The Basel Problem and Summing Rational Functions over Integers

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Cover Page Footnote
I thank Dr. Chandrasheel Bhagwat for our fruitful discussions and his inputs on the manuscript. I am also grateful to the anonymous reviewer for their helpful inputs.
The Basel Problem and Summing Rational Functions over Integers

By Pranjal Jain

Abstract. We provide a general method to evaluate convergent sums of the form \( \sum_{k \in \mathbb{Z}} R(k) \) where \( R \) is a rational function with complex coefficients. The method is entirely elementary and does not require any calculus beyond some standard limits and convergence criteria. It is inspired by a geometric solution to the famous Basel Problem given by Wästlund (2010), so we begin by demonstrating the method on the Basel Problem to serve as a pilot application. We conclude by applying our ideas to prove Euler’s factorisation for \( \sin x \) which he originally used to solve the Basel Problem.

1 Introduction

The Basel Problem, first solved by Euler in 1734, asks for a closed form for the sum of reciprocal squares. Euler proved that

\[
\sum_{k \in \mathbb{N}} \frac{1}{k^2} = \frac{\pi^2}{6}
\]

Since then a myriad of proofs have been produced for this fact using techniques such as Taylor series [1], Taylor series in conjunction with integrals [2, 3] and trigonometric identities [4, 5]. Euler’s own solution (as well as his generalisation to \( \sum_{k \in \mathbb{N}} \frac{1}{k^m} \) for even \( m \)) is detailed in [6]. [7] provides a geometric solution which relies on a physical interpretation for the sum and presents a tangible link to circles, hence explaining the occurrence of \( \pi \). However, the key identity used in [7] is the same as that used in [5], albeit proved geometrically. [8] also provides a geometric relation to circles and builds upon the integral used in [1].

The method we present here is primarily inspired by the video adaptation [9] of the solution presented in [7], in particular by the idea of thinking of the real number line as a limit of circles with growing radius. To frame this formally, we consider the points \( \frac{n}{\pi} + 0i \) and \( \frac{n}{\pi} e^{\frac{i\pi}{2}(2k+1)} \) (\( 0 \leq k < n \)) on the circle of radius \( \frac{n}{\pi} \) centred at the origin. The idea

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in [9] is essentially to geometrically evaluate the sum

\[ A_n = \sum_{k=0}^{n-1} \frac{1}{n} e^{\frac{in}{\pi} (2k+1) - \frac{n}{\pi}} \]

and then argue that this must approach \( \sum_{k \in \mathbb{Z}} \frac{1}{(2k+1)^2} \) as \( n \to \infty \). This is done by rewriting using a phase shift as follows.

\[ A_n = \sum_{k=-\left\lfloor \frac{n}{2} \right\rfloor}^{\left\lceil \frac{n}{2} \right\rceil} \frac{1}{n} e^{\frac{in}{\pi} (2k+1) - \frac{n}{\pi}} \]

The arc from \( \frac{n}{\pi} + 0i \) to \( \frac{n}{\pi} e^{\frac{in}{\pi} (2k+1)} \) for small \( \frac{|k|}{n} \) is on one hand of length precisely

\[ \left| \frac{\pi}{n} (2k+1) \right| \frac{n}{\pi} = |2k+1|, \]

and on the other hand is almost a straight line. Hence, for small \( \frac{|k|}{n} \) we have

\[ \left| \frac{n}{\pi} e^{\frac{in}{\pi} (2k+1)} - \frac{n}{\pi} \right| \approx |2k+1| \]

Taking \( n \to \infty \), the set of values of \( k \) for which this approximation ‘works well’ keeps growing. Hence, we intuitively expect

\[ \lim_{n \to \infty} A_n = \sum_{k \in \mathbb{Z}} \frac{1}{(2k+1)^2} \]

**Remark 1.1.** The values of \( k \) for which the above approximation does not work are also those for which \( \left| \frac{n}{\pi} e^{\frac{in}{\pi} (2k+1)} - \frac{n}{\pi} \right| \) is large, and so \( \frac{1}{\left| \frac{n}{\pi} e^{\frac{in}{\pi} (2k+1)} - \frac{n}{\pi} \right|^2} \) is small. Hence it is reasonable to expect that terms corresponding to such values of \( k \) can be ignored.
Figure 1: Assuming that the radius of the circle is $\frac{6}{\pi}$, $A_6$ is the sum of squared reciprocals of lengths of these line-segments. The smallest line-segments closely approximate the corresponding arcs.

From here, one can use the relation $\frac{2}{3} \sum_{k \in \mathbb{Z}} \frac{1}{(2k+1)^2} = \sum_{k \in \mathbb{N}} \frac{1}{k^2}$ and solve the Basel Problem. Our main modification is to drop the absolute value in the expression for $A_n$. To see that one can still evaluate $\sum_{k \in \mathbb{Z}} \frac{1}{(2k+1)^2}$ using this modified $A_n$, we note that for small $\frac{|k|}{n}$ we have

$$ne^{i\pi k (2k+1)} - n \approx i\pi(2k + 1)$$

since the derivative of $e^z$ at $z = 0$ is 1. Hence, if we modify the definition of $A_n$ to

$$A_n = \frac{1}{n^2} \sum_{k=0}^{n-1} \frac{1}{\left[e^{i\pi(2k+1)} - 1\right]^2}$$

then we would expect by the same intuition as before that

$$\sum_{k \in \mathbb{Z}} \frac{1}{(2k+1)^2} = \lim_{n \to \infty} (i\pi)^2 A_n$$

Now the obvious question which arises is whether this modified expression for $A_n$ allows us to evaluate its limit. It turns out that $A_n = -\frac{2n^2}{4n}$, hence yielding $\sum_{k \in \mathbb{Z}} \frac{1}{(2k+1)^2} = \frac{\pi^2}{4}$ if we are to rely on the above argument. This expression for $A_n$ is obtained as part of proving Theorem 3.1. Now, notice that this idea can be generalised a lot – since $ne^{i\pi(2k+1)} - n \approx i\pi(2k + 1)$, we can take any continuous function $f : \mathbb{C} \to \mathbb{C}$ and (subject to convergence) convert the sum $\sum_{k \in \mathbb{Z}} \frac{1}{f(2k+1)}$ into the limit

$$\lim_{n \to \infty} \sum_{k=0}^{n-1} \frac{1}{f\left(\frac{n}{i\pi} \left[e^{i\pi(2k+1)} - 1\right]\right)}$$
Hence, $\sum_{k \in \mathbb{Z}} \frac{1}{f(2k+1)}$ can be evaluated if the above limit can be evaluated – evaluating this limit for various polynomial choices of $f$ is essentially what we will mostly focus on. We formally state the above idea as follows, for $f$ sufficiently nice.

**Definition 1.2 (Nice function).** Say that a continuous function $f : \mathbb{C} \rightarrow \mathbb{C}$ is ‘nice’ if it satisfies the following.

1. $\exists R \geq 0$ such that $|f(z_1)| \geq |f(z_2)|$ whenever $|z_1| \geq |z_2| > R$.
2. $\sum_{k \in \mathbb{Z}} \frac{1}{|f(2k+1)|}$ converges.
3. $f$ has only finitely many roots (if any).

**Remark 1.3.** It is easy to see from the definition that $\lim_{|z| \to \infty} \frac{1}{f(z)} = 0$. Furthermore all polynomials of degree at least 2 are nice, which is important for our purposes.

**Proposition 1.4.** Let $f : \mathbb{C} \rightarrow \mathbb{C}$ be a nice function. Then

$$\sum_{k \in \mathbb{Z}} \frac{1}{f(2k+1)} = \lim_{n \to \infty} \frac{1}{f \left( \frac{n}{\pi} \left[ e^{i \pi (2k+1)} - 1 \right] \right)}$$

The idea used in [5] is essentially the same – the following approximation is used.

$$2k+1 \approx \frac{2n}{\pi} \left| \sin \frac{\pi (2k+1)}{2n} \right|$$

However, the result along the lines of **Proposition 1.4** which this approximation yields cannot be applied as widely as the one we present since the limit obtained cannot be evaluated for as large a variety of choices of $f$.\(^1\) To demonstrate how generally our method is applicable, we state our main results.

**Theorem 1.5.** Let $\alpha \in \mathbb{C}$ ($\alpha$ not an odd integer) and $m > 1$ be an integer. Then

$$\sum_{k \in \mathbb{Z}} \frac{1}{(2k+1-\alpha)^m} = \frac{-(i\pi)^m}{(1+e^{i\pi\alpha}) (m-1)!} \sum_{t=0}^{m-1} \sum_{s=0}^{t} (-1)^s \left( 1 + e^{i\pi\alpha} \right)^{-t} \left( \frac{t}{s} \right) (s+1)^{m-1}$$

\(^1\)A similar occurrence of the classic theme ‘simplification via complexification’ can be seen in the solution given by [2]. Via Taylor series expansion of the integrand, we obtain

$$\int_{-\infty}^{0} \ln (1 + e^x) \, dx = \sum_{k \in \mathbb{N}} \frac{(-1)^{k+1}}{k^2} = \frac{1}{2} \sum_{k \in \mathbb{N}} \frac{1}{k^2}$$

This is a dead-end most would have encountered when trying to solve the Basel Problem, since there is no elementary antiderivative for the integrand. The trick used in [2] deals with this problem by replacing $x$ by $ix$ (and changing the integration bounds accordingly), which makes it easy to find the antiderivative of the imaginary part of the integrand.
**Theorem 1.6.** Let $\alpha, \beta \in \mathbb{C}$ so that $(x - \alpha)^2 + \beta$ has no odd integer roots. Then

$$
\sum_{k \in \mathbb{Z}} \frac{1}{(2k + 1 - \alpha)^2 + \beta} = \frac{\pi e^{i\pi\alpha} \sinh(\pi\sqrt{\beta})}{\sqrt{\beta}(1 + 2 e^{i\pi\alpha} \cosh(\pi\sqrt{\beta}) + e^{2i\pi\alpha})}
$$

Where the case of $\beta = 0$ is covered by taking limits.

**Remark 1.7.** The denominator in the right side can be factorised as follows in terms of the roots $r_1$ and $r_2$ of $(x - \alpha)^2 + \beta$.

$$
\left(1 + 2 e^{i\pi\alpha} \cosh(\pi\sqrt{\beta}) + e^{2i\pi\alpha}\right) \left(1 + e^{i\pi\alpha + \pi\sqrt{\beta}}\right) = \left(1 + e^{i\pi r_1}\right)\left(1 + e^{i\pi r_2}\right)
$$

**Remark 1.8.** [5] and [7] proved Theorem 1.6 for $\beta = 0$. Some special sums of this form over positive integers are considered in [10].

Using these two results, we give a method to evaluate $\sum_{k \in \mathbb{Z}} R(k)$ for any rational function $R$ with complex coefficients and denominator degree at least two more than the numerator degree. From here it also follows how one can evaluate $\sum_{k \in \mathbb{Z}} s(k)R(k)$, where $s : \mathbb{Z} \to \mathbb{C}$ is any periodic function. Furthermore, as a by-product of combining Theorem 1.5 for $\alpha = 0$ with the well-known relation between $\sum_{k \in \mathbb{Z}} \frac{1}{(2k + 1)^m}$ and $B_m$ (the $m$-th Bernoulli number) for $m > 1$, we obtain the following identity for $B_m$.

**Corollary 1.9.** The Bernoulli number $B_m$ ($m > 1$) is given by

$$
B_m = \frac{m}{2(2^m - 1)} \sum_{t=0}^{m-1} \sum_{s=0}^{t} (-1)^s 2^{-t} \binom{t}{s} (s + 1)^{m-1}
$$

We conclude by using our method to prove the insightful factorisation of $\sin z$ first given by Euler (without proof) to solve the Basel Problem.

**Theorem 1.10.** Let $z \in \mathbb{C}$. Then

$$
\sin z = z \prod_{k \in \mathbb{N}} \left(1 - \frac{z^2}{k^2 \pi^2}\right)
$$

## 2 Some preliminaries

### 2.1 Notation

Let $\omega_m = e^{\frac{2\pi i}{m}}$, a primitive $m$-th root of unity. We will sometimes use the notation $\text{cis} \theta = e^{i\theta}$ to avoid complicated exponents. Some sets which are useful for the combinatorics in...
the proof of Theorem 1.5 are as follows.

\[ [n] = \{1, \ldots, n\} \]
\[ [n]_0 = \{0\} \cup [n] \]
\[ K_t = \{ \kappa \in \mathbb{N}^t \mid k_1 + \ldots + k_t = m \} \] (components of \( \kappa \))
\[ R_t = \{ \rho \in [m - 1]^t \mid r_1, \ldots, r_t \text{ are distinct} \} \] (components of \( \rho \))
\[ R'_t = \{ \rho' \subset [m - 1] \mid |\rho'| = t \} \]

\( K_t \) is the set of ordered partitions of \( m \) into \( t \) positive integers, \( R_t \) is the set of \( t \)-tuples with distinct elements from \( \{0, \ldots, m-1\} \), and \( R'_t \) is the set of \( t \)-subsets of \( \{0, \ldots, m-1\} \). As above, \( k_1, \ldots, k_t \) and \( r_1, \ldots, r_t \) will be used to denote the components of \( \kappa \in K_t \) and \( \rho \in R_t \) respectively. For \( \rho' \in R'_t \), we will denote its elements by \( r'_1, \ldots, r'_t \) in ascending order. The choice of \( \kappa, \rho \) and \( \rho' \) will be clear whenever these conventions are used.

### 2.2 Preliminary results

Let \( t \in \mathbb{N} \) and \((A, +)\) be an additive abelian group (such as a group of polynomials under addition). Suppose \( f, g : \mathcal{P}([t]) \to A \) are maps from the power set \( \mathcal{P}([t]) \) of \([t]\) to \( A \) satisfying

\[
g(S) = \sum_{T \subseteq S} f(T) \quad \forall S \subseteq [t] \]

If \( f \) were unknown and \( g \) were known, the above can be thought of as a system of linear equations for the unknowns \( f(T) \). The principle of Möbius Inversion gives an explicit solution as follows.

\[
f(T) = \sum_{S \subseteq T} (-1)^{|T| - |S|} g(S) \quad \forall T \subseteq [t] \]

where \( |T| \) is the cardinality of \( T \). In particular, for \( T = [t] \) we have

\[
f([t]) = \sum_{S \subseteq [t]} (-1)^{t - |S|} g(S) \]

Let \( \{B_m\}_{m \geq 2} \) be the Bernoulli numbers. It is known that \( B_m = 0 \) when \( m \) is odd. For \( m \) even it is known that

\[
\sum_{k \in \mathbb{N}} \frac{1}{k^m} = \frac{-2^{m-1}(i\pi)^m}{m!} B_m
\]

A proof of this identity can be found in [11]. Hence, for \( m \) even we obtain

\[
\sum_{k \geq 0} \frac{1}{(2k + 1)^m} = \frac{2^{m-1} - 1}{2^m} \sum_{k \in \mathbb{N}} \frac{1}{k^m} = \frac{-(2^m - 1)(i\pi)^m}{2 \cdot m!} B_m
\]

Doubling this sum by allowing \( k \) to be negative, we have the following for \( m \) even.

\[
\sum_{k \in \mathbb{Z}} \frac{1}{(2k + 1)^m} = \frac{-(2^m - 1)(i\pi)^m}{m!} B_m \tag{1}
\]
Now, observe that the above holds for \( m \) odd as well – in the left side, all terms cancel and we obtain 0, and the right side is also 0 since \( B_m = 0 \) for \( m \) odd. Hence, (1) holds for all \( m \geq 2 \).

**Remark 2.1.** The reader may justifiably conclude at this point that the mention of (1) in this paper as a preliminary result is rather circular. To address this, we mention that our use of (1) will only be to obtain an expression for the Bernoulli numbers in terms of the infinite summation \( \sum_{k \in \mathbb{Z}} \frac{1}{(2k+1)^m} \), and not the other way around. Hence, any formula we obtain for \( \sum_{k \in \mathbb{Z}} \frac{1}{(2k+1)^m} \) in this paper will also give a formula for the Bernoulli numbers.

The following result is useful for justifying taking term-wise limits in infinite summations. It follows at once from the Dominated Convergence Theorem for the counting measure on \( \mathbb{Z} \), but an elementary proof can be found in [5].

**Lemma 2.2 (Tannery’s Theorem).** Consider the finite sum

\[
S_n = \sum_{|k| < \frac{n}{2}} t_k(n)
\]

with \( t_k(n) \in \mathbb{C} \). If \( \lim_{n \to \infty} t_k(n) = t_k \) and \( |t_k(n)| < M_k \) for each \( n > 2|k| \) with \( \sum_{k \in \mathbb{Z}} M_k < \infty \), then

\[
\lim_{n \to \infty} S_n = \sum_{k \in \mathbb{Z}} t_k
\]

We shall require also a product version of Lemma 2.2 to justify taking term-wise limits in products. One may find it tempting to use logarithms for this purpose, but that would require minute care when choosing branches (since we are working over \( \mathbb{C} \)). Hence, we instead adopt an elementary real analysis approach.

**Lemma 2.3 ([12]).** Let \( a_1, a_2, \ldots > 0 \) and \( s_k = a_1 + \ldots + a_k \). Then \( \sum_{k \in \mathbb{N}} a_k \) converges if and only if \( \sum_{k \in \mathbb{N}} \frac{a_k}{s_k} \) converges.

**Proof.** The forward implication is obvious. For proving the reverse implication we will prove its contrapositive. Hence, suppose \( \sum_{k \in \mathbb{N}} a_k \) diverges and we will show that \( \sum_{k \in \mathbb{N}} \frac{a_k}{s_k} \) also diverges. Since the \( a_i \)'s are positive, the fact that their sum diverges means that it goes to \( \infty \). Hence, for each \( k \in \mathbb{N} \) there exists some \( r(k) > k \) such that \( s_{r(k)} \geq 2s_k \). This yields

\[
\frac{a_{k+1}}{s_{k+1}} + \ldots + \frac{a_{r(k)}}{s_{r(k)}} \geq \frac{a_{k+1} + \ldots + a_{r(k)}}{s_{r(k)}}
\]

\[
= \frac{s_{r(k)} - s_k}{s_{r(k)}}
\]

\[
\geq \frac{1}{2}
\]
Hence, we may write
\[
\sum_{k=1}^{\infty} \frac{a_k}{s_k} = \frac{a_1}{s_1} + \frac{r(1)}{2} + \frac{r(r(1))}{2} + \ldots = \infty
\]

Lemma 2.4. Let \( r_1, r_2, \ldots > 0 \). Then \( \prod_{k \in \mathbb{N}} (1 + r_k) \) converges if and only if \( \sum_{k \in \mathbb{N}} r_k \) converges.

Proof. Let \( s_k = (1 + r_1)(1 + r_2) \ldots (1 + r_k) \), \( a_1 = s_1 > 0 \) and \( a_k = s_k - s_{k-1} > 0 \) for \( k > 1 \). By Lemma 2.3, \( \lim_{k \to \infty} s_k \) converges if and only if \( \sum_{k \in \mathbb{N}} \frac{a_k}{s_k} \) converges. Now, we have
\[
\sum_{k \in \mathbb{N}} \frac{a_k}{s_k} = \frac{a_1}{s_1} + \sum_{k=2}^{\infty} \frac{s_k - s_{k-1}}{s_k} = 1 + \sum_{k=2}^{\infty} \left( 1 - \frac{1}{1 + r_k} \right) = 1 + \sum_{k=2}^{\infty} \frac{r_k}{1 + r_k}
\]
Since \( r_k > 0 \), the above converges if and only if \( \sum_{k=2}^{\infty} r_k \) converges. \( \square \)

The product version of Lemma 2.2 now follows.

Lemma 2.5. Consider the finite product
\[
P_n = \prod_{k=1}^{n} (1 + r_k(n))
\]
with \( r_k(n) \in \mathbb{C} \). If \( \lim_{n \to \infty} r_k(n) = r_k \) and \( |r_k(n)| < R_k \) for each \( n \geq k \) with \( \sum_{k \in \mathbb{N}} R_k < \infty \), then
\[
\lim_{n \to \infty} P_n = \prod_{k \in \mathbb{N}} (1 + r_k)
\]

Proof. Let \( t_1(n) = 1 + r_1(n) \) and for \( 1 < k \leq n \), define
\[
t_k(n) = \prod_{i=1}^{k} [1 + r_i(n)] = \prod_{i=1}^{k-1} [1 + r_i(n)] - \prod_{i=1}^{k-1} [1 + r_i(n)] = r_k(n) \prod_{i=1}^{k-1} [1 + r_i(n)]
\]
Let
\[
t_k = \lim_{n \to \infty} t_k(n) = r_k \prod_{i=1}^{k-1} (1 + r_i)
\]
Hence, we have
\[ P_n = \sum_{k=1}^{n} t_k(n) \]

Hence, if we can find bounds \( M_k \) with \( t_k(n) < M_k \forall n \geq k \) and \( \sum_{k \in \mathbb{N}} M_k < \infty \) then a ‘one-sided’ version of Lemma 2.2 would yield
\[ \lim_{n \to \infty} P_n = \sum_{k \in \mathbb{N}} t_k(n) = \prod_{k \in \mathbb{N}} (1 + r_k) \]

For this, note that since \( \sum_{k \in \mathbb{N}} R_k < \infty \) and \( R_k > 0 \), we may conclude by Lemma 2.4 that \( \prod_{k \in \mathbb{N}} (1 + R_k) = M' < \infty \). Hence, we have
\[
|t_k(n)| = |r_k(n)| \prod_{i=1}^{k-1} |1 + r_i(n)|
< R_k \prod_{i=1}^{k-1} (1 + R_i) \text{ (using triangle inequality)}
< R_k M'
\]

Hence \( M_k = R_k M' \) works. \( \Box \)

3 The core idea and a pilot application – the Basel Problem

Proof of Proposition 1.4. The circles \( C_n = \{ z \in \mathbb{C} \mid \| z + \frac{n}{n^2} \| = \frac{n}{n} \} \) (\( n \in \mathbb{N} \)) intersect only at the origin. Since \( f \) has only finitely many roots, this means that we may assume \( n \) to be sufficiently large so that \( A_n \) and \( S_n \) as defined below exist (i.e. no term is \( \frac{1}{0} \)).

\[
A_n = \sum_{k=0}^{n-1} \frac{1}{f \left( \frac{n}{n^2} \left[ e^{\frac{i\pi}{n}}(2k+1) - 1 \right] \right)}
\]

\[
S_n = \sum_{|k| < \frac{n}{2}} \frac{1}{f \left( \frac{n}{n^2} \left[ e^{\frac{i\pi}{n}}(2k+1) - 1 \right] \right)}
\]

We will first show that \( \lim_{n \to \infty} A_n - S_n = 0 \) and then apply term-wise limits on \( S_n \) using Lemma 2.2. Clearly, the \( k \)-th term of \( A_n \) does not change if \( k \) is replaced by \( k + jn \) for any \( j \in \mathbb{Z} \), i.e. class of \( k \) modulo \( n \) determines the value of the \( k \)-th term in \( A_n \). Since any set \( D \subset \mathbb{Z} \) of \( n \) consecutive integers has exactly one representative of each class of integers modulo \( n \), we have
\[
A_n = \sum_{k \in D} \frac{1}{f \left( \frac{n}{n^2} \left[ e^{\frac{i\pi}{n}}(2k+1) - 1 \right] \right)}
\]
In particular, we have

\[
A_n = \begin{cases} 
\sum_{k=1-\frac{n}{2}}^{\frac{n}{2}} f\left(\frac{n}{i\pi} e^{\frac{i\pi}{n}(2k+1)} - 1\right) & n \text{ even} \\
\sum_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} f\left(\frac{n}{i\pi} e^{\frac{i\pi}{n}(2k+1)} - 1\right) & n \text{ odd}
\end{cases}
\]

Hence, for \( n \) even we get

\[
A_n - S_n = \frac{1}{f\left(\frac{n}{i\pi} e^{\frac{i\pi}{n}(n+1)} - 1\right)} \left(\text{the } k = \frac{n}{2} \text{ term remains}\right)
\]

\[
= \frac{1}{f\left(\frac{n}{i\pi} e^{\frac{i\pi}{n}} + 1\right)}
\]

This error term goes to 0 as \( n \to \infty \) since \( f \) is nice. For \( n \) odd, \( A_n = S_n \). Hence, we may now work with \( S_n \) instead of \( A_n \).

Figure 2: The circles \( C_n \). The points marked on the imaginary axis are spaced \( \frac{1}{n} \) units apart.
Define \( t_k(n) \) for \( n > 2|k| \) as

\[
t_k(n) = \frac{1}{f\left(\frac{n}{i\pi} \left[ e^{\frac{i\pi}{n}(2k+1)} - 1 \right] \right)}
\]

It is clear that the limit of each term is given by \( \lim_{n \to \infty} t_k(n) = t_k = \frac{1}{f\left(\frac{2k+1}{2}\right)} \) (by continuity of \( f \)), so to apply Lemma 2.2 we only need to show that the \( M_k \)’s exist. For this, we observe

\[
e^{i\theta} - 1 = (\cos \theta - 1) + i \sin \theta
\]

\[
= -2\sin^2 \frac{\theta}{2} + 2i \sin \frac{\theta}{2} \cos \frac{\theta}{2}
\]

\[
= 2i e^{i\frac{\theta}{2}} \sin \frac{\theta}{2}
\]

\[
\Rightarrow |e^{i\theta} - 1| = 2 \left| \sin \frac{\theta}{2} \right|
\]

\[
\geq \frac{|\theta|}{2} \quad \text{(for } |\theta| \leq \pi)\]

Since \( n > 2|k| \), this yields

\[
|e^{\frac{i\pi}{n}(2k+1)} - 1| \geq \frac{\pi}{2n} |2k + 1|
\]

Hence, we may bound the argument of \( f \) from below.

\[
\left| \frac{n}{i\pi} \left[ e^{\frac{i\pi}{n}(2k+1)} - 1 \right] \right| \geq \frac{|2k + 1|}{2n} = \frac{|k|}{2n}
\]

Hence, the first condition of niceness yields the following for sufficiently large \( |k| \).

\[
\left| f\left(\frac{n}{i\pi} \left[ e^{\frac{i\pi}{n}(2k+1)} - 1 \right] \right) \right| \geq f\left(\frac{2k + 1}{2}\right)
\]

\[
\Rightarrow t_k(n) \leq \frac{1}{f\left(\frac{2k+1}{2}\right)}
\]

We may now choose \( M_k \) as

\[
M_k = \frac{1}{f\left(\frac{2k+1}{2}\right)}
\]

The bound \( t_k(n) \leq M_k \) might not be true for some finitely many values of \( k \) (since we assumed \( |k| \) is sufficiently large), so the limit of terms corresponding to these values can be taken separately. It can be seen that \( \sum M_k < \infty \) (sum over values of \( k \) for which \( M_k \) is defined) using the first two conditions of niceness. \( \square \)
Now we are ready to start evaluating summations.

**Theorem 3.1.**

\[
\sum_{k \in \mathbb{Z}} \frac{1}{(2k+1)^2} = \frac{\pi^2}{4}
\]

**Proof.** Define \( A_n \) as follows.

\[
A_n = \frac{1}{n^2} \sum_{k=0}^{n-1} \frac{1}{\left( e^{i\pi (2k+1)/n} - 1 \right)^2}
\]

By Proposition 1.4 (with \( f(x) = x^2 \)), we wish to show that \( \lim_{n \to \infty} A_n = \frac{-1}{4} \). We will in fact find a closed-form for \( A_n \), which is \( A_n = -\frac{n^2}{4} \). For this, consider the polynomial \( p(x) \) defined below. \( p(x) \) is constructed so the ratio of the coefficients of \( x^2 \) and \( x^0 \) respectively gives \(-n^2A_n\).

\[
p(x) = \prod_{k=0}^{n-1} \left( x^2 - \left( e^{i\pi (2k+1)/n} - 1 \right)^2 \right)
\]

\[
= \prod_{k=0}^{n-1} \left( x - 1 + e^{i\pi (2k+1)/n} \right) \left( x + 1 - e^{i\pi (2k+1)/n} \right)
\]

\[
= \prod_{k=0}^{n-1} \left( -e^{i\pi/n} \right) \left( 1-x \right) e^{-i\pi/n} - e^{2i\pi/k} \left( x + 1 \right) e^{-i\pi/n} - e^{2i\pi/n}
\]

\[
= (-1)^n \left( 1 - x \right)^n - (-1 + x)^n - 1
\]

\[
= (-1)^n \left( 2 - nx + \binom{n}{2} x^2 - \ldots \right) + \left( 2 + nx + \binom{n}{2} x^2 + \ldots \right)
\]

Now the ratio of coefficients of \( x^2 \) and \( x^0 \) can be seen to be

\[-n^2A_n = \frac{2\binom{n}{2} - n^2 + 2\binom{n}{2}}{4} = \frac{n(n-2)}{4} \]

\[\square\]

### 4 A first generalisation

We will now present a simple generalisation of the method used in the proof of Theorem 3.1. Although the result it yields does turn an infinite sum into a finite computation (hence ‘evaluating’ the infinite sum), the form of the answer is not very wieldy. The work required to get a better form is done as part of proving Theorem 1.5.

**Theorem 4.1.** Let \( \alpha \in \mathbb{C} \) (\( \alpha \) not an odd integer) and \( m > 1 \) an integer. Then

\[
\sum_{k \in \mathbb{Z}} \frac{1}{(2k+1-\alpha)^m} = \frac{-(i\pi)^m a_m,\alpha}{(1 + e^{i\pi\alpha})^m}
\]
where \( a_{m,\alpha} \) is the coefficient of \( x^m \) in the polynomial

\[
j_{m,\alpha}(x) = \prod_{r=0}^{m-1} \left( 1 + e^{i\pi \alpha} \sum_{k=0}^{m} \frac{1}{k!} \omega_m^{kr} x^k \right)
\]

**Proof.** Let \( a = \alpha i \pi \) and define \( A_n \) analogous to the proof of Theorem 3.1.

\[
A_n = \frac{1}{n^m} \sum_{k=0}^{n-1} \frac{1}{e^{i\pi (2k+1)} - 1 - \frac{a}{n}}^m
\]

By Proposition 1.4 (with \( f(x) = (x-\alpha)^m \)), we have

\[
\sum_{k \in \mathbb{Z}} \frac{1}{(2k+1-\alpha)^m} = \lim_{n \to \infty} (i\pi)^m A_n
\] (2)

Hence, we may now focus on obtaining an expression for \( A_n \) which allows us to evaluate the desired limit. As in the proof of Theorem 3.1, define the polynomial \( p(x) \) as follows so that \(-n^m A_n\) is given by the ratio of coefficients of \( x^m \) and \( x^0 \). We will ignore leading coefficients by absorbing them into a formal constant \( c \) since they do not affect the ratio we are after.

\[
c p(x) = \prod_{k=0}^{n-1} \left( x^m - \left[ e^{i\pi (2k+1)} - 1 - \frac{a}{n} \right]^m \right)
\]

\[
= \prod_{k=0}^{n-1} \prod_{r=0}^{m-1} \left( x - \omega_m^r \left[ e^{i\pi (2k+1)} - 1 - \frac{a}{n} \right] \right)
\]

\[
= \prod_{k=0}^{n-1} \prod_{r=0}^{m-1} \left( \omega_m^r e^{i\pi} \right) \left[ \left( \omega_m^{-r} x + 1 + \frac{a}{n} \right) e^{-i\pi} - e^{2i\pi} \right]
\]

\[
= \prod_{r=0}^{m-1} \left[ - \left( \omega_m^{-r} x + 1 + \frac{a}{n} \right)^n - 1 \right] \quad \text{(by switching the order of products)}
\]

\[
= \prod_{r=0}^{m-1} \left[ (\omega_m^{-r} x + 1 + \frac{a}{n})^n + 1 \right]
\]

\[
= \prod_{r=0}^{m-1} \left[ 1 + \sum_{k=0}^{n} \left( \frac{n}{k} \right) \omega_m^{-kr} \left( 1 + \frac{a}{n} \right)^{n-k} x^k \right]
\]

Since we only care about the coefficients of \( x^m \) and \( x^0 \), we may reduce \( p(x) \) modulo \( x^{m+1} \), i.e. ignore terms with \( x^k \) for \( k > m \). We can also replace \( \omega_m^{-1} \) by \( \omega_m \) by symmetry in the values of \( r \). Let \( q(x) \) be the polynomial obtained on making these simplifications.

\[
q(x) = \prod_{r=0}^{m-1} \left[ 1 + \sum_{k=0}^{m} \left( \frac{n}{k} \right) \omega_m^{kr} \left( 1 + \frac{a}{n} \right)^{n-k} x^k \right]
\]
Since the ratio of coefficients of \(x^m\) and \(x^0\) in \(q(x)\) gives \(-n^m A_n\), (2) yields

\[
\sum_{k \in \mathbb{Z}} \frac{1}{(2k + 1 - \alpha)^m} = \lim_{n \to \infty} \frac{-(i\pi)^m}{n^m} \left[ \text{ratio of coefficients of } x^m \text{ and } x^0 \text{ in } q(x) \right]
\]

\[
= -(i\pi)^m \lim_{n \to \infty} \left[ \text{ratio of coefficients of } x^m \text{ and } x^0 \text{ in } q\left(\frac{x}{n}\right) \right] \quad (3)
\]

Since \(\lim_{n \to \infty} q\left(\frac{x}{n}\right)\) converges to \(j_{m,a}(x)\) as defined in the theorem statement, by (3) we have

\[
\sum_{k \in \mathbb{Z}} \frac{1}{(2k + 1 - \alpha)^m} = -(i\pi)^m \left[ \text{ratio of coefficients of } x^m \text{ and } x^0 \text{ in } j_{m,a}(x) \right]
\]

The coefficient of \(x^0\) in \(j_{m,a}(x)\) is \((1 + e^{i\pi a})^m\) so the claimed result follows. \(\square\)

We now expand the expression for \(j_{m,a}(x)\) to obtain an explicit expression for the solution provided by Theorem 4.1 using some combinatorial manipulation of summations.

**Proof of Theorem 1.5.** By Theorem 4.1 it suffices to evaluate \(a_{m,a}\), the coefficient of \(x^m\) in \(j_{m,a}(x)\). Expanding the expression for \(j_{m,a}(x)\) yields

\[
a_{m,a} = \sum_{t=1}^{m} e^{i\pi a t} \left(1 + e^{i\pi a}\right)^{m-t} \sum_{\kappa \in K_t} \frac{\omega_m}{k_1! \cdots k_t!} k_1 r_1' + \cdots + k_t r_t'
\]

\[
= \sum_{t=1}^{m} e^{i\pi a t} \left(1 + e^{i\pi a}\right)^{m-t} \sum_{\kappa \in K_t} \frac{1}{k_1! \cdots k_t!} \text{cis} \left[ \frac{2\pi}{m} (k_1 r_1' + \cdots + k_t r_t') \right]
\]

\[
= \sum_{t=1}^{m} e^{i\pi a t} \left(1 + e^{i\pi a}\right)^{m-t} \sum_{\kappa \in K_t} \frac{1}{k_1! \cdots k_t!} \text{cis} \left[ \frac{2\pi}{m} (k_1 r_1 + \cdots + k_t r_t) \right] \quad (4)
\]

Now, we will focus on the following quantity for each \(t \in \{1, \ldots, m\}\) and \(\kappa \in K_t\).

\[
q(t, \kappa) = \sum_{\rho \in R_t} \text{cis} \left[ \frac{2\pi}{m} (k_1 r_1 + \cdots + k_t r_t) \right] \quad (5)
\]

We will prove the following by induction on \(t\).

\[
q(t, \kappa) = (-1)^{t+1}(t-1)! m \quad (6)
\]

If \(t = 1\), then \(\kappa = (m)\) and \(\rho = (r_1)\). Hence we have the following.

\[
q(1, \kappa) = \sum_{\rho \in R_1} \text{cis} \left[ \frac{2\pi}{m} k_1 r_1 \right] = \sum_{r_1=0}^{m-1} \text{cis} \left( \frac{2\pi}{m} m r_1 \right) = m
\]
Now suppose the claim is true for \( t - 1 \), and we will prove it for \( t \) (where \( 1 < t \leq m \)). We have

\[
q(t, \kappa) = \sum_{\rho \in \mathbb{R}_{t-1}} \operatorname{cis} \left( \frac{2\pi}{m} (k_1 r_1 + \ldots + k_t r_t) \right)
\]

\[
= \sum_{\rho \in \mathbb{R}_{t-1}} \left( \sum_{r=0}^{m-1} \operatorname{cis} \left( \frac{2\pi}{m} (k_1 r_1 + \ldots + k_{t-1} r_{t-1} + k_t r) \right) - \sum_{j=1}^{t-1} \operatorname{cis} \left( \frac{2\pi}{m} (k_1 r_1 + \ldots + k_{t-1} r_{t-1} + k_t r_j) \right) \right)
\]

(7)

Consider these two summations independently. First we will evaluate

\[
\sum_{r=0}^{m-1} \operatorname{cis} \left( \frac{2\pi}{m} (k_1 r_1 + \ldots + k_{t-1} r_{t-1} + k_t r) \right)
\]

Let \( k_1 r_1 + \ldots + k_{t-1} r_{t-1} = N \in \mathbb{Z} \). Hence, the above summation becomes

\[
\sum_{r=0}^{m-1} \operatorname{cis} \left( \frac{2N\pi}{m} + \frac{2k_t\pi}{m} r \right)
\]

\[
= \operatorname{cis} \left( \frac{2N\pi}{m} \right) \sum_{r=0}^{m-1} \operatorname{cis} \left( \frac{2k_t\pi}{m} r \right)
\]

By definition we know that \( 0 < k_t \), and since \( t > 1 \) we also have \( k_t < m \). Hence the above summation is 0, which yields

\[
\sum_{r=0}^{m-1} \operatorname{cis} \left( \frac{2\pi}{m} (k_1 r_1 + \ldots + k_{t-1} r_{t-1} + k_t r) \right) = 0
\]

Hence, by (7) we obtain

\[
q(t, \kappa) = -\sum_{\rho \in \mathbb{R}_{t-1}} \sum_{j=1}^{t-1} \operatorname{cis} \left( \frac{2\pi}{m} (k_1 r_1 + \ldots + k_{t-1} r_{t-1} + k_t r_j) \right)
\]

\[
= -\sum_{\rho \in \mathbb{R}_{t-1}} \sum_{j=1}^{t-1} \operatorname{cis} \left( \frac{2\pi}{m} (k_1 r_1 + \ldots + (k_j + k_t) r_j + \ldots + k_{t-1} r_{t-1}) \right)
\]

\[
= -\sum_{j=1}^{t-1} \sum_{\rho \in \mathbb{R}_{t-1}} \operatorname{cis} \left( \frac{2\pi}{m} (k_1 r_1 + \ldots + (k_j + k_t) r_j + \ldots + k_{t-1} r_{t-1}) \right)
\]

\[
= -\sum_{j=1}^{t-1} q(t-1, (k_1, \ldots, k_j + k_t, \ldots, k_{t-1})]
\]

\[
= -(t-1)(-1)^t(t-2)!m \quad \text{(by induction hypothesis)}
\]

\[
= (-1)^t+1(t-1)!m
\]
This completes the proof of (6). Now, plugging (5) into (4) yields

\[
a_{m,\alpha} = \sum_{t=1}^{m} \frac{e^{i\pi t} (1 + e^{i\pi\alpha})^{m-t}}{t!} \sum_{\kappa \in K_t} \frac{q(t, \kappa)}{k_1! \cdots k_t!}
\]

\[
= \sum_{t=1}^{m} \frac{e^{i\pi t} (1 + e^{i\pi\alpha})^{m-t}}{t} \sum_{\kappa \in K_t} \frac{(-1)^{t+1} m}{k_1! \cdots k_t!} \quad \text{(by (6))}
\]

\[
= \frac{1}{(m-1)!} \sum_{t=1}^{m} \frac{(-1)^{t+1} e^{i\pi t} (1 + e^{i\pi\alpha})^{m-t}}{t} \sum_{\kappa \in K_t} \left( \frac{m}{k_1, \ldots, k_t} \right)
\]

(8)

Recall that the inner summation over \( \kappa \) is over all ordered partitions \( \kappa = (k_1, \ldots, k_t) \) of \( m \) into positive integers\(^2\). Hence, the inner sum is precisely what we would get by plugging \( x_1 = \ldots = x_t = 1 \) into the polynomial \( f(x_1, \ldots, x_t) \) obtained by ignoring those terms in \( (x_1 + \ldots + x_t)^m \) which do not contain at least one of the variables. To evaluate this, define

\[
g(x_1, \ldots, x_t) = (x_1 + \ldots + x_t)^m
\]

With each \( S \subset [t] \) associate the polynomial \( g(S) \) obtained by setting \( x_i = 0 \) in \( g(x_1, \ldots, x_t) \) for each \( i \notin S \). Formally,

\[
g(S) = \left( \sum_{i \in S} x_i \right)^m
\]

Likewise, let \( f(S) \) be the polynomial obtained by plugging \( x_i = 0 \) in \( f(x_1, \ldots, x_t) \) for each \( i \notin S \). Hence,

\[
g(S) = \sum_{T \subset S} f(T) \quad \forall S \subset [t]
\]

By Möbius Inversion (cf. §2.2), we obtain

\[
f(x_1, \ldots, x_t) = f([t]) = \sum_{S \subset [t]} (-1)^{|S|} g(S)
\]

Now plugging \( x_1 = \ldots = x_t = 1 \), we obtain

\[
\sum_{\kappa \in K_t} \left( \frac{m}{k_1, \ldots, k_t} \right) = \sum_{S \subset [t]} (-1)^{|S|} |S|^m
\]

\[
= \sum_{s=0}^{t} (-1)^{t-s} \binom{t}{s} s^m
\]

**Remark 4.2.** \( K_t = \emptyset \) for \( t > m \), so in that case the left side of the above equation would be 0. Hence this would yield the identity \( \sum_{s=0}^{t} (-1)^{t-s} \binom{t}{s} s^m = 0 \) for \( t > m \).

\(^2\)If instead we were working with ordered partitions into non-negative integers, the inner sum would have simply been the multinomial expansion of \( t^m = (1 + \ldots + 1)^m \).
Plugging this into (8) yields

\[ a_{m, \alpha} = \frac{1}{(m-1)!} \sum_{t=1}^{m} \frac{(-1)^{t+1} e^{i\pi \alpha} t e^{i\pi \alpha}}{t} \sum_{s=0}^{m-t} (-1)^{t-s} \left( \frac{t}{s} \right) s^{m} \]

\[ = \frac{1}{(m-1)!} \sum_{t=1}^{m} e^{i\pi \alpha} (1 + e^{i\pi \alpha})^{m-t} \sum_{s=0}^{m} (-1)^{s-1} \left( \frac{t-1}{s-1} \right) s^{m-1} \]

\[ = \frac{1}{(m-1)!} \sum_{t=0}^{m-1} \sum_{s=0}^{t} (-1)^{s} \left( \frac{t}{s} \right) e^{i\pi \alpha (t+1)} (1 + e^{i\pi \alpha})^{m-t-1} (s+1)^{m-1} \]

Now that we have evaluated \( a_{m, \alpha} \), the claim follows by Theorem 4.1.

**Proof of Corollary 1.9.** Combine Theorem 1.5 for \( \alpha = 0 \) with (1).

**5 Summing a rational function over integers**

We wish to eventually evaluate sums of the form

\[ \sum_{k \in \mathbb{Z}} \frac{P(k)}{Q(k)} \]

where \( P \) and \( Q \) are polynomials, \( Q \) has no integer roots and \( \deg(Q) - \deg(P) \geq 2 \). To conform with the form required by Proposition 1.4, we may instead evaluate sums of the form

\[ \sum_{k \in \mathbb{Z}} \frac{P(2k + 1)}{Q(2k + 1)} \]

where \( Q \) has no odd integer roots. As with polynomials, we see that \( \frac{Q}{P} : \mathbb{C} \to \mathbb{C} \) is a nice function and so Proposition 1.4 may be applied. One may be tempted to use the same strategy as in the proofs of Theorem 3.1 and Theorem 4.1 by constructing a polynomial whose coefficients give us the answer, but that will not work in this general setting directly.\(^3\) Hence we first consider the special case Theorem 1.6, which will allow us to tackle the general scenario using partial fraction decomposition.

**Proof of Theorem 1.6.** Using Theorem 1.5 for \( m = 2 \), the case of \( \beta = 0 \) follows. Hence, suppose \( \beta \neq 0 \) and let \( a = i\pi \alpha \), \( b = -(i\pi)^2 \beta \) and define \( A_n \) as follows.

\[ A_n = \frac{1}{n^2} \sum_{k=0}^{n-1} \frac{1}{e^{\frac{i\pi}{n}(2k+1)} - 1 - \frac{a}{n}} - \frac{b}{n^2} \]

\(^3\) This is because we relied on being able to find the roots of the resultant polynomial and then collapse it using roots of unity, but this will not be possible for the generalisation we are after. In fact, even the case of \( \deg(Q) = 2 \) covered by Theorem 1.6 requires a bit more than a direct application of our method.
By Proposition 1.4 (with \( f(x) = (x - \alpha)^2 + \beta \)), we have
\[
\sum_{k \in \mathbb{Z}} \frac{1}{(2k + 1 - \alpha)^2 + \beta} = \lim_{n \to \infty} (i\pi)^2 A_n \tag{9}
\]
Consider the following polynomial \( p(x) \), defined so that the ratio of the coefficients of \( x^1 \) and \( x^0 \) respectively gives \(-n^2 A_n\). A formal constant is introduced as before.
\[
c p(x) = \prod_{k=0}^{n-1} \left( x - \left[ e^{i\pi(2k+1)} - 1 - \frac{a}{n} \right]^2 + \frac{b}{n^2} \right)
= \prod_{k=0}^{n-1} \left[ y - 1 - \frac{a}{n} + e^{i\pi(2k+1)} \right] \left[ y + 1 + \frac{a}{n} - e^{i\pi(2k+1)} \right]
\]
\[
= \left( -1 \left[ \frac{1 + a}{n} - y \right]^n + 1 \right) \left( - \left[ \frac{1 + a}{n} + y \right]^n + 1 \right)
= \left[ 1 + \left( \frac{1 + a}{n} - y \right)^n \right] \left[ 1 + \left( \frac{1 + a}{n} + y \right)^n \right]
\]
\[
= 1 + \left[ 1 + \frac{a}{n} - y \right]^n + \left[ 1 + \frac{a}{n} + y \right]^n + \left[ 1 + \frac{a}{n} \right]^2 - y^2 \right]^n
= 1 + 2 \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{k} \left( 1 + \frac{a}{n} \right)^{n-2k} \left( \frac{b}{n^2} \right)^k + \left[ 1 + \frac{a}{n} \right]^2 - \frac{b}{n^2} - x \right]^n \tag{11}
\]

We are only concerned with the coefficients of \( x^1 \) and \( x^0 \), so let these be \( u \) and \( v \) respectively. \( v \) can be obtained by simply plugging \( y = \sqrt[n]{\frac{b}{n}} \) (i.e. \( x = 0 \)) in (10), so we have
\[
v = \left[ 1 + \left( 1 + \frac{a}{n} - \sqrt[n]{\frac{b}{n}} \right) \right] \left[ 1 + \left( 1 + \frac{a}{n} + \sqrt[n]{\frac{b}{n}} \right) \right]
\]
\[
\Longrightarrow \lim_{n \to \infty} v = \left( 1 + e^{a - \sqrt[n]{b}} \right) \left( 1 + e^{a + \sqrt[n]{b}} \right) = 1 + 2 e^a \cosh \sqrt[n]{b} + e^{2a} \tag{12}
\]
Expanding (11) yields
\[
u = 2 \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} \left( 1 + \frac{a}{n} \right)^{n-2k} \left( \frac{b}{n^2} \right)^k - n \left[ \left( 1 + \frac{a}{n} \right)^2 - \frac{b}{n^2} - x \right]^n
\]

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The summation in this expression is

\[
2 \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \left( \frac{n}{2k} \right) \left( 1 + \frac{a}{n} \right)^{n-2k} \left( \frac{b}{n^2} \right)^{k-1} = n \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \left( \frac{n-1}{2k-1} \right) \left( 1 + \frac{a}{n} \right)^{n-2k} \left( \frac{b}{n} \right)^{k-1}
\]

\[
= n \frac{n}{\sqrt{b}} \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \left( \frac{n-1}{2k-1} \right) \left( 1 + \frac{a}{n} \right)^{n-2k} \left( \frac{\sqrt{b}}{n} \right)^{2k-1}
\]

\[
= \frac{n^2}{2\sqrt{b}} \left[ \left( 1 + \frac{a}{n} + \frac{\sqrt{b}}{n} \right)^{n-1} - \left( 1 + \frac{a}{n} - \frac{\sqrt{b}}{n} \right)^{n-1} \right]
\]

Plugging this into the expression for \( u \), we obtain

\[
\lim_{n \to \infty} \frac{u}{n^2} = \frac{e^a \sinh \sqrt{b}}{\sqrt{b}}
\]  

(13)

We know that \( \lim_{n \to \infty} A_n = \lim_{n \to \infty} \frac{u}{n^2 v} \), so if \( \lim_{n \to \infty} v \neq 0 \) then by (9), (12) and (13) we obtain

\[
\sum_{k \in \mathbb{Z}} \frac{1}{(2k+1-\alpha)^2 + \beta} = \frac{-(i\pi)^2 e^a \sinh \sqrt{b}}{\sqrt{b} \left( 1 + 2 e^a \cosh \sqrt{b} + e^{2a} \right)}
\]

Plugging in \( a = i\pi \alpha \) and \( b = -(i\pi)^2 \beta \) (and taking the same branch for \( \sqrt{b} \) everywhere), we obtain the desired result. Now, note that if \( \lim_{n \to \infty} v = 0 \), then by (12) at least one of \( a - \sqrt{b} \) and \( a + \sqrt{b} \) is an odd integer multiple of \( i\pi \). However this would mean that \( (x-\alpha)^2 + \beta \) has an odd integer root, which we assumed is not the case. \( \square \)

Now we will consider the general form stated at the beginning of this section, namely

\[
\sum_{k \in \mathbb{Z}} \frac{P(2k+1)}{Q(2k+1)}
\]

To evaluate this sum we write \( Q \) in terms of its roots as \( Q(x) = \prod_{i=1}^{d} (x - \alpha_i)^{r_i} \) where \( r_j \in \mathbb{N} \). If \( d = 1 \), then by Theorem 1.5 we are done. If \( d > 1 \), we apply partial fraction decomposition on \( \frac{P}{Q} \) to write

\[
\frac{P(x)}{Q(x)} = \sum_{i=1}^{d} \sum_{j=0}^{r_i} \frac{c_{ij}}{(x - \alpha_i)^j}, c_{ij} \in \mathbb{C}
\]

For \( j > 1 \), we can evaluate

\[
\sum_{k \in \mathbb{Z}} \frac{c_{ij}}{(2k+1 - \alpha_i)^j}
\]
using Theorem 1.5. For \( j = 0 \) we have \( \sum_{i=1}^{d} c_{i0} = 0 \) since \( \deg(P) < \deg(Q) \), so the \( j = 0 \) terms may be ignored. Likewise since \( \deg(P) < \deg(Q) - 1 \), we have \( \sum_{i=1}^{d} c_{i1} = 0 \). Hence,

\[
\begin{align*}
c_{d1} &= - \sum_{i=1}^{d-1} c_{i1} \\
\implies \sum_{i=1}^{d} \frac{c_{i1}}{x - \alpha_i} &= \sum_{i=1}^{d-1} \frac{c_{i1}}{x - \alpha_i} - \frac{c_{i1}}{x - \alpha_d} \\
&= \sum_{i=1}^{d-1} \frac{c_{i1}(\alpha_i - \alpha_d)}{(x - \alpha_i)(x - \alpha_d)}
\end{align*}
\]

Terms of this form can be summed using Theorem 1.6, so this concludes the \( j = 1 \) case. Hence, Theorem 1.5 and Theorem 1.6 together with partial fraction decomposition allow us to evaluate all sums of the form

\[
\sum_{k \in \mathbb{Z}} \frac{P(2k+1)}{Q(2k+1)}
\]

when \( Q \) has no odd integer roots and \( \deg(Q) - \deg(P) \geq 2 \). Using this, we can evaluate all sums of the form

\[
\sum_{k \in \mathbb{Z}} \frac{P(k)}{Q(k)}
\]

when \( Q \) has no integer roots and \( \deg(Q) - \deg(P) \geq 2 \). We can generalise even further to sums of the form

\[
\sum_{k \in \mathbb{Z}} s(k) \frac{P(k)}{Q(k)}
\]

where \( P \) and \( Q \) are as before and \( s : \mathbb{Z} \to \mathbb{C} \) is a periodic function. This can be done by grouping together all terms from a single period of \( s \) to convert the problem into one which involves only rational function terms, and then applying the above method. For example, consider

\[
\sum_{k \in \mathbb{Z}} e^{\frac{2\pi ik}{3}}
\]

Since \( k \mapsto e^{\frac{2\pi ik}{3}} \) is a 3-periodic map, we group together terms corresponding to \( k = 3k', k = 3k' + 1 \) and \( k = 3k' + 2 \) as \( k' \) varies over \( \mathbb{Z} \) to obtain

\[
\sum_{k \in \mathbb{Z}} e^{\frac{2\pi ik}{3}} = \sum_{k' \in \mathbb{Z}} \frac{1}{(3k')^2 + 1} + \frac{e^{\frac{2\pi i}{3}}}{(3k' + 1)^2 + 1} + \frac{e^{\frac{4\pi i}{3}}}{(3k' + 2)^2 + 1}
\]

It is unnecessary to combine these fractions into one, since we will later apply partial fraction decomposition anyway. In fact, this is already in a form which Theorem 1.6 can handle. A similar situation occurs when combining terms in the general form
\[
\sum_{k \in \mathbb{Z}} s(k) \frac{P(k)}{Q(k)}. \]
We can also use this method to compute alternating sums involving reciprocals of linear terms (i.e. sums which do not converge absolutely) such as \[ \sum_{k \in \mathbb{Z}} \frac{(-1)^k}{k}, \]
since grouping terms leads to a quadratic in the denominator of the summand (which Theorem 1.6 can handle).

A big limitation of this method is that it cannot always handle sums over \( \mathbb{N} \), such as the series \( \ln 2 = \sum_{k \in \mathbb{N}} \frac{(-1)^{k+1}}{k} \) obtained from the Maclaurin series of \( \ln \).

This is not surprising since every sum which can be evaluated using this method will be a finite rational combination of the exponential function, square root function and integer powers of \( \pi \) (by Theorem 1.5 and Theorem 1.6).

### 6 Euler’s ‘factorisation’ for \( \sin z \)

Euler’s original method for solving the Basel Problem was based on the idea that if the Maclaurin series for \( \sin z \) is treated as a ‘polynomial’ with infinite terms, then its roots over \( \mathbb{C} \) will be \( 0, \pm \pi, \pm 2\pi, \ldots \), each with multiplicity 1 (since \( \frac{d}{dz} \sin z \) at each of the roots is non-zero). Hence, supposing that an analogue of the Fundamental Theorem of Algebra holds for such a series, he concluded that

\[
\sin z = \Lambda z \prod_{k \in \mathbb{N}} \left( 1 - \frac{z^2}{k^2 \pi^2} \right)
\]

where \( \Lambda \) is a proportionality constant. Since \( \lim_{z \to 0} \frac{\sin z}{z} = 1 \), he concluded that \( \Lambda = 1 \). Hence,

\[
\sin z = z \prod_{k \in \mathbb{N}} \left( 1 - \frac{z^2}{k^2 \pi^2} \right)
\]

(14)

He then expanded this product and equated the coefficient of \( z^3 \) to that in the Maclaurin series of \( \sin z \) to obtain the solution to the Basel Problem.

\[
\sum_{k \in \mathbb{N}} \frac{-1}{k^2 \pi^2} = \frac{-1}{3!}
\]

Equating coefficients for the rest of the terms gave him the general formula

\[
\sum_{k_1 < \ldots < k_m} \frac{1}{k_1^2 \ldots k_m^2} = \frac{\pi^m}{m!}
\]

where \( m \in \mathbb{N} \) is even. From here, \( \sum_{k \in \mathbb{N}} \frac{1}{k^m} \) could be evaluated inductively for \( m \) even.

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\(^4\) If the summand is an even function (such as \( \frac{1}{k^2} \)) then such sums can be dealt with.
He was not, however, able to justify his factorisation of \( \sin z \) rigorously. Eventually, Weierstrass generalised this idea rigorously for entire functions in his famous factorisation theorem.

We note that Euler’s idea of constructing a ‘polynomial’ in terms of its roots whose coefficients give the answer to the Basel Problem is very similar to our proof of **Theorem 3.1**. So similar, in fact, that the methods we have used here can also prove (14) and other related identities.

**Proof of Theorem 1.10.** We have

\[
\sin z = \frac{e^{iz} - e^{-iz}}{2i} = \lim_{n \to \infty} \frac{(1 + iz/n)^n - (1 - iz/n)^n}{2i}
\]

We will now factorise the expression inside the limit. Also, assume for convenience that \( n \) is odd.

\[
\frac{(1 + iz/n)^n - (1 - iz/n)^n}{2i} = \frac{1}{2i} \prod_{k=0}^{n-1} \left[ 1 + \frac{iz}{n} - e^{\frac{2\pi k}{n}} \left( 1 - \frac{iz}{n} \right) \right]
\]

\[
= \frac{1}{2i} \prod_{k=0}^{n-1} \left[ 1 - e^{\frac{2\pi k}{n}} + \frac{iz}{n} \left( 1 + e^{\frac{2\pi k}{n}} \right) \right]
\]

\[
= \frac{1}{2i} \prod_{k=1}^{n-1} \left[ 1 + \frac{iz}{n} \left( 1 + e^{\frac{2\pi k}{n}} \frac{1 + e^{2\pi k/n}}{1 - e^{2\pi k/n}} \right) \right] \left( 1 - e^{\frac{2\pi k}{n}} \right)
\]

\[
= \frac{1}{2i} \prod_{k=1}^{n-1} \left[ 1 + \frac{iz}{n} \frac{1 + e^{\frac{2\pi k}{n}}}{1 - e^{\frac{2\pi k}{n}}} \right] \left( 1 - e^{\frac{2\pi k}{n}} \right)
\]

\[
= \frac{1}{2i} \prod_{|k| \leq \frac{n}{2}} \left[ 1 + \frac{iz}{n} \frac{1 + e^{\frac{2\pi k}{n}}}{1 - e^{\frac{2\pi k}{n}}} \right] \left( 1 - e^{\frac{2\pi k}{n}} \right)
\]

\[
= \frac{1}{2i} \prod_{0 < k < \frac{n}{2}} \left[ 1 + \frac{iz}{n} \frac{1 + e^{\frac{2\pi k}{n}}}{1 - e^{\frac{2\pi k}{n}}} \right] \left[ 1 + \frac{iz}{n} \frac{1 + e^{-\frac{2\pi k}{n}}}{1 - e^{-\frac{2\pi k}{n}}} \right]
\]

\[
= \frac{1}{2i} \prod_{0 < k < \frac{n}{2}} \left[ 1 + \frac{iz}{n} \frac{1 + e^{\frac{2\pi k}{n}}}{1 - e^{\frac{2\pi k}{n}}} \right]^2
\]
Taking $n \to \infty$, we have

$$\sin z = z \lim_{n \to \infty} \prod_{n \text{ odd}, 0 < k < \frac{n}{2}} \left[ 1 + \frac{z^2}{n^2} \left( 1 + e^{\frac{2i\pi k}{n}} \right) \left( 1 - e^{\frac{2i\pi k}{n}} \right) \right]$$

$$= z \prod_{k \in \mathbb{N}} \left( 1 - \frac{z^2}{k^2\pi^2} \right)$$

In the last step we used Lemma 2.5 to apply term-wise limits. The bounds on terms required to apply Lemma 2.5 can be found using methods used to define $M_k$ in the proof of Proposition 1.4.

Such factorisation can also be obtained for $\cos z$, $1 + e^z$ etc. using the above elementary method.

References


[5] Hofbauer, Josef (2002). A Simple Proof of $1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}$ and Related Identities. The American Mathematical Monthly, 109, 196–200.


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