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Abstract

Classical minimal surface theory can be thought of as dealing with the shapes of soap films stretched across wires in Euclidean space $\mathbb{R}^3$. This article will examine such structures in an abstract three-dimensional space, the Lie Group $SO(3)$. This is the space of possible rotations in $\mathbb{R}^3$, where each rotation is expressed as three angles: two to indicate the axis of rotation and one to indicate the amount of rotation. The properties of the space $SO(3)$ may result in minimal surfaces that behave differently than they do in $\mathbb{R}^3$.

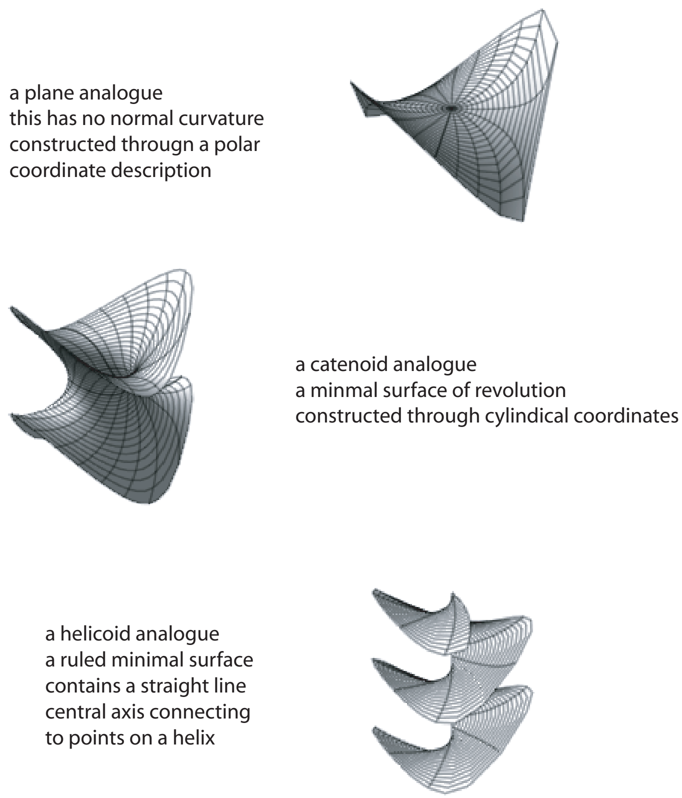

Figure 1: Analogues of planes, catenoids, and helicoids in $SO(3)$

1 Introduction

Minimal surfaces in $\mathbb{R}^3$ are surfaces which minimize the area for all surfaces with the same boundary. The goal here will be to generalize the idea of a minimal surface for the Lie group $SO(3)$, and attempt to find surfaces which fit this idea. As such, it will be necessary to first describe $SO(3)$, which will include providing the definition of a Lie group and a description of the tangent space of $SO(3)$. Part of this investigation is how the algebraic properties of $SO(3)$ as a group influence the geometric contruction of minimal surfaces. To determine how to define the criterion for a surface in $SO(3)$ being minimal, the geometric properties of curvature and mean curvature will be discussed, and an explanation of Riemannian metrics will be provided, followed by a metric for $SO(3)$. After establishing which metric will be used, it will be possible to discuss curvature in $SO(3)$, which will allow the defining of what is meant by a "minimal surface in $SO(3)$."

In an attempt to construct such surfaces, the exponential map for $SO(3)$ will be introduced. This is the algebraic structure that was found to be most useful. Use of the exponential map will facilitate the construction of surfaces in $SO(3)$ which are analogous to classical minimal surfaces in $\mathbb{R}^3$ - specifically the plane, the helicoid, and the catenoid.

2 Defining $SO(3)$

$SO(3)$ is defined as the space of all 3×3 matrices with the properties that $A^T A = AA^T = I$ and $\det A = 1$. This section will define a way of describing $SO(3)$ as a Lie group. It will begin by providing the definition for *Lie group*, after which a way of representing elements of $SO(3)$ will be provided. Lastly, it will provide a definition for the tangent space of a manifold and a description for the tangent space of $SO(3)$ at the identity.

2.1 Lie Groups

In order to define what is meant by describing $SO(3)$ as a *Lie Group*, the terms *smooth manifold* and *group* must first be defined.

Definition 1 *Manifold:*
A manifold *M is a topological space such that for every point $p \in M$ there is an open set U which contains p, and a homeomorphism $\varphi : U \to \mathbb{R}^n$. M is then n-dimensional.*

Definition 2 *Smooth Manifold:*
A manifold M is called smooth if given two overlapping charts (maps $\varphi_U : U \to \mathbb{R}^n$ and $\varphi_V : V \to \mathbb{R}^n$ with $U \cap V \neq \emptyset$) the transition function $\psi_{UV} : \varphi_U(U \cap V) \to \varphi_V(U \cap V)$ is a smooth function from $\mathbb{R}^n$ to $\mathbb{R}^n$ with a smooth inverse function.

$SO(3)$ is a smooth 3-dimenional manifold. That is, it can be parameterized with 3 variables. To see that, begin with 3×3 matrix with 9 parameters, represented by $\begin{bmatrix} \mathbf{x} & | & \mathbf{y} & | & \mathbf{z} \end{bmatrix}$, where the column vectors $\mathbf{x}, \mathbf{y}, \mathbf{z}$ are 3-vectors. For this matrix to be an element of $SO(3)$, $\mathbf{x}$ must be unit lenth, so it then has 2 free parameters. Likewise, $\mathbf{y}$ must be unit length and orthogonal to $\mathbf{x}$, so $\mathbf{y}$ has 1 free parameter. Finally, $\mathbf{z}$ must be unit length and orthogonal to $\mathbf{x}$ and $\mathbf{y}$, leaving it with no free parameters. Thus $SO(3)$ can be parameterized with 3 variables.

Definition 3 *Group:*
A set G with a binary operation $*$ such that

- *$*$ is associative.*
- *There exists a unique element $e \in G$ such that for any $a \in G$, $e * a = a * e = a$. That is, e is the* identity *element of G.*
- *For any element $a \in G$, there exits a unique $b \in G$ such that $b*a = a*b = e$, where e is the identity element of G. This b is denoted a^{-1}, and is the* inverse *of a.*

With the operation of matrix multiplication, that $SO(3)$ satisfies 1), 2), and 3) is obvious, with the identity e of $SO(3)$ being the 3×3 identity matrix and $A^{-1} = A^T$. That $SO(3)$ is closed under this operation is less obvious. Take some $A, B \in SO(3)$ and consider $(AB)^T(AB) = B^T A^T AB = B^T IB = B^T B = I$. Thus for any $A, B \in SO(3), AB \in SO(3)$. Therefore, $SO(3)$ is a group with the operation of matrix multiplication.

Defining a *Lie Group* is now quite simple.

Definition 4 *Lie Group:*
Let G be a smooth manifold. G is a Lie Group *if*

- *G is a group under some binary operation.*
- *The group operations $G \times G \to G$, $(x, y) \mapsto xy$ and $G \to G$, $x \mapsto x^{-1}$ are both differentiable functions.*

It can be shown that the maps $SO(3) \times SO(3) \to SO(3), (A, B) \mapsto AB$, and $SO(3) \to SO(3), A \mapsto A^T = A^{-1}$ are differentiable by writing them as maps on the coordinates, that is $\mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ and $\mathbb{R}^3 \to \mathbb{R}^3$.

2.1.1 Left Translation and Right Translation

Lie groups also have *left translation* and *right translation* maps.

Definition 5 *Left- and Right-Translation:*
Let G be a Lie group and fix $a \in G$. Then the map

$$L_a : G \to G \quad \text{given by} \quad L_a(g) = ag$$

is a left-translation. Similarly, the map

$$R_a : G \to G \quad \text{given by} \quad R_a(g) = ga$$

is a right-translation.

The left and right translations are analoguous to the translation of position in $\mathbb{R}^3$ by a vector $T_v(a) = a + v$. These provide a natural manner of moving infornation at one point in a Lie group to another. The distinct left and right translations are required because the group operation on the Lie group is not necessarily commutative. These also are differentiable maps $G \to G$, and their derivatives will be considered later.

2.2 Representing $SO(3)$

The Lie group $SO(3)$ consists of all 3×3 orthogonal matrices with the operation of standard matrix multiplication. An orthogonal matrix is simply a matrix A such that $A^T A = AA^T = I$. Equivalently, an orthogonal matrix is a matrix with orthogonal column vectors of unit length. This is the space of all possible rotations in $\mathbb{R}^3$.

In choosing a way to represent elements of $SO(3)$, three coordinates (θ, ϕ, ψ) are required. Each element of $SO(3)$ will be repesented by a 3×3 matrix $\mathbf{A}(\theta, \phi, \psi)$. Setting the first column vector as

$$\mathbf{u}_1 = \langle \cos\theta\cos\phi, \sin\theta\cos\phi, \cos\phi \rangle,$$

we have $\|\mathbf{u}_1\| = 1$, and at $(\theta, \phi, \psi) = (0, 0, 0)$ that $\mathbf{u}_1 = \langle 1, 0, 0 \rangle$. For a second orthogonal unit vector, choosing

$$\mathbf{u}_2 = \langle -\sin\theta, \cos\theta, 0 \rangle,$$

we have $\mathbf{u}_1 \cdot \mathbf{u}_2 = 0$, $\|\mathbf{u}_2\| = 1$, and at $(\theta, \phi, \psi) = (0, 0, 0)$, $\mathbf{u}_2 = \langle 0, 1, 0 \rangle$. A third orthogonal unit vector is easily found by taking $\mathbf{u}_3 = \mathbf{u}_1 \times \mathbf{u}_2$, that is

$$\mathbf{u}_3 = \mathbf{u}_1 \times \mathbf{u}_2 = \langle -\cos\theta\sin\phi, -\sin\theta\sin\phi, \cos\phi \rangle.$$

We have by properties of the cross-product that $\|\mathbf{u}_3\| = 1$, $\mathbf{u}_1 \cdot \mathbf{u}_3 = 0$, $\mathbf{u}_2 \cdot \mathbf{u}_3 = 0$ and when $(\theta, \phi, \psi) = (0, 0, 0)$ that $\mathbf{u}_3 = \langle 0, 0, 1 \rangle$.

Finally, to account for the third angle in the rotation, define

$$\mathbf{A}(\theta, \phi, \psi) = [\mathbf{u}_1 | \cos\psi\mathbf{u}_2 + \sin\psi\mathbf{u}_3 | -\sin\psi\mathbf{u}_2 + \cos\psi\mathbf{u}_3].$$

The angle ψ represent how much that basis $\mathbf{u}_2$, $\mathbf{u}_3$ is rotated about the $\mathbf{u}_1$ axis. We have thus represented an element of $SO(3)$ with the preimage of $\mathbf{A}$ in $\mathbb{R}^3$ being

$$a = (\theta, \phi, \psi).$$

In this representation, the angles (θ, ϕ, ψ) are restricted to the ranges $-\pi < \theta < \pi$, $-\frac{1}{2}\pi < \phi < \frac{1}{2}\pi$, and $-\pi < \psi < \pi$. We note in this representation the group identity e (the identity matrix) is given by $e = (0, 0, 0)$.

2.3 The Tangent Space of $SO(3)$

One of the defining aspects of any smooth manifold is its tangent space. To look at the tangent space of $SO(3)$, it is necessary to define the tangent space for a general manifold; to do this, begin with the definition for the tangent vector of a curve in a manifold.

Definition 6 *Tangent Vector*
Let M be a smooth manifold, and $p \in M$. Let $c : \mathbb{R} \to M$ be a curve with $c(0) = p$ and derivative c'. Then $c'(0)$ is a tangent vector to M at p.

The tangent space at a point $p \in M$, then, is the set of all possible tangent vectors to curves in M at p.

Definition 7 *Tangent Space*
Let M be a smooth manifold. For each point $p \in M$, the tangent space to M at p, denoted T_pM, is the vector space consisting of all tangent vectors of M at p. That is, the space of all vectors $c'(0)$ where $c : \mathbb{R} \to M$ is a curve with $c(0) = p$ and derivative c'.

To find a way of describing the tangent space of a point in $SO(3)$, consider a curve $c : \mathbb{R} \to SO(3)$ represented by $\mathbf{A}(\theta(t), \phi(t), \psi(t))$. From the fact that $\mathbf{A}^T\mathbf{A} = I$, it follows that

$$\begin{aligned} \frac{d}{dt}\left(\mathbf{A}^T\mathbf{A}\right) &= \mathbf{A}^T\frac{d}{dt}\mathbf{A} + \frac{d}{dt}\left(\mathbf{A}^T\right)\mathbf{A} \\ &= \mathbf{A}^T\frac{d}{dt}\mathbf{A} + \left(\frac{d}{dt}\mathbf{A}\right)^T\mathbf{A} = 0. \end{aligned}$$

At the identity, $A = A^T = I$, so

$$\mathbf{A}^T\frac{d}{dt}\mathbf{A} + \left(\frac{d}{dt}\mathbf{A}\right)^T\mathbf{A} = \frac{d}{dt}\mathbf{A} + \left(\frac{d}{dt}\mathbf{A}\right)^T = 0.$$

Therefore, the tanget space to $SO(3)$ at the identity is the set of all antisymmetric matrices, that is matrices of the form

$$\begin{bmatrix} 0 & a & -b \\ -a & 0 & c \\ b & -c & 0 \end{bmatrix},$$

for any $a, b, c \in \mathbb{R}$.

2.4 Derivatives of the Translation Maps

Now that the tangent space has been defined, the derivatives of the left and right translation maps can be considered. For L_a and R_a, these are

$$dL_a : T_g SO(3) \to T_{ag} SO(3) \quad \text{and} \quad dR_a : T_g SO(3) \to T_{ga} SO(3)$$

given by

$$dL_a(\mathbf{v}) = a\mathbf{v} \quad \text{and} \quad dR_a(\mathbf{v}) = \mathbf{v}a,$$

respectively.

The derivatives of left and right translation provided a manner of identifying the tangent space at any point with the tangent space at the identity. We simply look at a vector $\mathbf{v} \in T_e SO(3)$ and its translate $dL_g(\mathbf{v}) \in T_g SO(3)$ as the same vector under left translation. Likewise, we consider the vector $\mathbf{v} \in T_e SO(3)$ and its translate $dR_g(\mathbf{v})) \in T_g SO(3)$ as the same vector under right translation.

3 The Exponential Map

In order to construct analoges to classical minimal surfaces in $\mathbb{R}^3$, it also is useful to find the *exponential map* for $SO(3)$, which require first defining the *Lie algebra* $\mathfrak{g}$ of $SO(3)$.

Definition 8 *Lie Algebra:*
For a Lie group G, the Lie algebra $\mathfrak{g}$ is the tangent space to the identity element of G, with the additional structure of a Lie bracket: a binary operation on the Lie algebra satisfying antisymmetry and the Jacobi identity.

So for $SO(3)$, $\mathfrak{g} = T_e SO(3)$.

The exponential map of a Lie group G provides a way of generating curves in G from lines in $\mathfrak{g}$. This provides analogues to straight lines in G, and, potentially analogues to plane.

From matrix algebra, the exponential map for matrices is

$$\mathbf{exp}(X) = I + X + \frac{1}{2}X^2 + \frac{1}{6}X^3 + \cdots = \sum_{k=0}^{\infty} \frac{X^k}{k!}.$$

$SO(3)$ is a subgroup of $GL(3, \mathbb{R})$, the group of all invertible 3×3 matrices, so the exponential map of $SO(3)$ uses this matrix exponential. As shown previously, in $SO(3)$, an abritrary element $\mathbf{v} \in \mathfrak{g}$ can be represented as

$$\mathbf{v} = \begin{bmatrix} 0 & a & -b \\ -a & 0 & c \\ b & -c & 0 \end{bmatrix}.$$

We note that

$$\mathbf{v}^2 = \begin{bmatrix} -\left(a^2 + b^2\right) & bc & ac \\ bc & -\left(a^2 + c^2\right) & ab \\ ac & ab & -\left(b^2 + c^2\right) \end{bmatrix};$$

and furthermore

$$\mathbf{v}^3 = -\left(a^2+b^2+c^2\right)\mathbf{v} \quad \text{and} \quad \mathbf{v}^4 = -\left(a^2+b^2+c^2\right)\mathbf{v}^2,$$

so

$$\mathbf{v}^{2n+1} = (-1)^n \left(a^2+b^2+c^2\right)^n \mathbf{v} \quad \text{and} \quad \mathbf{v}^{2n+2} = (-1)^n \left(a^2+b^2+c^2\right)^n \mathbf{v}^2.$$

This means that the exponential map for $SO\left(3\right)$ simplifies to

$$\begin{aligned}
\mathbf{exp}\left(\mathbf{v}\right) &= \sum_{k=0}^{\infty} \frac{\left(\mathbf{v}\right)^k}{k!} \\
\mathbf{exp}\left(\mathbf{v}\right) &= I + \left(\sum_{i=0}^{\infty} \frac{\left(-\alpha\right)^i}{\left(2i+1\right)!}\right)\mathbf{v} + \left(\sum_{j=0}^{\infty} \frac{\left(-\alpha\right)^j}{2\left(j+1\right)!}\right)\mathbf{v}^2 \\
\mathbf{exp}\left(\mathbf{v}\right) &= I + \frac{\sin\left(\alpha\right)}{\alpha}\mathbf{v} + \frac{1-\cos\left(\alpha\right)}{\alpha^2}\mathbf{v}^2,
\end{aligned}$$

where $\alpha^2 = a^2+b^2+c^2$.

It is useful to consider $\mathbf{exp}\left(s\mathbf{v}\right) = \mathbf{exp}\left(\mathbf{v}, s\right)$, where $s \in \mathbb{R}$ and $v \in \mathfrak{g}$. This gives us

$$\begin{aligned}
\mathbf{exp}\left(\mathbf{v}, s\right) &= \sum_{k=0}^{\infty} \frac{\left(\mathbf{v}s\right)^k}{k!} \\
\mathbf{exp}\left(\mathbf{v}, s\right) &= I + \left(\sum_{i=0}^{\infty} \frac{\left(-\alpha s\right)^i}{\left(2i+1\right)!}\right)\mathbf{v} + \left(\sum_{j=0}^{\infty} \frac{\left(-\alpha s\right)^j}{2\left(j+1\right)!}\right)\mathbf{v}^2 \\
\mathbf{exp}\left(\mathbf{v}, s\right) &= I + \frac{\sin\left(\alpha s\right)}{\alpha}\mathbf{v} + \frac{1-\cos\left(\alpha s\right)}{\alpha^2}\mathbf{v}^2,
\end{aligned}$$

where $\alpha^2 = a^2+b^2+c^2$. The mapping $\mathbf{exp}((\mathbf{v}, s)$ can be interpreted as mapping a vector $v \in \mathfrak{g}$ to a curve in G, and such a curve can be viewed as analogous to a "straight line." In any Lie group, the exponential map defines a curve beginning at e and going in the direction of $\mathbf{v} \in \mathfrak{g}$. Such a curve can be translated by left or right translation to a curve starting at any pont.

4 Curvature and Mean Curvature in $\mathbb{R}^3$

In $\mathbb{R}^3$, the curvature of an embedded surface is defined using the "plane" curvature of curves on the surface.

Definition 9 *Normal Principal Curvatures:*
Given a surface $S = S\left(x^1, x^2\right) = \langle S_1\left(x^1,x^2\right), S_2\left(x^1,x^2\right), S_3\left(x^1,x^2\right)\rangle$, *at a point* $p \in S$, *let* N *be a unit normal vector to the surface and* $\mathbf{v}$ *be a unit vector in the tangent plane* T_pS. *We can then define a plane curve* c *as the*

intersection of the surface S and the plane P containing the point p, the unit normal vector N and the tangent vector $\mathbf{v}$. The curvature of this curve is called the normal curvature of the surface in the direction $\mathbf{v}$. The minimum and maximum curvatures among all such curves are the principal curvatures κ_1, κ_2 *at the point p.*

The mean curvature of the surface is defined to be the average of the principal curvatures.

Definition 10 *Mean Curvature:*
The mean curvature H of a surface at a point p is the average of the principal curvatures of the surface at p, $H = \frac{1}{2}(\kappa_1 + \kappa_2)$.

From Euler, the mean curvature is also the average of the normal curvature in any two perpendicular directions.

5 Minimal Surfaces in $\mathbb{R}^3$

Definition 11 *Minimal Surface (Calculus of Variations)*
A minimal surface $S(x_1, x_2)$ *in $\mathbb{R}^3$ is a surface which locally minimizes the surface area functional*

$$A(S) = \iint_{\Delta S} \sqrt{det(g_{ij})} dx_1 dx_2,$$

where $g_{ij} = \frac{\partial S}{\partial x_i} \cdot \frac{\partial S}{\partial x_j}$.

There is an alternative geometric definition in terms of curvature, which depends upon the mean curvature of S. It is this view of a minimal surface that we will use in our investigations.

Definition 12 *Minimal Surface (Geometry)*
A surface S in $\mathbb{R}^3$ is minimal if and only if $H(S) = 0$.

6 Riemannian Metrics

To determine whether a surface embedded in a smooth manifold such as $SO(3)$ is minimal requires a *Riemannian metric* with which distances in the manifold can be measured.

Definition 13 *Riemannian Metric:*
On a manifold M, a Riemannian metric (henceforth simply "metric") g is a choice of inner product on T_pM, where the map $g_p : T_pM \times T_pM \to \mathbb{R}$ can be represented by a positive definite, symmetric matrix $\left[g_{ij}\right]$.

A metric describes the "shape" of the manifold.

Because $SO(3)$ has a group structure, it is preferable for the metric to describe it to be compatible with the algebraic structure. On a Lie group G, such a metric can be defined by choosing an inner product on T_eG, and translating it elsewhere. This is similar to the standard Euclidean metric on $\mathbb{R}^3$.

Definition 14 *Left-, Right-, and Bi-Invariant Metrics:*
Let G be a Lie group, and g be a metric for M. Then g is left-invariant if for any $p, q \in M$ and $u, v \in T_pM$,

$$g_p(u, v) = g_{pq}\left(\left(dL_q\right)_p(u), \left(dL_q\right)_p(v)\right),$$

where $L_q(p)$ is the left translation map $L_q(p) = qp$. Similarly, g is right-invariant if for any $p, q \in M$ and $u, v \in T_pM$,

$$g_p(u, v) = g_{qp}\left(\left(dR_q\right)_p(u), \left(dR_q\right)_p(v)\right),$$

where $R_q(p)$ is the right translation map $R_q(p) = pq$. If g is both left-invariant and right-invariant, it is called bi-invariant.

6.1 The Bi-Invariant Metric for $SO(3)$

From [Ar], [Fr], and other sources, every compact Lie group has a unique bi-invariant metric. $SO(3)$ is a compact Lie group, because it can be represented as a closed and bounded as a subset of $\mathbb{R}^9$. Using the definitions of left- and right-invariance, one can show that the metric generated from the standard inner product $\langle , \rangle$ on $T_eSO(3)$, $[g_{ij}]_e = I$, is the unique bi-invariant metric for $SO(3)$. At the point (θ, ϕ, ψ), this metric is given by:

$$[g_{i,j}] = \begin{bmatrix} 1 & 0 & \sin\phi \\ 0 & 1 & 0 \\ \sin\phi & 0 & 1 \end{bmatrix}.$$

7 Curvature and Minimal Surfaces in Manifolds

The same geometric definition of a minimal surface can be used for surfaces in manifolds, but requires the use of generalized methods for the steps involved in computing the principal curvatures. In order to compute the mean curvature of a surface, one must first construct a unit normal vector to the surface.

7.1 Finding a Unit Normal Vector

Let $S\left(u^1, u^2\right)$ be a surface in a manifold M with coordinates $\left(x^1, x^2, x^3\right)$ and metric g_{ij}. Define $\frac{\partial}{\partial u^k} = \frac{\partial S}{\partial u^k}$.

In order to define the normal vector to the surface in a way analoguous to the cross product in $\mathbb{R}^3$, it is useful to first let

$$\begin{aligned} \delta^{123} = \delta^{231} = \delta^{312} &= 1, \\ \delta^{132} = \delta^{213} = \delta^{321} &= -1, \\ \text{and all other} \quad \delta^{ijk} &= 0. \end{aligned}$$

Note that the cross product in $\mathbb{R}^3$ is given by $(\mathbf{u} \times \mathbf{v})^k = \delta^{ijk} \mathbf{u}^i \mathbf{v}^j$. Similarly, the vector $N \in T_pM$ defined by

$$N^i = \delta^{ijk} \left(g_{jm} \frac{\partial S^m}{\partial u^1} \right) \left(g_{kn} \frac{\partial S^n}{\partial u^2} \right)$$

is orthogonal to both $\frac{\partial}{\partial u^1}$ and $\frac{\partial}{\partial u^2}$ in T_pM. Then, the unit normal ν to the surface can be defined as

$$\nu = \frac{N}{\sqrt{N^i g_{ij} N^j}}.$$

7.2 Taking the Derivative with Respect to the Metric

Next, in order to take the derivative of ν with respect to the metric, define the inverse metric

$$g^{ij} = (g_{ij})^{-1},$$

and the Christoffel symbols

$$\Gamma^k_{ij} = \frac{1}{2} g^{km} \left(\frac{\partial g_{jm}}{\partial x^i} + \frac{\partial g_{im}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^m} \right).$$

Then the covariant derivative of ν in $SO(3)$ in the direction of $\frac{\partial}{\partial u^l}$ is

$$\nabla_l (\nu)^k = \frac{\partial \nu^k}{\partial u^l} + \Gamma^k_{ij} \nu^j \left(\frac{\partial}{\partial u^l} \right)^i.$$

This can be written in terms of the basis $\frac{\partial}{\partial u^1}$, $\frac{\partial}{\partial u^2}$, and ν as

$$\nabla_l (\nu)^k = a_1 \frac{\partial}{\partial u^1} + a_2 \frac{\partial}{\partial u^2} + a_3 \nu$$

for some a_1, a_2, a_3 in $\mathbb{R}^3$. The covariant derivative of ν along the surface in the direction of $\frac{\partial}{\partial u^l}$ is simply the $b\nu$ component of this, and can be found by taking

$$D_l (\nu) = \nabla_l (\nu) - \left(\nabla_l (\nu)^i g_{ij} \nu^j \right) \nu.$$

This method is equivalent to the definition of the covariant derivatives using the induced metric on the surface.

The covariant derivative can be done in a direction via $D_X(\nu) = D_l(\nu) X^l$ where the vector $X = X^l \frac{\partial}{\partial u^l}$.

7.3 Mean Curvature

The mean curvature is given by solving the eigenvalue equation

$$g\left(D_X\left(\nu\right), X\right) = \lambda g\left(X, X\right),$$

where in this case $X \in T_pM$. In order to do this, define the 2×2 matrices $A = A_{kl}$ and $B = B_{kl}$ as

$$A_{kl} = \left(\frac{\partial}{\partial u^k}\right)^i g_{ij} \left(\frac{\partial}{\partial u^l}\right)^j$$
$$B_{kl} = D_k\left(\nu\right)^i g_{ij} \left(\frac{\partial}{\partial u^l}\right)^j.$$

The metric eigenvalue problem above then becomes the matrix eigenvalue problem

$$B\,X = \lambda A\,X$$

or

$$A^{-1}\,B\,X = \lambda\,X.$$

The mean curvature can then be computed via the trace of the matrix $a == A^{-1}B$, or

$$H = \frac{1}{2}\left(a_{11} + a_{22}\right).$$

Our goal is then to construct surfaces with $H \equiv 0$.

This provides a partial differential equation for a minimal surface as $H = 0$. The equation involves partial derivatives of the surface given by $S(u^1, u^2) = (x^1(u^1, u^2), x^2(u^1, u^2), x^3(u^1, u^2))$. For example, consider the special case of a surface defined as a function of two coordinates, $z = S\left(x, y\right)$ in $\mathbb{R}^3$ with the standard Euclidean metric. Then, using the upward pointing normal, the equation $H \equiv 0$ results in

$$H = \frac{\left(1 + \left(\frac{\partial S}{\partial x}\right)^2\right)\frac{\partial^2 S}{\partial y^2} - 2\frac{\partial S}{\partial x}\frac{\partial S}{\partial y}\frac{\partial^2 S}{\partial xy} + \left(1 + \left(\frac{\partial S}{\partial y}\right)^2\right)\frac{\partial^2 S}{\partial x^2}}{\left(1 + \left(\frac{\partial S}{\partial x}\right)^2 + \left(\frac{\partial S}{\partial y}\right)^2\right)^{\frac{3}{2}}}.$$

Of course, the equation we are primarily interested in is much more complicated, as we express it in terms of $(\theta(u^1, u^2), \phi(u^1, u^2), \psi(u^1, u^2))$.

8 Classical Minimal Surfaces Analogues in $SO\left(3\right)$

In this section, we first describe the geometric construction of the three classical minimal surfaces; the plane, the catenoid and the helicoid. We then show how to construct analoguous minimal surfaces in $SO(3)$ using the exponential map.

These surfaces can be directly verified to be minimal surfaces by showing the mean curvature is equal to zero by direct computation. We show geometrically why these are minimal surfaces by computing the principal curvatures or at least the normal curvatures in two orthogonal directions.

8.1 Planes

The simplest type of minimal surface in $\mathbb{R}^3$ is a plane, as it has no curvature whatsover. The normal vector is constant throughout the surface. We will consider a special construction of planes in $\mathbb{R}^3$ that we can generalize to $SO(3)$ through the exponential map.

8.1.1 Construction of Planes in $\mathbb{R}^3$

One can construct a plane in $\mathbb{R}^3$ geometrically, represented by the image of the vector-valued function $f\,(r,t) = r\langle \cos t, \sin t, 0\rangle$ on $t \in (0, 2\pi)$ and $r \in \mathbb{R}_0^+$. This is the standard polar coordinate description of the plane $z = 0$.

8.1.2 Construction of Plane-Analogues in $SO\,(3)$

An analoguous construction in $SO\,(3)$ can be obtained using the exponential map. Let

$$\mathbf{v}(t) = \begin{bmatrix} 0 & \cos t & -\sin t \\ -\cos t & 0 & 0 \\ \sin t & 0 & 0 \end{bmatrix},$$

describe a unit vector at the identity to the coordinate plane $\psi = 0$. We look at the surface $X(r,t) = \mathbf{exp}(r\mathbf{v}(t)) = \mathbf{exp}(\mathbf{v}(t), r)$ as the polar coordinate analogue of a plane in $SO\,(3)$ where $r \in \mathbb{R}_0^+ = \{x : x \geq 0\}$. We can get other planes in $SO(3)$ by using left and right translations. For instance, the surface $g\,\mathbf{exp}(\mathbf{v}(t), r)\,g^{-1}$ is a rotation of the $\mathbf{exp}(\mathbf{v}(t), r)$ that contains the origin.

8.1.3 Mean Curvature of Plane-Analogues in $SO\,(3)$

Let $X(r,t) = \mathbf{exp}(r\,\mathbf{v}(t))$ be our desired plane-analogue where $\mathbf{v}(t)$ given above describes a unit tangent vector at the identity to the coordinate plane $\psi = 0$. We note that the vector $\mathbf{v}(t)$ is a unit vector at the identity, so using the definition of the exponential map, we have

$$X(r,t) = I + \sin(r)\,\mathbf{v}(t) + (1 - \cos(r))\,\mathbf{v}(t)^2.$$

This can be expressed in matrix form as

$$X = \begin{bmatrix} \cos(r) & \sin(r)\cos(t) & -\sin(r)\cos(t) \\ -\sin(r)\cos(t) & 1-(1-\cos(r))\cos(t)^2 & (1-\cos(r))\cos(t)\sin(t) \\ \sin(r)\sin(t) & (1-\cos(r))\cos(t)\sin(t) & 1-(1-\cos(r))\sin(t)^2 \end{bmatrix}.$$

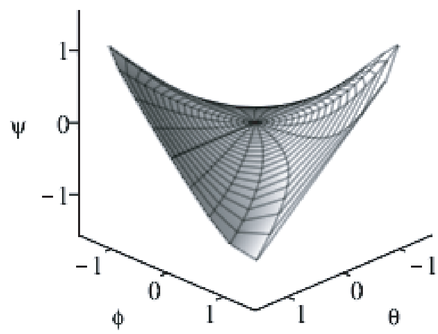

Figure 2: Example of a plane analogue in $SO(3)$.

Computing X_r directly from the definition of the exponential map gives

$$\begin{aligned} X_r &= \cos(r)\,\mathbf{v}(t) + \sin(r)\,\mathbf{v}(t)^2 \\ &= (\cos(r)\,I + \sin(r)\,\mathbf{v}(t))\ \mathbf{v}(t). \end{aligned}$$

The fact that $\mathbf{v}^3 = -\mathbf{v}$ for a unit vector in $T_eSO(3)$ simplifies the calculation above, noting that

$$\begin{aligned} \mathbf{exp}(r\mathbf{v})\,\mathbf{v} &= (I + \sin(r)\,\mathbf{v} + (1-\cos(r))\,\mathbf{v})\ \mathbf{v} \\ &= \mathbf{v} + \sin(r)\,\mathbf{v}^2 + (1-\cos(r))\,\mathbf{v}^3 \\ &= \mathbf{v} + \sin(r)\,\mathbf{v}^2 - (1-\cos(r))\,\mathbf{v} \\ &= \cos(r)\,\mathbf{v} + \sin(r)\,\mathbf{v}^2. \end{aligned}$$

Therefore,

$$X_r = X\mathbf{v}(t).$$

This is the derivative of the left translation map L_X, $dL_{\mathbf{X}}(\mathbf{v}) = X\,\mathbf{v}$. It can also be written as the derivative of the right translation map $dR_X(\mathbf{v}) = \mathbf{v}\,X$.

The computation of X_t proceeds similarly, but now $\mathbf{v}$ must be differentiated. We then have

$$X_t = \sin(r)\,\mathbf{v}'(t) + (1-\cos(r))\ (\mathbf{v}(t)\mathbf{v}'(t) + \mathbf{v}'(t)\,\mathbf{v}(t))\,,$$

where $\mathbf{v}'(t)$ represents the derivative of the matrix $\mathbf{v}(t)$, since matrix multiplication is not commutative. This can expanded in matrix form as

$$X_t = \sin(r)\begin{bmatrix} 0 & -\sin(t) & -\cos(t) \\ \sin(t) & 0 & 0 \\ \cos(t) & 0 & 0 \end{bmatrix} + (1-\cos(r))\begin{bmatrix} 0 & 0 & 0 \\ 0 & \sin(2t) & \cos(2t) \\ 0 & \cos(2t) & -\sin(2t) \end{bmatrix}.$$

Furthermore, this can be written as a left multiplication by X by considering $X_t = X\,\mathbf{w} = L_X(\mathbf{w})$ as X_t should be an element in $T_X SO(3)$ and equivalent to some element $\mathbf{w}$ in $T_e SO(3)$. The element $\mathbf{w}$ is given by $\mathbf{w} = X^{-1}X_t$ or

$$\mathbf{w} = \begin{bmatrix} 0 & -\sin(r)\sin(t) & -\sin(r)\cos(t) \\ \sin(r)\sin(t) & 0 & -(1-\cos(r)) \\ \sin(r)\cos(t) & 1-\cos(r) & 0 \end{bmatrix}.$$

One can also write this in terms of right multiplication.

Notice that X_r and X_t are orthogonal with respect to the bi-invariant metric on $SO(3)$. This is because the vector $\mathbf{v}$ is orthogonal to $\mathbf{w}$ at $T_e SO(3)$, and the metric is left-invariant. To see that the vectors $\mathbf{v}$ and $\mathbf{w}$ are orthogonal at e, take the dot product of the vectors $\mathbf{v}$ and $\mathbf{w}$ as

$$\langle \cos(t), \sin(t), 0\rangle \cdot \langle \sin(r)\sin(t), -\sin(r)\cos(t), 1-\cos(r)\rangle$$

using that the metric at e in the coordinates (θ, ϕ, ψ) is the standard inner product. To find the unit normal vector to the surface, we note that the unit vector of $\mathbf{w}$ is

$$\tfrac{1}{\sqrt{2-2\cos(r)}}\,\langle \sin(r)\sin(t), -\sin(r)\cos(t), 1-\cos(r)\rangle.$$

Thus the unit normal vector ν to the surface can be formed as left translation of the vector N given by

$$\mathbf{N} = \tfrac{1}{\sqrt{2-2\cos(r)}}\,\langle \sin(t)\,(1-\cos(r)), -\cos(t)\,(1-\cos(r)), -\sin(r)\rangle.$$

We simply compute the standard cross product of $\mathbf{v}$ and the unit vector of $\mathbf{w}$ in $T_e SO(3)$, and use left-translation and the left-invariance of the metric.

A direct calculation of the covariant derivative T_r (the unit vector of X_r) in the direction of X_r (increasing r) and the covariant derivative T_t (the unit vector of X_t) in the direction of X_t (increasing t) shows that

$$\nabla_{X_r} T_r = \mathbf{0} \quad \text{and} \quad \nabla_{X_\theta} T_t = -\kappa(r)\,\|X_t\|\,T_r$$

where $\kappa(r)$ is the curvature of the curve $c(t) = X(r,t)$ and given by

$$\kappa(r) = \frac{1}{2}\sqrt{\frac{1+\cos(r)}{1-\cos(r)}}.$$

We note that $\|X_t\| = \sqrt{2-2\cos(r)}$ because $\|\mathbf{w}\| = \sqrt{2-2\cos(r)}$ and that $T_r = X\,\mathbf{v}(t)$, as $\|\mathbf{v}(t)\| = 1$. The calculation $\nabla_{X_r} T_r = 0$ implies that the curvature of the curve $c(r) = X(r,t)$ is zero, as the unit tangent T_r does not change along the curve with respect to the metric.

> The curvature of a curve $c(t)$ in a manifold M, is given by the magnitude of $\nabla_{c'(t)}T$ divided by the magnitude of $c'(t)$ where $T = c'(t)/\|c'(t)\|$ is the unit tangent vector of the curve c. The normal curvature of a curve on a surface is then component of the curvature vector $\frac{1}{\|c'(t)\|}\nabla_{c'(t)}T$ in the direction of the normal to the surface.

This computation shows that the normal curvature of the surface $X(r,t)$ is zero. The normal curvature in two mutually perpendicular directions is zero. We have the curve $c(r) = \mathbf{exp}(r\,\mathbf{v}(t))$ with zero curvature, and thus normal curvature equal to zero. Furthermore, we have the curve $c(t) = \mathbf{exp}(r\,\mathbf{v}(t))$ with curvature vector in the tangent plane, and thus with normal curvature equal to zero.

We note that the curve $c(t) = \mathbf{exp}(r\,\mathbf{v}(t))$ is the analogue of a circle of radius r centered at the origin in this plane. Furthermore, we have shown that the curvature of the circle depends only on the radius, but in a different manner than in Euclidean space. Also note that the translation of a circle is still a circle, as the translation does not affect the calculations.

It is also a straight forward calculation to compute the covariant derivatives of T_r in the direction of increasing t and the covariant derivative of T_t in the direction of increasing r. We get from these calculations that

$$\nabla_{X_t}T_r = \kappa(r)\,\|X_t\|\,T_t \quad \text{and} \quad \nabla_{X_r}T_t = \mathbf{0}.$$

This shows that the vector T_t does not change in the direction of X_r (increasing r)

To understand the impact of these computations, it is useful to use that $[T_r, T_t, \nu]$ is a orthonormal basis. Expressing

$$\begin{aligned} \nabla_{X_r}V &= \alpha_{11}\,T_r + \alpha_{12}\,T_t + \alpha_{13}\,\nu \\ \nabla_{X_t}V &= \alpha_{21}\,T_r + \alpha_{22}\,T_t + \alpha_{23}\,\nu \end{aligned}$$

where V is a vector field on the surface and α_{ij} are scalars, we have

$$\begin{aligned} \nabla_{X_r}T_r &= A_{12}\,T_t + A_{13}\,\nu, & \nabla_{X_t}T_r &= A_{22}\,T_t + A_{23}\,\nu, \\ \nabla_{X_r}T_t &= B_{11}\,T_r + B_{13}\,\nu, & \nabla_{X_t}T_t &= B_{21}\,T_r + B_{23}\,\nu, \\ \nabla_{X_r}\nu &= C_{11}\,T_r + C_{12}\,T_t, & \nabla_{X_t}\nu &= C_{21}\,T_r + C_{22}\,T_t. \end{aligned}$$

This is because $\nabla_l g(U,U) = 2\,g(U,\nabla_l U) = 0$ for a unit vector, so $\nabla_l U$ does not have a component in the U direction, meaning that $A_{11} = 0$, $A_{21} = 0$, $B_{12} = 0$, $B_{22} = 0$, $C_{13} = 0$, and $C_{23} = 0$. We have therefore shown directly from $\nabla_{X_r}T_r$ and $\nabla_{X_t}T_r$ that $A_{11} = 0$, $A_{12} = 0$, $A_{13} = 0$, $A_{21} = 0$, $A_{23} = 0$ and $A_{22} = \kappa(r)\,\|X_t\|$. Likewise, we have directly shown from $\nabla_{X_r}T_t$ and $\nabla_{X_t}T_t$ that $B_{11} = 0$, $B_{12} = 0$, $B_{13} = 0$, $B_{23} = 0$ and $B_{21} = -\kappa(r)\,\|X_t\|$.

We can compute the coefficients C_{ij} using the compatibility of the covariant derivative with the metric,

$$\nabla g(X,Y) = g(\nabla X, Y) + g(X, \nabla Y).$$

Looking at

$$\nabla_{X_r} g(T_r, \nu) = g(\nabla_{X_r} T_r, \nu) + g(T_r, \nabla_{X_r}\nu),$$

it follows that $g(T_r, \nabla_{X_r}\nu) = C_{11} = 0$ because $g(T_r, \nu) = 0$ and $\nabla_{X_r} T_r = 0$. Likewise looking at

$$\nabla_{X_t} g(T_t, \nu) = g(\nabla_{X_t} T_t, \nu) + g(T_t, \nabla_{X_t}\nu)$$

leads to $g(T_t, \nabla_{X_t}\nu) = C_{22} = 0$ since $g(T_t, \nu) = 0$ and $\nabla_{X_t} T_t = -\kappa T_r$ so $C_{tt} = 0$. Looking at

$$\nabla_{X_r} g(T_t, \nu) = g(\nabla_{X_r} T_t, \nu) + g(T_t, \nabla_{X_r}\nu)$$

leads to $g(T_t, \nabla_{X_r}\nu) = C_{12} = 0$ since $g(T_t, \nu) = 0$ and $\nabla_{X_r} T_t = 0$ implying $C_{rt} = 0$. Finally, looking at

$$\nabla_{X_t} g(T_r, \nu) = g(\nabla_{X_t} T_r, \nu) + g(T_r, \nabla_{X_t}\nu)$$

leads to $g(T_r, \nabla_{X_t}\nu) = C_{21} = 0$, because $g(T_r, \nu) = 0$ and $\nabla_{X_t} T_r = \kappa T_t$, so $C_{tr} = 0$. This shows $\nabla_{X_r}\nu = 0$ and $\nabla_{X_t}\nu = 0$. Thus, the normal curvatures in every direction is equal to zero, just like on a plane.

8.2 Catenoids

8.2.1 Construction of Catenoids in $\mathbb{R}^3$

In $\mathbb{R}^3$, a catenoid is defined as a surface of revolution obtained by rotating a catenary curve, such as that given by the function $y(x) = a\cosh\left(\frac{x}{a}\right)$ for some $a \in \mathbb{R}$, about its directrix, the x-axis. So for a catenoid $\mathbf{X}$ in $\mathbb{R}^3$, at any point $p \in \mathbf{X}$, one of the principal curvatures at p will be given by defining a plane curve in $\mathbf{X}$ from the unit normal vector at p and the unit tangent vector at p that is parallel to the directrix of the original catenary curve. The intersection of this curve with $\mathbf{X}$ is a catenary curve in $\mathbf{X}$ which contains p and has the same directrix as the original catenary curve used to define $\mathbf{X}$. Using a unit tangent vector at p orthogonal to that used to define the previous curve, along with the unit normal to $\mathbf{X}$ at p, one can define another plane curve perpendicular to the previous one at p. Using Meusnier's theorem, the curvature of this curve is given by the reciprocal of the radius of the circle $y(x_0)$ of revolution that it lies on times the cosine of the angle between the normal of the surface and the normal of circle in the plane $x =$ constant. It is a straight forward calculation to show that the curvatures add to zero.

The curvature of the catenary curve $\langle x, a\cosh(\frac{x}{a}, 0)\rangle$ is $\kappa = \frac{1}{a}\text{sech}(x/a)^2$. The curvature of the circle with radius in the plane $x = x_0$, $c = \langle x_0, r\cos(t), r\sin(t)\rangle$

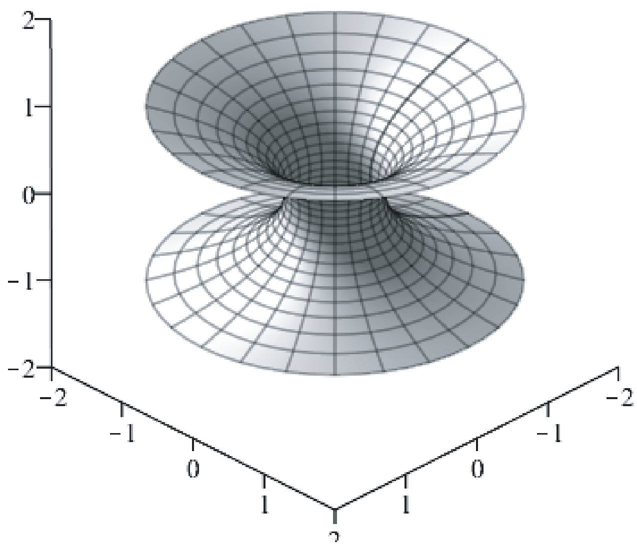

Figure 3: A Catenoid in $\mathbb{R}^3$ with axis of revolution the z-axis

with radius $r = a\cosh(x/a)$ is $\kappa = \frac{1}{a}\operatorname{sech}(x/a)$. The angle between the normal of the circle $\langle 0, -r\cos(t), -r\sin(t)\rangle$ when $t = 0$ and the normal to the catenary is $\arccos(0 - 1/\cosh(x/a))$, so that the normal curvature in the direction $\langle 0, 1, 0\rangle$ is $-\frac{1}{a}\operatorname{sech}(x/a)^2$. This verifies that the catenood is a minimal surface.

8.2.2 Construction of Catenoid-Analogues in $SO(3)$

Rather than defining a curve first and then defining a surface from it, this construction is performed by defining first a surface of rotation formed by an undetermined curve and then using the mean curvature equation to determine the curve. This curve will be an analogue to a catenary curve in $SO(3)$, and the surface will be an analogue to a catenoid.

Let

$$\mathbf{v}_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_1(t) = \begin{bmatrix} 0 & \cos\omega t & -\sin\omega t \\ -\cos\omega t & 0 & 0 \\ \sin\omega t & 0 & 0 \end{bmatrix}.$$

The matrix representation of a catenoid-analogue is then

$$\mathbf{X}(s,t) = \mathbf{exp}\left(\mathbf{v}_0, s\right)\mathbf{exp}\left(\mathbf{v}_1(t), u\left(s\right)\right).$$

Then, computing the mean curvature H of $\mathbf{X}$, the equation $(H)_{t=0} = 0$, we obtain the equation for a catenoid-analogue in $SO(3)$. This is a second order ODE. We obtain a catenoid-analogue surface by solving numerically with initial conditions $u(0) = u_0$, $\frac{d}{ds}(u(s))_{s=0} = 0$. We can change the axis of revolution by using left and right translations.

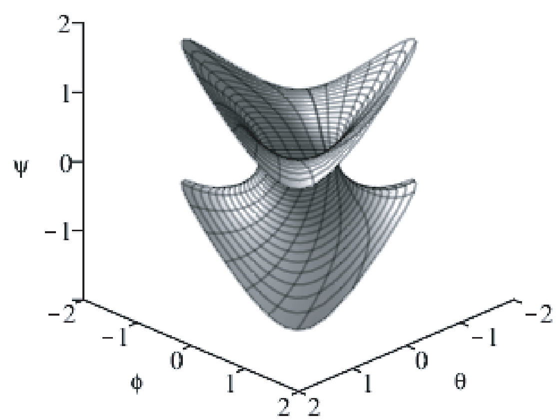

Figure 4: An Example of a Catenoid-Analogues in $SO(3)$.

We are essentially constructing a circle of radius $u(s)$ in our plane analogue and then translating by s in the normal direction of the plane. The calculation below mimics the calculation in $\mathbb{R}^3$ using the change of metric, and the fact that the curvature of the circle changes.

8.2.3 Mean Curvature of Catenoid-Analogues in $SO(3)$

For convenience, let $\mathbf{F}(s) = \mathbf{exp}(\mathbf{v}_0, s)$ and $\mathbf{G}(s,t) = \mathbf{exp}(\mathbf{v}_1, u(s))$. To see that $\mathbf{X}$ has mean curvature $H = 0$, consider

$$\nabla_{\frac{\partial}{\partial t}}\left(\frac{\partial}{\partial t}(\mathbf{X})\right) \quad \text{and} \quad \nabla_{\frac{\partial}{\partial s}}\left(\frac{\partial}{\partial s}(\mathbf{X})\right).$$

The curve $c_1(t) = \mathbf{X}(s_0, t)$ is the analogue of a circle of radius $u(s_0)$, and the curve $c_2(s) = \mathbf{X}(s, 0)$ is the analogue of a catenary curve. We need to compute the curvature of these two curves.

We first consider the curve $c(t)$. We note that the matrix form of the vector X_t is given by

$$\frac{\partial}{\partial t}\mathbf{X} = \frac{\partial}{\partial t}(\mathbf{F}(s)\mathbf{G}(s,t)) = \mathbf{F}(s)\frac{\partial}{\partial t}\mathbf{G}(s,t).$$

We further note that

$$\nabla_{\frac{\partial}{\partial t}}\left(\frac{\partial}{\partial t}\mathbf{X}\right) = \mathbf{F}(s)\,\nabla_{\frac{\partial}{\partial t}}\frac{\partial}{\partial t}\mathbf{G}(s,t).$$

The function $\mathbf{G}(s,t) = \mathbf{exp}(\mathbf{v}_1(t), u(s))$. This calculation is the curvature of the circle of radius $u(s)$ in the plane analogue computation, and therefore is given by

$$\begin{aligned}\nabla_{\frac{\partial}{\partial t}}(T_t) &= -\mathbf{F}(s)\kappa(u(s))\sqrt{2-2\cos(u(s))}\,\mathbf{G}(s,t)\,\mathbf{v}_1(t)\\ &= -\kappa(u(s))\sqrt{2-2\cos(u(s))}\,\mathbf{X}\,\mathbf{v}_1(t),\end{aligned}$$

where T_t is the unit vector of $\frac{\partial}{\partial t}\mathbf{X}$. This is a direct consequence of $\mathbf{G}(s,t)$ being the analogue of a circle of $u(s)$. The normal curvature in the direction $\mathbf{X}_t$, then, is computed by Meusnier's theorem by computing the component of curvature vector $\frac{1}{\|\mathbf{X}_t\|}\nabla_{\frac{\partial}{\partial t}}T_t$ in the normal direction of the surface, $\nu^k = \delta^{ijk}g_{im}\mathbf{X}_t^m g_{jn}\mathbf{X}_s^n$.

Computing the curvature of $c_2(s) = \mathbf{X}(s,0) = \mathbf{F}(s)\,G(s,0)$, we apply the Liebniz rule we get the matrix form of $\mathbf{X}_s$ as

$$\frac{\partial}{\partial s}\mathbf{X} = \left(\frac{\partial}{\partial s}F\right)\,\mathbf{G}(s,t) + \mathbf{F}(s)\,\left(\frac{\partial}{\partial s}\mathbf{G}(s,t)\right).$$

Then, we use the exponential rule

$$\frac{\partial}{\partial r}\,\mathbf{exp}(\mathbf{v},r) = \mathbf{exp}(\mathbf{v},r)\,\mathbf{v} = \mathbf{v}\,\mathbf{exp}(\mathbf{v},r)$$

to get

$$\frac{\partial}{\partial s}\mathbf{X} = \mathbf{exp}(\mathbf{v}_0,s)\,\mathbf{v}_0\,\mathbf{exp}(\mathbf{v}_1(t),u(s)) + u'(s)\,\mathbf{exp}(\mathbf{v}_0,s)\,\mathbf{v}_1(t)\,\mathbf{exp}(\mathbf{v}_1(t),u(s)).$$

This can also be written as

$$\frac{\partial}{\partial s}\mathbf{X} = \mathbf{v}_0\,\mathbf{X} + u'(s)\,\mathbf{X}\,\mathbf{v}_1(t).$$

The length of the $\frac{\partial}{\partial s}\mathbf{X}$ is $\sqrt{1+(u'(s))^2}$, which follows from looking at

$$\mathbf{F}(s)^{-1}\left(\frac{\partial}{\partial s}\mathbf{X}\right)\,\mathbf{G}(s,t)^{-1} = \mathbf{v}_0 + u'(s)\,\mathbf{v}_1(t)$$

and using the left and right invariance of the metric. This implies that the unt tangent vector of the analogue to the catenary curve is then

$$T_s = \frac{1}{\sqrt{1+(u'(s))^2}}\,\mathbf{v}_0\,\mathbf{X}(s,t) + \frac{u'(s)}{\sqrt{1+(u'(s))^2}}\,\mathbf{X}(s,t)\,\mathbf{v}_1(t).$$

The covariant derivative of T_s in the direction of $\frac{\partial}{\partial s}$ then gives the curvature of the catenary curve. It is worth noting that the covariant derivative is entirely in the direction of the normal to the surface, that is

$$\nabla_{\frac{\partial}{\partial s}}T_s = \kappa_c(s,u(s),u'(s),u''(s))\,\|\mathbf{X}_s\|\,\nu,$$

where $\kappa_c(s)$ is the curvature of this catenary curve. This is because this catenary curve is a line of curvature by construction; its curvature is always a principal curvature. Computing the covariant derivative to determine the curvature is a tedious calculation performed using Maple (a computer algebra system). We note that κ_c depends on the first and second derivative of $u(s)$ as a result of the linearity of the covariant derivative that

$$\nabla_{\frac{\partial}{\partial s}}(\mathbf{Y}+\mathbf{Z}) = (\nabla_{\frac{\partial}{\partial s}}\mathbf{Y}) + (\nabla_{\frac{\partial}{\partial s}}\mathbf{Z})$$

and the generalized Liebniz rule that

$$\nabla_{\frac{\partial}{\partial s}}(\varphi\,\mathbf{Y}) = \frac{\partial\varphi}{\partial s}\mathbf{Y} + \partial\,\nabla_{\frac{\partial}{\partial s}}\mathbf{Y}.$$

where φ is a scalar function and $\mathbf{Y}$, $\mathbf{Z}$ are vectors. The derivative of the scalar factors $\frac{1}{\sqrt{1+(u'(s))^2}}$ and $\frac{u'(s)}{\sqrt{1+(u'(s))^2}}$ both are proportional to $u''(s)$. It depends on s and $u(s)$ from the dependence of position on $u(s)$.

The catenoid is obtained by solving

$$\kappa_c(s, u(s), u'(s), u''(s)) - \kappa(u(s))\cos(\alpha(s)) = 0,$$

where α is the angle between the vector $X\,\mathbf{v}_1(t)$ and the normal vector ν at the point $X(s,t)$. The equation is highly nonlinear involving terms of $\cos(u(s))$ and $\sin(u(s))$ along with $cos(s)$ and $\sin(s)$. Some example solution curves are given below for this equation. We solved using the initial condition $u(0) = u_0$ and $u'(0) = 0$. This can be done without loss of generality, as we can obtain other solutions by translation.

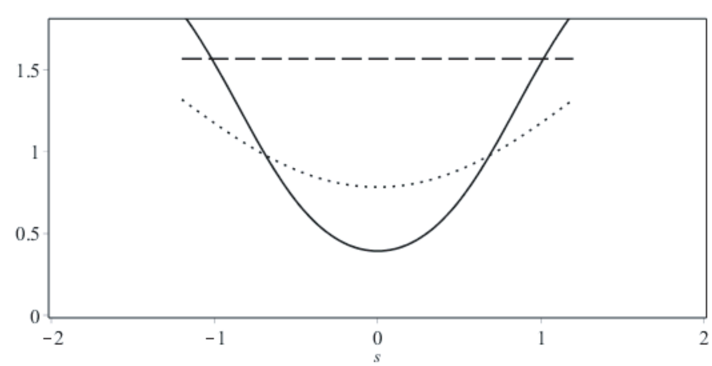

Figure 5: Some example catenary-curves in $SO(3)$ with different values for $u(0)$

8.3 Helicoids

8.3.1 Construction of Helicoids in $\mathbb{R}^3$

Begin by choosing an axis in $\mathbb{R}^3$, say the line $(0, 0, z)$. Then, take a vector at the origin perpendicular to this line, such as $r\langle 1, 0, 0\rangle$, and define a vector-valued function as a translation of this vector along the axis while rotating it about the axis: $\mathbf{X}(r,t) = \langle r\cos(\omega t), r\sin(\omega t), t\rangle$ for $r \in \mathbb{R}_0^+, t \in \mathbb{R}$ and a real constant ω. This would be a helicoid about the z-axis. Notice the curve $c(t) = \langle r\cos(\omega t), r\sin(\omega t), t\rangle$ for fixed $r > 0$ is just a helix.

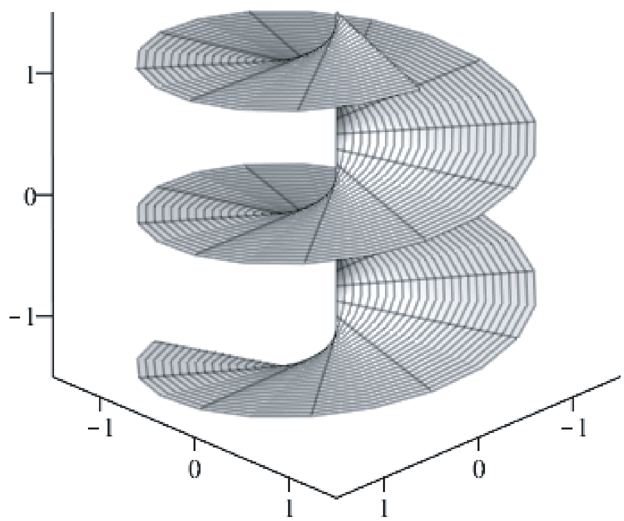

Figure 6: Example of a Helicoid in $\mathbb{R}^3$

8.3.2 Construction of Helicoid-Analogues in $SO(3)$

We construct a helicoid analogue in basically the same manner as we constructed a catenoid analogue. We imitate the construction in $\mathbf{R}^3$. Let $\mathbf{v}_0$ represent the axis of of the helicoid, and $\mathbf{v}_1(t)$ represent the rotation around the axis. In matrix form for $SO(3)$, we choose

$$\mathbf{v}_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_1(t) = \begin{bmatrix} 0 & \cos\omega\theta & -\sin\omega\theta \\ -\cos\omega\theta & 0 & 0 \\ \sin\omega\theta & 0 & 0 \end{bmatrix},$$

where ω is a positive constant. The matrix representation of a helicoid-analogue about the ψ-axis is given by

$$\mathbf{X}(r,t) = \mathbf{exp}\left(\mathbf{v}_0, t\right) \mathbf{exp}\left(\mathbf{v}_1(t), r\right).$$

This is a translation and rotation of an analogue to a straight line along and about a perpendicular straight-line-analogue.

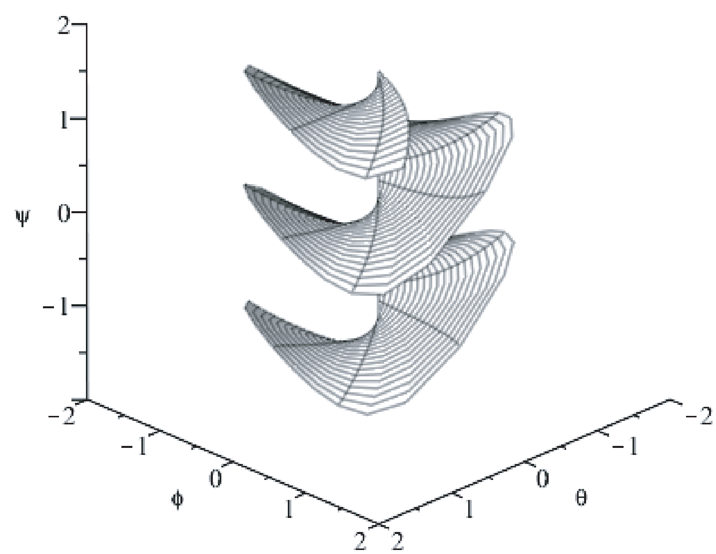

Figure 7: Example of a Helicoid Analogue in $SO(3)$

8.3.3 Mean Curvature of Helicoid-Analogues in $SO\,(3)$

Let $\mathbf{v}_0, \mathbf{v}_1(t)$ and $\mathbf{X}(r,t)$ be as in the construction for a helicoid analogue in $SO\,(3)$ given above. For convenience, let $\mathbf{F}\,(t) = \mathbf{exp}\,(\mathbf{v}_0, t)$ and $\mathbf{G}\,(r,t) = \mathbf{exp}\,(\mathbf{v}_1(t), r)$.

To see that $\mathbf{X}$ is minimal in $SO(3)$, begin by considering for any point $p \in \mathbf{X}$ the tangent vectors

$$\tfrac{\partial}{\partial t}(\mathbf{X}) \quad \text{and} \quad \tfrac{\partial}{\partial r}(\mathbf{X}).$$

Noting that $\mathbf{X} = \mathbf{F}(t)\,\mathbf{G}(r,t)$ by definition, and applying Leibniz's rule to get

$$\begin{aligned}
&\tfrac{\partial}{\partial t}(\mathbf{X}) = \tfrac{\partial}{\partial t}(\mathbf{F}(t))\,\mathbf{G}(r,t) + \mathbf{F}(t)\tfrac{\partial}{\partial t}(\mathbf{G}(r,t)) \quad \text{and} \\
&\tfrac{\partial}{\partial r}(\mathbf{X}) = \tfrac{\partial}{\partial r}(\mathbf{F}(t))\,\mathbf{G}(r,t) + \mathbf{F}(t)\tfrac{\partial}{\partial r}(\mathbf{G}(r,t)) = \mathbf{F}(t)\tfrac{\partial}{\partial r}(\mathbf{G}(r,t))
\end{aligned}$$

respectively.

Noting again that

$$\frac{\partial}{\partial r}(\mathbf{exp}(\mathbf{v}(t), r)) = \mathbf{v}(t)\,\mathbf{exp}(\mathbf{v}(t), r) = \mathbf{exp}(\mathbf{v}(t), r)\,\mathbf{v}(t),$$

we have that

$$\begin{aligned}
&\frac{\partial}{\partial t}(\mathbf{X}) = \mathbf{v}_0\mathbf{F}(t)\mathbf{G}(r,t) + \mathbf{F}\frac{\partial}{\partial t}(\mathbf{G}(r,t)) \quad \text{and} \\
&\frac{\partial}{\partial r}(\mathbf{X}) = \mathbf{F}(t)\mathbf{G}(r,t)\mathbf{v}_1(t).
\end{aligned}$$

We compute $\frac{\partial}{\partial t}\mathbf{G}(r,t)$ by noting that $\mathbf{G}(r,t)$ is a circle of radius r centered at the identity in our construction of the plane analogue. We thus have

$$\frac{\partial}{\partial t}\mathbf{G}(r,t) = \mathbf{G}(r,t)\mathbf{w}(r,t)$$

where

$$\mathbf{w}(r,t) = \begin{bmatrix} 0 & -\sin(r)\sin(t) & -\sin(r)\cos(t) \\ \sin(r)\sin(t) & 0 & -(1-\cos(r)) \\ \sin(r)\cos(t) & 1-\cos(r) & 0 \end{bmatrix}.$$

This gives

$$\frac{\partial}{\partial t}\mathbf{X} = \mathbf{v}_0\,\mathbf{X} + \mathbf{X}\,\mathbf{w} \quad \text{and} \quad = \frac{\partial}{\partial r}\mathbf{X} = \mathbf{X}\,\mathbf{v}_1$$

We have by definition $\mathbf{v}_0$ and $\mathbf{v}_1$ orthogonal, and therefore by manipulating the left and right invariance of $\mathbf{v}_0\,\mathbf{X} = \mathbf{F}\,\mathbf{v}_0\,\mathbf{G}$ and $\mathbf{X}\,\mathbf{v}_1 = \mathbf{F}\,\mathbf{v}_1\mathbf{G}$, we have $\mathbf{v}_0\mathbf{X}$ orthogonal to $\mathbf{X}\mathbf{v}_1$. From the plane analogue construction, we know that $\mathbf{X}\mathbf{w}$ and $\mathbf{X}\,\mathbf{v}_1$ are orthogonal. The vectors $\frac{\partial}{\partial t}\mathbf{X}$ and $\frac{\partial}{\partial r}\mathbf{X}$ form an orthogonal basis of the tangent plane.

For any point $p \in \mathbf{X}$, the curve of constant t in $\mathbf{X}$ passing through p is simply a straight-line-analogue in $SO(3)$, and so has normal curvature of 0. Then for the mean curvature at p to be 0, the curve of constant r in $\mathbf{X}$ passing through p must also have zero normal curvature. To see that it does, we look at the vector $\frac{\partial}{\partial t}X$ more carefully. First, we note that $\mathbf{v}_0$ and $\mathbf{w}$ are orthogonal, and we can write $\mathbf{v}_0\mathbf{X} = \mathbf{F}\,\mathbf{v}_0\,\mathbf{G}$ and $\mathbf{X}\,\mathbf{w} = \mathbf{F}\,\mathbf{w}\,\mathbf{G}$ so $\mathbf{v}_0\mathbf{X}$ and $\mathbf{X}\mathbf{w}$ are orthogonal.

This implies the length of $\frac{\partial}{\partial t}\mathbf{X}$ depends on r but is independent of t. A direct calculation gives the length of

$$\frac{\partial}{\partial t}\mathbf{X} = \sqrt{1 + 2(\omega^2 - \omega) - 2(\omega^2 - \omega)\cos(r)}.$$

This is important in computing the covariant derivative.

The constant length of $\frac{\partial}{\partial t}\mathbf{X}$ implies that is covariant derivative has components in the direction of $\frac{\partial}{\partial r}\mathbf{X}$ and ν. A direct calculations of the covariant derivative of $\frac{\partial}{\partial t}\mathbf{X}$ by definition in the direction of $\frac{\partial}{\partial t}\mathbf{X}$ yields

$$\nabla_{\frac{\partial}{\partial t}} \frac{\partial}{\partial t}\mathbf{X} = -(\omega^2 - \omega)\sin(s)\, \frac{\partial}{\partial r}\mathbf{X}.$$

Thus showing there is no normal curvature, as the curvature vector of the curve $c(t) = X(s,t)$ with s held constant is given by

$$\frac{1}{\|\frac{\partial}{\partial t}\mathbf{X}\|^2}\, \nabla_{\frac{\partial}{\partial t}} \frac{\partial}{\partial t}\mathbf{X} = -\kappa_h(r)\, \frac{\partial}{\partial r}\mathbf{X},$$

with $\kappa_h(r)$ being the curvature of this helix-analogue,

$$\kappa_h(r) = \frac{(\omega^2 - \omega)\sin(r)}{1 + 2(\omega^2 - \omega) - 2(\omega^2 - \omega)\cos(r)}.$$

The normal curvature is given by

$$\frac{1}{\|\frac{\partial}{\partial t}\mathbf{X}\|^2}\, g(\nabla_{\frac{\partial}{\partial t}} \frac{\partial}{\partial t}\mathbf{X}, \nu) = 0.$$

Thus the helicoid is a minimal surface as the mean curvature equals zero.

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