_1 + \cdots + t_{s+1}$ is divisible by n. There are only a finite number of bad choices for the linear fractional transformation L.

For the remainder of this paper we will assume conditions 1-4 above, unless otherwise noted. We shall call $(a_1, a_2, \ldots, a_s)$ the branch points of S, t_i the local degree at a_i, and $(t_1, t_2, \ldots, t_s)$ the multi-degree of S. If condition 4 holds we call $(t_1, t_2, \ldots, t_s)$ a *complete multi-degree.*

Remark 2.1. If $deg(f)$ is not divisible by n then we add the additional branch point $a_{s+1} = \infty$, and set $t_{s+1} = r$ defined as above. Then 1-4 all hold. We call t_{s+1} the local degree at ∞ and $(t_1, t_2, \ldots, t_{s+1})$ is a complete multi-degree. It is convenient to also consider $y^n = f(x)$ when $f(x)$ is a rational function. A denominator factor of the form $(x - a_i)^{t_i}$, $0 < t_i < n$, contributes $-t_i$ to the multi-degree. Using the birational equivalence transformations above, the contribution $-t_i$ changes to $n - t_i$ when $f(x)$ is converted to normal polynomial form.

We may determine when two cyclic n-gonal surfaces are birationally equivalent using the following proposition.

Proposition 2.2. *Suppose that two cyclic n-gonal surfaces with the same multi-degree have branch points $(a_1, a_2 \ldots, a_s)$ and $(b_1, b_2 \ldots, b_s)$. Then the surfaces are conformally equivalent if there is an $L \in PSL_2(\mathbb{C})$ and a permutation $\vartheta \in \Sigma_s$, preserving multi-degrees ($t_{\vartheta i} = t_i$), so that*

$$b_i = L(a_{\vartheta i}).$$

for all i.

Proof. In the proof of equation 2.6 we can assume that $r = 0$ and then observe

$$\prod_{i=1}^{s} (x - L(a_i))^{t_i} = \prod_{i=1}^{s} (x - L(a_{\vartheta i}))^{t_{\vartheta i}} = \prod_{i=1}^{s} (x - b_i)^{t_i}$$

□

2.3. Cyclic n-gonal actions. If ω is a n^{th} root of unity, then $(x, y) \to (x, \omega y)$ is an automorphism of S which fixes the points $(a_i, 0)$ and no others in the finite part of S. Let C be the cyclic group of automorphisms obtained by letting ω range over all n^{th} roots of unity. The action of C on S, and its lift to S^ν, is called a *cyclic n-gonal action.* The map $\pi : S^\nu \to S \to \mathbb{P}^1$, $(x, y) \to x$ is a quotient map for the projection $S^\nu \to S^\nu/C$, and is called the *cyclic n-gonal morphism.* The degree of ramification of π over a_i is $m_i = n/gcd(t_i, n)$. In fact there are $d_i = gcd(t_i, n)$ points lying over a_i, and at each such point P the stabilizer of the C action, C_P, is the unique subgroup of C of order m_i. The quotient group C/C_P transitively permutes the points lying over $\pi(P)$. The map is unramified over ∞ because n divides t and there are n distinct points over ∞. Let w be the generator of C corresponding to $\omega = \exp(2\pi i/n)$. At any point $\widetilde{a}_j$ in S^ν lying over lying over a_j, the rotation number of w, i.e., the differential $dw_{|\widetilde{a}_j}$ is $\exp\left(\frac{2\pi i t_j}{n}\right)$.

Now, if S is any closed Riemann surface with a conformal automorphism w such that $S/\langle w\rangle \backsimeq \mathbb{P}^1$, then using standard field theory, it can be shown that the function field $\mathbb{C}(S) = \mathbb{C}\left(x, \sqrt[n]{f(x)}\right)$, for some $f(x)$. Hence, S has a plane model of the form given in equation 2.2, satisfying conditions 1-4 above. We state this as a proposition.

PROPOSITION 2.3. *A closed Riemann surface is a cyclic n-gonal surface if and only if either of the following two equivalent conditions hold.*

(1) *The surface S has a conformal automorphism w of order n such that $S/\langle w\rangle$ has genus zero*
(2) *The surface S has a plane model of the form given in equation 2.2 and conditions 1-4 in subsection 2.2 are satisfied.*

2.4. Cyclic p-gonal and superelliptic surfaces.

DEFINITION 2.4. A *cyclic p-gonal surface* is any closed Riemann surface S with a conformal automorphism w of prime order p such that $S/\langle w\rangle$ has genus zero or alternatively has a plane model of the form

$$y^p = f(x) = \prod_{i=1}^{s}(x-a_i)^{t_i}$$

where the a_i and t_i satisfy

(1) the a_i are distinct,
(2) $0 < t_i < p$, and
(3) p divides $t = t_1 + \cdots + t_s = deg(f)$.

REMARK 2.5. The singularities of the plane model of S are all cusps except possibly a point at infinity with p smooth branches. Moreover, in the genus formula 2.3 all $d_i = 1$ and hence the genus is given by

$$\sigma = \frac{1}{2}(s-2)(p-1). \tag{2.7}$$

Observe that σ only depends on the number of branch points and not on the multi-degree.

Finally, we give the standard definition of a superelliptic surface (see [**18**] for instance) and a generalization of superelliptic introduced discussed in [**8**] and [**9**].

DEFINITION 2.6. A *superelliptic surface* is any p-gonal surface with a plane smooth model of the form $y^p = f(x)$ where $f(x)$ is square free and p does not divide the degree of $f(x)$.

DEFINITION 2.7. A generalized superelliptic surface is any n-gonal surface with a plane model of the form $y^n = f(x)$ where the multi-degree $(t_1, \ldots, t_s)$ of $f(x)$ satisfies

(1) $0 < t_i < n$,
(2) $gcd(n, t_j) = 1$ for all s,
(3) n divides $t = t_1 + \cdots + t_s = deg(f)$.

REMARK 2.8. A superelliptic surface is simply a p-gonal surface given by Definition 2.4 where the multi-degree is $(1, 1, \ldots, 1, t_\infty)$ with $0 < t_\infty < p$. For a generalized superelliptic S the stabilizer C_P of a point $P \in S$ satisfies $C_P = C$ or $C_P = \langle 1 \rangle$. This is guaranteed by condition 2 of the definition. A p-gonal surface is automatically a generalized superelliptic surface but need not be superelliptic. For genus 3 there are two 7-gonal surfaces: the superelliptic surface given by $y^7 = x(x-1)$ with multi-degree $(1, 1, 5)$ and the non-superelliptic surface $y^7 = x(x-1)^2$ with multi-degree $(1, 2, 4)$.

3. The action point of view of p-gonal surfaces

We now use the cyclic action point of view to describe p-gonal surfaces. Here we will describe group theoretically a cyclic p-gonal surface as a branch set and a class of generating vectors of C. This allows us to handle equivalence without constructing birational maps. To this end, we introduce Fuchsian groups which captures both the conformal and automorphic structure of the surface.

3.1. Covering actions by Fuchsian groups. A (co-compact) Fuchsian group Γ, a discrete group acting on the hyperbolic plane $\mathbb{H}$, has a presentation by hyperbolic and elliptic generators and relations:

$$\text{generators}: \{\alpha_i, \beta_i, \gamma_j, 1 \le i \le \sigma, 1 \le j \le s\}$$

$$\text{relations}: \prod_{i=1}^{\sigma} [\alpha_i, \beta_i] \prod_{j=1}^{s} \gamma_j = \gamma_1^{m_1} = \cdots = \gamma_s^{m_s} = 1$$

with $m_i \geq 2$. The signature of Γ is

$$\mathcal{S}(\Gamma) = (\sigma : m_1, \ldots, m_s)$$

which we shorten to $(m_1, \ldots, m_s)$, when the genus is zero. Here are important invariants of and facts about Fuchsian groups.

- The *genus of* Γ: $\sigma(\Gamma) = \sigma$ is the genus of $\mathbb{H}/\Gamma$. If $s = 0$ then Γ is torsion free and usually denoted by Π, as it is isomorphic to $\pi_1(S)$.
- The *area of a fundamental region* of Γ in $\mathbb{H}$ is given by $A(\Gamma) = 2\pi\mu(\Gamma)$ where, $\mu(\Gamma) = 2(\sigma - 1) + \sum_{j=1}^{s} \left(1 - \frac{1}{m_j}\right)$.
- The *Teichmüller dimension* $d(\Gamma)$ of Γ, the dimension of the Teichmüller space of Fuchsian groups with signature $\mathcal{S}(\Gamma)$, is given by

$$d(\Gamma) = 3(\sigma - 1) + s.$$

- If $\gamma \in \Gamma$ fixes a point $z \in \mathbb{H}$ the γ is conjugate to a power of some γ_i.

For any group G acting conformally on a surface S we have pair of Fuchsian groups $\Pi \trianglelefteq \Gamma$ such that Π is torsion free, $S \backsimeq \mathbb{H}/\Pi$, and the action of G on S is given by the action of Γ/Π on $\mathbb{H}/\Pi$. The isomorphism is given by an exact sequence.

$$\Pi \hookrightarrow \Gamma \overset{\eta}{\twoheadrightarrow} G \tag{3.1}$$

such that the order of elliptic elements is preserved under η. The map η is called a *surface-kernel epimorphism*, and the isomorphism between Γ/Π and G is denoted by $\overline{\eta}: \Gamma/\Pi \longleftrightarrow G$. Also observe that $S/G \backsimeq \mathbb{H}/\Gamma$. All the proceeding is summarized in the following commutative diagram.

$$\begin{array}{ccc} \mathbb{H} & \overset{\pi_\Pi}{\to} & S \\ \pi_\Gamma \downarrow & & \downarrow \pi_G \\ \mathbb{H}/\Gamma & \backsimeq & S/G \end{array} \tag{3.2}$$

Now assume Γ is an arbitrary Fuchsian group with $\sigma = 0$, but not necessarily arising from a group action as in the exact sequence 3.1. We have $\mathbb{H}/\Gamma \backsimeq \mathbb{P}^1$ and our relations then simplify:

$$\prod_{j=1}^{s} \gamma_j = \gamma_1^{m_1} = \cdots = \gamma_s^{m_s} = 1. \tag{3.3}$$

Each γ_j fixes a unique point z_j such that π_Γ is ramified over $a_j = \pi_\Gamma(z_j) \in \mathbb{P}^1$ with degree m_j. In this case Γ is determined up to conjugacy by the branch points $(a_1, a_2, \ldots, a_s)$ and the signature $(m_1, \ldots, m_s)$. If $(a'_1, a'_2, \ldots, a'_s)$ is the branch set of a Γ' with signature $(m'_1, \ldots, m'_s)$ then Γ and Γ' are conjugate if and only if there is an $L \in PSL_2(\mathbb{C})$ and a permutation $\vartheta \in \Sigma_s$, such that

$$a'_i = L(a_{\vartheta i}),\ m'_i = m_{\vartheta i}.$$

3.2. Generating vectors. With η as in 3.1, set $c_j = \eta(\gamma_j)$, then the s-tuple $(c_1, \ldots, c_s)$ satisfies

$$\begin{array}{rcl} c_1 \cdot c_2 \cdots c_s & = & 1 \\ o(c_i) & = & m_i > 1 \\ G & = & \langle c_1, \ldots, c_s \rangle \end{array} \tag{3.4}$$

Such a tuple is called an $(m_1, \ldots, m_s)$-generating vector of G. Given such a vector then $c_j = \eta(\gamma_j)$ defines a surface kernel epimorphism and an action of G on $S = \mathbb{H}/\ker(\eta)$. If $\omega \in \mathrm{Aut}(G)$ then $c'_j = \omega(c_j)$ defines another generating vector such that the epimorphism $\eta' = \omega \circ \eta$ satisfies $c'_j = \eta'(\gamma_j)$ and $\ker(\eta) = \ker(\eta')$. We conclude that $\mathrm{Aut}(G)$-equivalent generating vectors define the same surface with $\mathrm{Aut}(G)$-equivalent G-actions.

Remark 3.1. If $G = \mathbb{Z}_n$ then generating vectors are in 1-1 correspondence with the s-tuples satisfying

(1) $0 < t_i < n$,
(2) $gcd(n, t_1, \ldots, t_s) = 1$
(3) $t_1 + \cdots + t_s = 0 \bmod n$.

Of course, these are simply the equations satisfied by multi-degrees in the previous section. The two multi-degrees $(t_1, \ldots, t_s)$ and $(et_1, \ldots, et_s) \bmod n$ are $\mathrm{Aut}(\mathbb{Z}_n)$-equivalent if $e \neq 0 \ mod\ n$ and these are the only equivalences. If $n = p$ is prime then $m_i = p$ for all i and 2 automatically holds. The number of generating vectors can be computed using inclusion-exclusion, see [**7**].

3.3. Conformally equivalent actions. We would like to determine when two conformal G-actions on surfaces are conformally equivalent. So assume that $(m_1, \ldots, m_s)$-actions of G on S_1 and S_2 are defined via surface kernel epimorphisms $\eta_1 : \Gamma_1 \to G$ and $\eta_2 : \Gamma_2 \to G$ then the conformal actions of G on S_1 and S_2 are conformally equivalent if and only if there is an $\phi \in \mathrm{Aut}(\mathbb{H})$, and $\omega \in \mathrm{Aut}(G)$ such that the following diagram commutes.

$$\begin{array}{ccccc} \Pi_1 & \hookrightarrow & \Gamma_1 & \overset{\eta_1}{\twoheadrightarrow} & G \\ \downarrow Ad_\phi & & \downarrow Ad_\phi & & \downarrow \omega \\ \Pi_2 & \hookrightarrow & \Gamma_2 & \overset{\eta_2}{\twoheadrightarrow} & G \end{array} \tag{3.5}$$

In the diagram, $\Pi_1 = \ker(\eta_1)$, $\Pi_2 = \ker(\eta_2)$, and $Ad_\phi(\gamma) = \phi\gamma\phi^{-1}$ denotes the adjoint action of ϕ on γ. There is an induced conformal equivalence $h : S_1 \leftrightarrow S_2$ which intertwines the G-actions on S_1 and S_2 :

$$\begin{array}{ccc} G & \overset{\overline{\eta_1}}{\hookrightarrow} & \mathrm{Aut}\,(S_1) \\ \downarrow \omega & & \downarrow Ad_h \\ G & \overset{\overline{\eta_2}}{\twoheadrightarrow} & \mathrm{Aut}\,(S_2) \end{array} \tag{3.6}$$

Next, we want to determine conformal equivalence of p-gonal surfaces simply in terms of the branch set and the multi-degree without producing the element ϕ, i.e., work directly on $\mathbb{P}^1$ and the generating vectors $(c_1, \ldots, c_s)$. To this end, we say that two generating vectors define topologically equivalent actions if there is a homeomorphism $\widetilde{h}$ of $\mathbb{H}$ normalizing Γ such that the following diagram commutes.

$$\begin{array}{ccc} \Gamma & \overset{\eta}{\twoheadrightarrow} & G \\ \downarrow Ad_{\widetilde{h}} & & \downarrow \omega \\ \Gamma & \overset{\eta'}{\twoheadrightarrow} & G \end{array} \tag{3.7}$$

In the case $\ker(\eta) = \ker(\eta') = \Pi$, there is a homeomorphism h of S and $\overline{h}$ of $\mathbb{P}^1$ such that such that the following diagrams are commutative.

$$\begin{array}{ccc} \mathbb{H} & \overset{\widetilde{h}}{\to} & \mathbb{H} \\ \downarrow \pi_\Pi & & \downarrow \pi_\Pi \\ S & \overset{h}{\to} & S \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathbb{H} & \overset{\widetilde{h}}{\to} & \mathbb{H} \\ \downarrow \pi_\Gamma & & \downarrow \pi_\Gamma \\ \mathbb{P}^1 & \overset{\overline{h}}{\to} & \mathbb{P}^1 \end{array} \tag{3.8}$$

Moreover, given an $\overline{h}$ on $\mathbb{P}^1$ mapping the branch set to itself, in an order-preserving manner, the covering $\widetilde{h}$ may be found. The map $\widetilde{h}$ is conformal if and only if $\overline{h}$ is conformal. A particular homeomorphism, that may be constructed, switches adjacent branch points inducing the switch $a_j \to a_{j+1}$, $a_{j+1} \to a_j$, $m_j \to m_{j+1}$, $m_{j+1} \to m_j$. In the superelliptic case the change in branching orders is always permissible. By lifting the appropriately chosen $\overline{h}$, we get $\gamma_j \to \gamma_j\gamma_{j+1}\gamma_j^{-1}$, $\gamma_{j+1} \to \gamma_j$, and hence $c_j \to c_jc_{j+1}c_j^{-1}$, $c_{j+1} \to c_j$. If G is abelian we simply get $c_j \to c_{j+1}$, $c_{j+1} \to c_j$. It can be shown that the totality of the homeomorphism action is induced by the switch maps and so the action on generating vectors is simply permutation. Thus we arrive at the following.

Proposition 3.2. *Let S be a cyclic p-gonal surface branched over $(a_1, a_2, \ldots, a_s)$ with generating vector (multi-degree) $(t_1, t_2, \ldots, t_s)$. If S' is another such surface with corresponding $(a'_1, a'_2, \ldots, a'_s)$ and $(t'_1, t'_2, \ldots, t'_s)$ then S and S' are conformally* *equivalent p-gonal surfaces if and only if there is $L \in PSL_2(\mathbb{C})$ and a permutation $\vartheta \in \Sigma_s$, and $e \neq 0$ mod p such that*

$$a_i' = L(a_{\vartheta i}), t_i' = et_{\vartheta i}.$$

REMARK 3.3. We note that the conformal equivalence can be constructed as a birational transformation, and so the Fuchsian group construction is not completely necessary. However, the statement extends to elementary abelian actions but there is no easy construction of birational maps. See [**7**].

4. Automorphism groups of cyclic n-gonal surfaces

There is a great deal of interest in the automorphism group $A = \mathrm{Aut}(S)$ of a cyclic n-gonal surface as these surfaces have tractable automorphism groups. Indeed, the automorphism group can generally be computed directly from the equations but the Fuchsian group methods are easier. Of special interest, is the normal case where $N = \mathrm{Nor}_A(C) = A$, which we discuss next. Automorphisms in $A - N$ are called *exceptional*. See [**8**], [**9**], and [**21**] for a discussion of these automorphisms.

4.1. The normal case. In the normal case, $K = N/C = A/C$ is an automorphism group of the sphere, one of five types of Platonic groups $\mathbb{Z}_k$, D_k, A_4, Σ_4, A_5. One "simply" solves an extension problem

$$C \hookrightarrow N \twoheadrightarrow K.$$

For large genus we automatically have $C \trianglelefteq A$ using Accola's theorem on strong branching [**1**]. In the next theorem we state the normality theorem in more general contexts because they all follow from Accola's work in exactly the same way.

THEOREM 4.1. *Let S be a cyclic n-gonal surface of genus σ with cyclic group C. Then the following hold.*

1. *If $n = p$ is a prime and $\sigma > (p-1)^2$, then C is normal in* $\mathrm{Aut}(S)$ [**1**].
2. *Suppose S is generalized superelliptic i.e., $gcd(n, t_i) = 1$ for all. Then, if $\sigma > (n-1)^2$, C is normal in* $\mathrm{Aut}(S)$ [**17**].
3. *Suppose the action of C is weakly malnormal i.e., for all $g \in \mathrm{Aut}(S)$ either $gCg^{-1} = C$ or $gCg^{-1} \cap C = \langle 1 \rangle$ Then, if $\sigma > (n-1)^2$, C is normal in* $\mathrm{Aut}(S)$ [**8**].

REMARK 4.2. Accola's method critically uses results from algebraic geometry and there is no group theoretic proof known to the author. For genus greater than 1 the hyperelliptic involution is always central. A result on centrality for superelliptic surfaces is given in Proposition 4.8. The involution is also unique, namely, for any surface S there can only be one involution ι such that $S/\langle \iota \rangle$ has genus zero. An analogue for p-gonal surfaces is that if $w_1, w_2 \in \mathrm{Aut}(S)$ of the same prime order p are such that $S/\langle w_1 \rangle$ and $S/\langle w_2 \rangle$ both have genus 0 then w_1 and w_2 are power-conjugate [**15**].

REMARK 4.3. Since $\sigma = \frac{1}{2}(s-2)(p-1)$ then the condition $\sigma > (p-1)^2$ is satisfied when $s > 2p$. So when p is of even moderate size, $N < A$, is possible for high degree polynomials.

Here are some works on the problem of determination of automorphism groups, in increasing generality on the properties of the cyclic action. See also [**9**].

- The case $n = 2$ (hyperelliptic case) has been studied extensively: Brandt, Bujulance, Etayo, Gamboa, Gromadzki, Martinez, Shaska – [**5**], [**10**], [**18**],
- The case where $n = 3$, (cyclic trigonal surfaces): Accola, Bujalance, Cirre, Costa, Duma, Gromadski, Izquierdo, Martinez, Radtke, Ying – [**2**], [**11**], [**14**], [**13**], [**23**].
- The case where $n = p$, for p a prime: Bartolini, Brandt, Costa, Gonzalez-Diez, Harvey, Izquierdo, Wootton – [**4**], [**5**], [**15**], [**16**], [**21**].
- General n where the cyclic n-gonal morphism $S \to S/C$ is fully ramified: Kontogeorgis [**17**].
- General n with weak malnormality conditions: Broughton & Wootton [**8**].
- The paper [**4**] includes some cases missed in previous works and in [**17**].

EXAMPLE 4.4. Here are some low genus p-gonal surfaces. The two surfaces with exceptional automorphisms ($N < A$) are the well-known Klein quartic and Bring's curve.

Table 1

genus	A	N	K	$\lvert A/N\rvert$	$\lvert C\rvert$	$(t_1, \ldots, t_s)$
3	$\mathbb{Z}_{14}$	$\mathbb{Z}_{14}$	$\mathbb{Z}_2$	1	7	$(1, 1, 5)$
3	$PSL_2(7)$	$\mathbb{Z}_3 \ltimes \mathbb{Z}_7$	$\mathbb{Z}_3$	8	7	$(1, 2, 4)$
4	Σ_5	$\mathbb{Z}_4 \ltimes \mathbb{Z}_5$	$\mathbb{Z}_4$	6	5	$(1, 2, 3, 4)$
4	$\mathbb{Z}_4 \ltimes \mathbb{Z}_5$	$\mathbb{Z}_4 \ltimes \mathbb{Z}_5$	$\mathbb{Z}_4$	1	5	$(1, 1, 4, 4)$
4	$\mathbb{Z}_{15} = \mathbb{Z}_3 \times \mathbb{Z}_5$	$\mathbb{Z}_3 \times \mathbb{Z}_5$	$\mathbb{Z}_3$	1	5	$(1, 1, 1, 2)$

The two normalizers $\mathbb{Z}_4 \ltimes \mathbb{Z}_5$ are not isomorphic.

4.2. Action on multi-degrees and examples. Let us finish this section by giving examples of defining equations and automorphism groups for $K = Z_\ell$ and $K = D_\ell$, where ℓ and p are relatively prime. The method we use is described in detail for all K in [**22**]. Before proceeding we need to know the adjoint action of K on C, and the K-action on the local degrees t_j.

The adjoint action, $Ad_g(x) = gxg^{-1}$ introduced in 3.3, defines a representation $Ad : N \to \mathrm{Aut}(C)$. Since C is abelian, Ad factors through K: $\overline{Ad} : K \to \mathrm{Aut}(C)$. Since $C = \mathbb{Z}_p$ has prime order $\mathrm{Aut}(C) = \mathbb{Z}_p^*$ is cyclic and this severely limits the K-action except in the case where $K = Z_\ell$. Next, let us see how K acts on the local degrees. Let $\widetilde{a}_i$ and $\widetilde{a}_j$ be points lying over branch points a_i and a_j, and suppose that there is an $h \in N$ such that $h(\widetilde{a}_i) = \widetilde{a}_j$ and hence the image $\overline{h} \in K$ satisfies $\overline{h}(a_i) = a_j$. The standard generator w of C fixes $\widetilde{a}_i$ and $\widetilde{a}_j$ and the rotation numbers of w are $dw_{|\widetilde{a}_i} = \exp(2\pi i t_i)$, $dg_{|\widetilde{a}_j} = \exp(2\pi i t_j)$. Since $h^{-1}wh = Ad_{h^{-1}}(w) = w^e$, for some e, then

$$\exp(2\pi i e t_i) = d(w^e)_{|\widetilde{a}_i} = dh^{-1} dw_{|\widetilde{a}_j} dh = dw_{|\widetilde{a}_j} = \exp(2\pi i t_j).$$

Thus

$$t_j = e t_i \bmod p, \text{ i.e., } t_j = \overline{Ad}(\overline{h}^{-1}) t_i \bmod p. \tag{4.1}$$

It follows that if $\overline{h}$ fixes a point then $\overline{Ad}(\overline{h}^{-1})$ acts trivially. The orbits of K on $\mathbb{P}^1$ will be important in what follows. We call an orbit *singular* if the points have non-trivial K-stabilizers, otherwise the orbits are *regular*.

EXAMPLE 4.5. Suppose $K = \mathbb{Z}_\ell$. We use the action of $K = \mathbb{Z}_\ell$ on $\mathbb{P}^1$ generated by $x \to \zeta x$ where $\zeta = exp(\frac{2\pi i}{\ell})$. The group N must be $\mathbb{Z}_\ell \times \mathbb{Z}_p = \mathbb{Z}_{\ell p}$ or $\mathbb{Z}_\ell \ltimes \mathbb{Z}_p$ with the $\mathbb{Z}_\ell$ action is generated by $j \to e'j$ on the additive group $\mathbb{Z}_p$, where $(e')^\ell = 1$ mod p, Pick e so that $ee' = 1$ mod p. The action has two fixed points at 0 and ∞. Suppose first, that none of the branch points is 0 or ∞. Then the branch points are made up of regular orbits of K of the form $\left\{a, a\zeta, \ldots, a\zeta^{\ell-1}\right\}$. Each such orbit contributes a factor to $f(x)$ of the form $\prod_{i=0}^{\ell-1}\left(x - a\zeta^i\right)^{e^i}$, possibly to some power. Thus

$$f(x) = \prod_{a \in B}\prod_{i=0}^{\ell-1}\left(x - a\zeta^i\right)^{e^i},$$

where B is some finite list, possibly with repeated elements. Since $1+e+\cdots+e^{\ell-1} = 0$ mod p then $deg(f) = 0$ mod p and so ∞ is not a branch point. If 0 or ∞ is a branch point then equation 4.1 shows that the $\mathbb{Z}_\ell$ action on $\mathbb{Z}_p$ must be trivial so $K = \mathbb{Z}_{\ell p}$ by relative primeness arguments. Note we may also argue that $K = \mathbb{Z}_{\ell p}$ by using ramification arguments whether p and ℓ are relatively prime or not. Then

$$f(x) = z^{t_0}\prod_{a \in B}\prod_{i=0}^{\ell-1}\left(x - a\zeta^i\right),$$

where $0 < t_0 < p$. If $t_\infty < p$ is the local degree at ∞ with then $t_0 + t_\infty + \ell b = 0$ mod p, Where $b = |B|$ is the number of repetitions of the factor $\prod_{i=0}^{\ell-1}\left(x - a\zeta^i\right)$.

EXAMPLE 4.6. Suppose $K = D_\ell$. We use the action of $K = D_\ell$ on $\mathbb{P}^1$ generated by $x \to \zeta x$ and $z \to \frac{1}{z}$. The group K must be $D_\ell \times \mathbb{Z}_p$ or $D_\ell \ltimes \mathbb{Z}_p$ with the non-trivial D_ℓ action generated by $j \to -j = (p-1)j$, the only non-trivial involution in $\mathrm{Aut}(C)$. The abelianization of D_ℓ is $\mathbb{Z}_2$ for ℓ odd and $\mathbb{Z}_2 \times \mathbb{Z}_2$ for ℓ even, so there are several non-trivial choices for $\overline{Ad}$. For simplicity let us focus on ℓ odd, so that $\overline{Ad}$ is trivial on the subgroup $\mathbb{Z}_\ell \lhd D_\ell$ The action of D_ℓ has three singular orbits (orbits with fixed points) $\left\{1, \zeta, \ldots, \zeta^{\ell-1}\right\}$, $\left\{-1, -\zeta, \ldots, -\zeta^{\ell-1}\right\}$, and $\{0, \infty\}$. A regular orbit has the form $\left\{a, a\zeta, \ldots, a\zeta^{\ell-1}\right\} \cup \left\{a^{-1}, a^{-1}\zeta, \ldots, a^{-1}\zeta^{\ell-1}\right\}$ for $a \neq \pm 1$. In the trivial action case each regular orbit contributes a factor of the form $\prod_{i=0}^{\ell-1}\left(x - a\zeta^i\right)\prod_{i=0}^{\ell-1}\left(x - a^{-1}\zeta^i\right)$ possibly to some power, the singular orbits can contribute z^{t_0}, and powers of $\prod_{i=0}^{\ell-1}\left(x - \zeta^i\right)$ and $\prod_{i=0}^{\ell-1}\left(x + \zeta^i\right)$. Now the transformation $z \to \frac{1}{z}$ interchanges 0 and ∞ and so the local degrees t_0 and t_∞ must be the same. The sum condition must satisfy

$$t_0 + t_\infty + b_r\ell + b_1\ell + b_{-1}\ell = 0 \bmod p$$

where b_r, b_1, b_{-1} are the number of repetitions of the factors

$$\prod_{i=0}^{\ell-1}\left(x - a\zeta^i\right)\prod_{i=0}^{\ell-1}\left(x - a^{-1}\zeta^i\right), \ \prod_{i=0}^{\ell-1}\left(x - \zeta^i\right), \text{ and } \prod_{i=0}^{\ell-1}\left(x + \zeta^i\right).$$

Now suppose that $\overline{Ad}$ is not trivial. Then the singular orbits $\left\{1, \zeta, \ldots, \zeta^{\ell-1}\right\}$, $\left\{-1, -\zeta, \ldots, -\zeta^{\ell-1}\right\}$ cannot contribute any factors since the stabilizers of these points have non-trivial $\overline{Ad}$ values. Regular orbits contribute factors of the type

$$\frac{\prod_{i=0}^{\ell-1}\left(x-a\zeta^i\right)}{\prod_{i=0}^{\ell-1}\left(x-a^{-1}\zeta^i\right)} \text{ or } \prod_{i=0}^{\ell-1}\left(x-a\zeta^i\right)\prod_{i=0}^{\ell-1}\left(x-a^{-1}\zeta^i\right)^{p-1}$$

and we see that $t_\infty = -t_0 \bmod p$. So we have $t_0 + p - t_0 + b_r\ell + (p-1)b_r\ell = 0 \bmod p$, which is no constraint at all.

REMARK 4.7. The general form of factors for any K-action will be

$$\left(\prod_{k\in K}\left(x-\overline{k}\cdot a\right)^{\overline{Ad}(\overline{k})}\right)^{\frac{1}{r}}$$

where r is the order of the stabilizer of the point $a \in \mathbb{P}^1$.

Shaska observed [**19**] that for many automorphism groups of low genus that if a cyclic prime order subgroup had a genus quotient then the subgroup was central. This generalizes the hyperelliptic case. It turns out that many of these surfaces are superelliptic. In fact, we have the following.

PROPOSITION 4.8. *Let S be a superelliptic surface with cyclic p-gonal subgroup C. Then C is central N.*

PROOF. The multi-degree has the form $(1,\ldots,1,t_\infty)$, and there must be at least two ones. But then it is not possible for any automorphism in K to satisfy equation 4.1 for all t_i unless the action is trivial. □

5. Families of p-gonal surfaces

In this section we very briefly discuss families of p-gonal surfaces. Given our cyclic p-gonal equation

$$y^p = f(x) = \prod_{i=1}^{s}(x-a_i)^{t_i} \tag{5.1}$$

we can look at families of surfaces in three different ways.

(1) For each complete multi-degree $T = (t_1, t_2, \ldots, t_s)$ consider the family of surfaces parameterized by the branch points $(a_1, a_2, \ldots, a_s) \in \mathbb{C}^s -$ *diagonals*. The genus of the surfaces so constructed has the constant value $\frac{(s-2)(p-1)}{2}$. If two multi-degrees are equivalent by the equivalence given in Remark 3.4 then a family of equivalent surfaces is determined.
(2) Each family described in (1) for a fixed T and p determines a family of surfaces, or equisymmetric stratum, in the moduli space of surfaces of genus $\frac{(s-2)(p-1)}{2}$. These equisymmetric strata are defined in [**6**]. The complex dimension of this family is $s-3$ as our next proposition shows. The union of these families for all T is called the p-gonal locus of the moduli space. For $p = 2$ this is the well known hyperelliptic locus of the moduli space. The trigonal ($p = 3$) locus for genus 4 is discussed in [**23**]. Some recent results on the topology of this branch locus for $p > 3$ are given in [**3**] and [**12**].

(3) Each polynomial $f(x)$ defines vector of coefficients in $\mathbb{C}^t$, where $t = deg(f)$. By expanding the right hand side of equation 5.1, we see that each multi-degree defines a locally closed subvariety of $\mathbb{C}^t$. If t is held fixed and s and T are allowed to vary, $\mathbb{C}^t$ is a union of these subvarieties (minus a small piece corresponding to reducible surfaces). The genus of the surface varies over different subvarieties. The set where all t_i equal 1 is open and dense and the genus of the surfaces is largest over this set. In [**19**] Shaska adopts this approach to determine equations of hyperelliptic surfaces with prescribed automorphism group.

PROPOSITION 5.1. *Let* $T = (t_1, t_2, \ldots, t_s)$ *be a complete multi-degree and* Σ_T *be the subgroup of permutations of* Σ_s *that preserve* T. *Let* $PSL_2(\mathbb{C}) \times \Sigma_T$ *act (partially) on* $\mathbb{C}^s - diagonals$ *by* $(L, \vartheta) \cdot (a_1, a_2, \ldots, a_s) = (L(a_{\vartheta 1}), \ldots, L(a_{\vartheta s}))$. *Then*

$$\mathcal{MC}_{n,T} = (\mathbb{C}^s - diagonals)/(PSL_2(\mathbb{C}) \times \Sigma_T)$$

of complex dimension $s-3$ *is "almost" a moduli space of* p*-gonal surfaces of multi-degree* T.

Rather than prove the statement, we make some remarks that justify the word almost.

(1) Every cyclic p-gonal action with multi-degree T is accounted for in the quotient space.
(2) The action of $PSL_2(\mathbb{C})$ is only partial and exceptional automorphisms (where $N < A$) need to be taken into account. The partial action can be fixed by looking at $\left(\mathbb{P}^1\right)^s$ but then the cyclic model needs to be fixed.
(3) Each $\mathcal{MC}_{n,T}$ corresponds to a moduli space of the same dimension, of Fuchsian groups determined by the signature $(p, p, \ldots, p)$, specifically there is a finite to one map $\mathcal{MC}_{n,T} \to \mathcal{M}_{(p,p,\ldots,p)}$.

EXAMPLE 5.2. Here is a table of multi-degrees for a small number of branch points, moduli space dimension $\mathfrak{m} = s - 3$, prime p, and genus σ.

	$p = 3$	$p = 5$
$s = 3, \mathfrak{m} = 0$	$\sigma = 1,\ (1,1,1)$	$\sigma = 2,\ (1,1,2)$
$s = 4, \mathfrak{m} = 1$	$\sigma = 2,\ (1,1,2,2)$	$\sigma = 4,\ (1,1,1,2), (1,1,4,4), (1,2,3,4)$

	$p = 7$	$p = 11$
$s = 3, \mathfrak{m} = 0$	$\sigma = 3,\ (1,1,5), (1,2,4)$	$\sigma = 5,\ (1,1,9)$
$s = 4, \mathfrak{m} = 1$	$\sigma = 6,\ (1,1,1,4), (1,1,6,6)$	$\sigma = 10,\ (1,1,1,8), (1,1,10,10)$

(1) The pure superelliptic case is always present, unless p divides s.
(2) All the cases potentially have automorphisms, depending on the position of the branch points.
(3) In the moduli space of surfaces of genus 4 the 5-gonal locus consists of three strata of dimension 1.

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