RESPONSE TO THE REVIEW OF STRONG RECOVERY IN GROUP SYNCHRONIZATION

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I wish to thank the Referee and the Rose-Hulman Undergraduate Mathematics Journal for their careful consideration of the paper and for their suggested revisions. I have incorporated the suggestions of the referee into a revised manuscript. Below I offer my responses to the referee, indicating the adjustments made to the manuscript. Comments of the referee are italicized.

1. Referee Comments

1.1. **Summary.** *This paper considers the problem of group synchronization: each vertex of a graph is labelled with an unknown group element $g \in G$. for each edge of the graph, an observation is given of the difference between the vertex labels, up to some noise. The "strong recovery" problem is to reconstruct all of the vertex labels (up to an overall global translation by a group element $g \in G$) from the observations.This is a well-known problem, with applications in x-ray crystallography, community detection in networks, and structure for motion, among others. The author provides two results (an information-theoretic achievability result, and an information-theoretic impossibility result) which to the best of my knowledge are new. The paper is clearly organized and well-written, and the results are elegantly proven. I would like to congratulate the author on a very nice first paper! I recommend acceptance after the author slightly strengthens Theorem 1 (see below).*

Response: I would like to thank the referee for their positive feedback. Remarks concerning Theorems 1 and 2 are addressed below.

1.2. **Remarks Concerning Theorem 1.**

- *In Theorem 1, the author proves that strong group synchronization on the complete graph is possible as long as there is β such that the edge flip probability satisfies $p_n < \beta < 1 - \sqrt{2}/2$. This result can be strengthened without much extra work, and it should be strengthened before the paper is published: Consider the group on two elements, $|G| = 2$. The flip probability condition can be weakened to $p_n < \beta < 1/2$. This is because in Lemma 1 the event C happens with probability 0. Therefore, $\mathbb{P}(Y_n(1, v)Y_n(v, 2) \neq X_n(1)^{-1}X_n(2)) \leq 2(p_n - p_n^2)$. Note that effectively this implies that reconstruction is possible as long as p_n is uniformly bounded (either above or below) away from $1/2$. So it becomes an interesting question for future work to see how close to $1/2$ we can take p_n and still get reconstruction. (By a similar argument to Abbe et al.'18, one can prove that strong reconstruction is impossible once $|p_n - 1/2| \ll \frac{\log n}{2n}$, but how tight is that?) Now generalize the above argument. Consider any finite group, as in the statement of the theorem. The flip probability condition probability can be weakened to $p_n < \beta < 1 - 1/|G|$. For simplicity, consider* *the following notation. Let $q = |G|p/(|G|-1)$. The noise kernel on G is equivalent to observing g with probability $1-q$, or observing noise $\sim \mathrm{Unif}[G]$ with probability q. Under this notation, the distribution of $Y_n(1,v)Y_n(v,2)$ is: equal to $X_n(1)^{-1}X_n(2)$ with probability $(1-q)^2$, and equal to noise $\sim \mathrm{Unif}[G]$ with probability $1-(1-q)^2$. This allows one to slightly tighten the analysis on the event $F_n(1,2)$ in the proof of Theorem 1. For any g, define $W_{n,g} = \sum_{v \in V_n \setminus \{1,2\}} f_{1,2}^{(g)}(v)$. Let $g^* = X_n(1)^{-1}X_n(2)$. Therefore, $F_n(1,2) \subset \{W_{g^*} > (n-2)/|G|\} \cap (\cap_{g \neq g^*}\{W_g \leq (n-2)/|G|\})$. This event can be seen to hold with exponentially small probability as $n \to \infty$, as long as q_n is uniformly bounded away from 1, which is the event that p_n is uniformly bounded away from $1-1/|G|$. Interestingly the argument I have just written above does not prove that one can recover when $p_n > \beta > 1-1/|G|$. However, that can be seen by instead taking the arg min as the estimator, instead of the arg max. So overall a few changes to Theorem 1 seem to show that strong synchronization is possible whenever the flip probability is uniformly bounded away from $1-1/|G|$.*

Response: Thank you for these suggestions. I have revised Theorem 1 and its proof along the lines of these suggestions. As a result, I am pleased that I have strengthened Theorem 1 so that it is sufficient for the noise p_n to be bounded away from the critical constant $p_c = 1 - 1/|\mathfrak{G}|$. Surprisingly, I found that the $\arg\max$ estimator worked for the case when $p_n > \beta > p_c$ as well! And I have unified the proof so that it does not require separate cases for subcritical and supercritical values of p_n.

1.3. **Remarks Concerning Theorem 2.**

- *In Theorem 2, the author proves that strong group synchronization is impossible when the observation graph is of bounded degree d, has a large independent set D_n, and the flip probability $p_n \in (0,1/2)$ grows as $p_n = \omega(|D_n|^{-1/d})$. I like this result, and the technique used to prove it is clever. My only other comment is that I do not understand why the graph is assumed to be directed. This seems like an unnecessary complication. In my opinion, it would be simpler to just take $Y_n(w,u) = Y_n(u,w)^{-1}$. Although the result would be a tiny bit weaker the same core ideas would come into play and the notation would be a bit clearer.*

  Response: We appreciate the referees feedback for Theorem 2. Although we have elected to maintain a higher level of generality, we have added remarks to the proof of the Theorem to inform the reader that the proof simplifies under the additional hypothesis that the graph is undirected and that $Y_n(w,u) = Y_n(u,w)^{-1}$.

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