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Cover Page Footnote

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On Solutions of First Order PDE with Two-Dimensional Dirac Delta Forcing Terms

By *Ian Robinson*

Abstract. We provide solutions of a first order, linear partial differential equation of two variables where the nonhomogeneous term is a two-dimensional Dirac delta function. Our results are achieved by applying the unilateral Laplace Transform, solving the subsequently transformed PDE, and reverting back to the original space-time domain. A discussion of uniqueness and existence of solutions, a derivation of solutions of the PDE coupled with a boundary and initial condition, as well as a few worked examples are provided.

1 Introduction

This paper concerns a mathematical field of study known as Partial Differential Equations (PDE), a vibrant topic which is intricately linked with functional analysis. Modern research in this field focuses primarily on certain qualities of solutions of PDE, such as existence, uniqueness, and stability, but may also focus on the PDE itself, such as whether a well-posed boundary value problem is guaranteed to exist on the domain of the PDE.

PDE are commonplace in several scientific professions, including the studies of physics, chemistry, and engineering. The reader may be familiar with some simple second order PDE, such as the heat equation and the wave equation, both of which are used to model observable physical phenomena. We investigate a first order PDE in two variables, x and t, which may physically be interpreted as a spatial variable and a temporal variable, respectively.

In section 1, we provide several elementary definitions necessary for understanding the remainder of this paper. We also provide the background and motivation for the problem that we address, including a brief discussion of the physical intuition behind the Dirac delta function. The most technical material in this paper will be found in section 2, which will introduce several new definitions from functional analysis and PDE theory. We make use of the Lax-Milgram Theorem as stated in [2], which was first proved by Peter Lax and Arthur Milgram in 1954 and has been further generalized since, in order

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to demonstrate the existence and uniqueness of weak solutions of our PDE. In section 3 and section 4, we provide a derivation of solutions of our PDE through the use of the Laplace Transform, which is followed by several examples.

In this paper, it is beneficial for the reader to have a firm understanding of basic multivariate calculus. An understanding of the terminology and techniques used in Ordinary Differential Equations (ODE) is also recommended, but not necessary. Please note that section 2 will include some segments that provide little explanation as to why each step in the mathematical process is taken as it is unreasonable to fully explain the more complicated concepts motivating the material covered there. Every attempt, however, shall be made to effectively explain the results found in other sections.

1.1 General Definitions

A Partial Differential Equation (PDE) is a relationship between an unknown function and its partial derivatives in some or all of the function's independent variables. In standard PDE notation, we frequently use subscripts to represent partial derivatives. For example, $u_t = \frac{\partial u}{\partial t}$, $u_{xx} = \frac{\partial^2 u}{\partial x^2}$, and $u_{xyz} = \frac{\partial}{\partial z}\frac{\partial}{\partial y}\frac{\partial u}{\partial x}$. The order of a partial differential equation is dictated by the highest order derivative that shows up in the equation. This paper mainly concerns a function u of two independent variables: x, a spatial variable, and t, a variable of time. We consider a first order PDE described by

$$\mathrm{F}\left(u(x,t), \frac{\partial u(x,t)}{\partial x}, \frac{\partial u(x,t)}{\partial t}, x, t\right) = 0, \tag{1.1}$$

where F is a generic function defined on some region R. A function $u(x,t)$ is said to be a classical solution of the PDE if it satisfies eq. (1.1) on all of R, meaning that u must be continuously differentiable in both of its variables on all of R as well. For example, consider the first order PDE

$$\frac{\partial}{\partial x} u(x,t) = 0. \tag{1.2}$$

We see that the partial derivative of $u(x,t)$ with respect to the independent variable x is equal to 0. Any student familiar with multivariate calculus will immediately be able to identify a solution to this equation, which is as follows:

$$u(x,t) = f(t), \tag{1.3}$$

where $f(t)$ is *any* continuously differentiable function of t. We can verify that this is true by plugging our result into our PDE. The partial derivative of a function of t with respect to x will always be zero. Thus, eq. (1.3) is a solution of eq. (1.2).

It is also helpful to consider the linearity of a PDE. A linear partial differential equation can be written in the form

$$\mathrm{L}(u) = f, \tag{1.4}$$

where L is a linear differential operator and f is a function of the independent variables of u. A linear operator L satisfies the following two qualities:

$$\mathrm{L}(f+g)=\mathrm{L}(f)+\mathrm{L}(g) \qquad\qquad \mathrm{L}(cf)=c\mathrm{L}(f), \tag{1.5}$$

where f and g are any two functions in the domain of L and c is any scalar. The derivative $\frac{d}{dx}$ is a very common linear operator. On the other hand, the operator $(\cdot)^2$ is nonlinear because $(f+g)^2 \neq f^2+g^2$ and $(cf)^2 \neq c(f^2)$. Any PDE that cannot be written in the form of eq. (1.4) is considered to be nonlinear. A nonlinear PDE of first order may further be classified as semilinear or quasilinear, but we will not define those terms here.

We need to be able to identify one final feature of a PDE. A PDE is said to be homogeneous if $u \equiv 0$ is a solution. Otherwise, we say that the PDE is nonhomogeneous. However, PDE are often paired with boundary conditions, a set of prescribed values that describe the behavior of u at the spatial boundaries of the domain. We refer to this pair as a Boundary Value Problem (BVP).

Consider the following PDE and their classifications:

$xu_t = u_{xx}+u_{yy}$	Second Order, Linear, Homogeneous
$u_t + uu_x = 0$	First Order, Nonlinear, Homogeneous
$u_{xtt} - 5u = x^2+y^2$	Third Order, Linear, Nonhomogeneous

Despite the x on the left side of the equation in our first example, the whole problem is still linear because x is not a function of u. In our second example, the uu_x term is nonlinear. Our third example is nonhomogeneous because if $u \equiv 0$, we have that $x^2+y^2=0$, which is only true when x and y are both zero. Hence, we call x^2+y^2 the nonhomogeneous term.

In addition to a BVP, another common problem in differential equations is an Initial Value Problem (IVP). These problems are common occurrences when u is dependent on time. For a PDE of two variables, x and t, an initial condition usually takes the form $u(x,0)=g(x)$, where g is a generic spatial distribution defined when $t=0$. If we pair a BVP with an initial condition, then we usually have everything needed to describe how that initial distribution changes with respect to space and time. This is not always guaranteed as many boundary value problems do not have a solution.

We begin our study by defining the one-dimensional Dirac delta function $\delta(x-x_0)$ as

$$\delta(x-x_0)=\begin{cases}\infty & \text{if } x=x_0\\ 0 & \text{if } x\neq x_0,\end{cases} \tag{1.6}$$

which we define to have the additional properties

$$\int_{-\infty}^{\infty} \delta(x-x_0)\,\mathrm{d}x = 1 \tag{1.7}$$

$$\int_{-\infty}^{\infty} f(x)\delta(x-x_0)\,\mathrm{d}x = f(x_0).$$

This is an extraordinarily naïve definition that we shall improve upon soon. For now, we will neglect the problematic usage of infinity and accept the definition as given. It is easy to see that the above integral of the Delta Dirac function is 0 on any interval that does not include x_0. The area of interest for eq. (1.7) is near $x = x_0$, so let us consider $x \in (x_0 - \epsilon, x_0 + \epsilon)$ where $\epsilon > 0$. On this interval, our integral becomes

$$\int_{x_0-\epsilon}^{x_0+\epsilon} \delta(x-x_0)\,\mathrm{d}x = 1. \tag{1.8}$$

We may also define the two-dimensional Delta Dirac function in a similar manner:

$$\delta(x-x_0, y-y_0) = \delta(x-x_0)\delta(y-y_0) = \begin{cases} \infty & \text{if } x = x_0 \text{ and } y = y_0 \\ 0 & \text{if } x \neq x_0 \text{ or } y \neq y_0. \end{cases} \tag{1.9}$$

Here, *or* is considered an *inclusive or*. Just as with its one-dimensional counterpart, we have that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x-x_0, y-y_0)\,\mathrm{d}x\,\mathrm{d}y = 1, \tag{1.10}$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\delta(x-x_0, y-y_0)\,\mathrm{d}x\,\mathrm{d}y = f(x_0, y_0),$$

where the only contribution to this quantity comes from the region directly about the point (x_0, y_0).

We now define the Heaviside step function $\mathrm{H}(x-x_0)$ as

$$\mathrm{H}(x-x_0) = \begin{cases} 0 & \text{if } x < x_0 \\ 1 & \text{if } x \geq x_0. \end{cases} \tag{1.11}$$

Depending on the source, the Heaviside step function may be defined differently at the point of discontinuity. With the methods employed here, the definition at this point is of little importance.

It is also known that

$$\int_{-\infty}^{t} \delta(x - x_0)\, \mathrm{d}x = \mathrm{H}(t - x_0). \tag{1.12}$$

This relationship between the Dirac delta function and the Heaviside step function may seem odd, especially given our naïve definition in eq. (1.6). However, this is more reasonable when we consider the Dirac delta "function" as a *distribution* instead. A distribution is a generalized function that conveniently allows us to compute derivatives and antiderivatives of functions that would otherwise not have such attributes. We may think of the Dirac delta distribution defined by

$$\delta(x - x_0) = \lim_{\epsilon \to 0} \frac{e^{-(\frac{x - x_0}{\epsilon})^2}}{|\epsilon|\sqrt{\pi}} \tag{1.13}$$

as a continuous approximation of the naïve function involving infinity. This distribution still satisfies the integral properties in eq. (1.7) regardless of the value of ϵ. However, as long as we are dealing with distributions, we abandon the ideas of the classical derivative and antiderivative and replace them with the *weak derivative* and *weak antiderivative*. Given our distribution-based infrastructure, it is now appropriate to call the Heaviside step function the weak antiderivative of the Dirac delta function.

1.2 Background

This paper is mainly aimed at describing solutions of a first order, linear PDE that has a two-dimensional Dirac delta function applied at (x_0, t_0) serving as the nonhomogeneous term. Thus, the main topic of this paper is a PDE of the form

$$p(x,t)u_t + q(x,t)u_x + r(x,t)u = y(x,t)\delta(x - x_0, t - t_0), \tag{1.14}$$

where $u \equiv u(x,t)$. When the coefficients are functions of both x and t, the problem becomes too general to solve with the methods employed in this paper. Thus, when we derive solutions of this PDE, we only consider eq. (1.14) when the coefficients are reduced to either constants or simply functions of one variable only. In later sections, we apply a boundary and initial condition in order to find a general solution of the associated BVP. By our definition in eq. (1.9), this PDE is homogeneous when either $x \neq x_0$ or $t \neq t_0$. An abrupt change is introduced to the system when the impulse is applied at (x_0, t_0), subsequently resulting in a nonhomogeneous problem. This abrupt change is typically interpreted in physically motivated problems to be some kind of immense force applied at one particular point over an extremely short amount of time, such as when a hammer hits the head of a nail. Regardless of the physical interpretation, the sudden shift from homogeneity to nonhomogeneity means that we will need to be careful with our analysis about the point of impulse and, consequently, at any point in time and space after the impulse is applied.

Another topic of interest that does not fit into the scope of this paper would be generalizing this theory to include a one-dimensional Dirac delta function as the nonhomogeneous term as opposed to the two-dimensional term that we work with. However, restricting our impulse to a specific point in one dimension necessitates that the impulse is applied at every point at which the other independent variable is defined. For example, the nonhomogeneous term $\delta(x-x_0)$ implies that an impulse is applied at a specific spatial point x_0. However, in the xt-plane, this impulse would be felt for all time. Impulses are generally reliant upon short time intervals, so this is not feasible in a physical sense. While solutions do exist for a problem such as this, they may not be physically realistic and therefore may have few real-world applications.

1.3 Definition of Laplace Transforms

We now define the transforms that we will use to solve the problem at hand. As given in [1], we define the one-dimensional Laplace Transform (1D-LT) as

$$\mathscr{L}\{f(t)\}(s) = \mathrm{F}(s) = \lim_{k\to\infty}\int_0^k f(t)e^{-st}\,\mathrm{d}t, \quad t\in\mathbb{R}: t\geq 0 \text{ and } s\in\mathbb{C}, \tag{1.15}$$

where $f(t)$ is a function of time and $\mathrm{F}(s)$ is a function of the complex variable s. For the sake of brevity, we use the improper integral form of the above definition more frequently in this paper. The Laplace Transform is a linear operator, meaning that we can take advantage of the properties in eq. (1.5). The Laplace Transform is frequently used to transform linear differential equations into algebraic problems. For example, this allows us to solve an IVP for a time-dependent ODE without having to find the family of solutions first as we would with other techniques. A table containing common one-dimensional Laplace Transforms in the time variable is given in the appendix. Consider the following example, where $a\in\mathbb{R}$ is a constant:

$$\begin{aligned}\mathscr{L}\{e^{at}\}(s) &= \int_0^\infty e^{at}e^{-st}\,\mathrm{d}t = \int_0^\infty e^{(a-s)t}\,\mathrm{d}t\\ &= \frac{e^{(a-s)t}}{a-s}\bigg|_0^\infty = \frac{1}{(a-s)e^{(s-a)t}}\bigg|_0^\infty\\ &= 0-\frac{1}{a-s} = \frac{1}{s-a}, \quad \text{if } \mathrm{Re}(s) > a.\end{aligned}$$

We require that the real part of s is greater than a so that our integral converges. If $a \geq \mathrm{Re}(s)$, then our integral would not converge. Note that if the real part of s approaches

a from the right, then the value of $F(s)$ grows towards infinity in the above example. If $s = a + 0i$, then we have a division by zero. Consequently, we call $(a, 0) \in \mathbb{C}$ a pole (a type of singularity) of $\mathscr{L}\{e^{at}\}$ because, in the complex plane, there is no finite value of $F(s)$ associated with the complex coordinate $(a, 0i)$.

Once the Laplace Transform has been applied to a differential equation, we may solve for the resulting $F(s)$ and apply the one-dimensional Inverse Laplace Transform (1D-ILT) in order to obtain a solution to the IVP. The 1D-ILT, also referred to as the Bromwich Integral, is defined in terms of the following line integral:

$$\mathscr{L}^{-1}\{F(s)\}(t) = f(t) = \frac{1}{2\pi i} \lim_{k\to\infty} \left\{ \int_{\gamma - ik}^{\gamma + ik} F(s) e^{st} \mathrm{d}s \right\}, \text{ for } t > 0, \tag{1.16}$$

where γ is a real number greater than the real part of each pole of $F(s)$. This computation can become extremely complicated and shouldn't be tackled directly, so we will instead rely on the Laplace Transform table found in the appendix. The Bromwich Integral also necessitates that $f(t)$ has a Laplace Transform and is thus of exponential order, but we shall not touch upon this condition. For more information about the definition of the Laplace Transform, please refer to [1].

While the Laplace Transform is usually used for functions of time, we make use of the more generalized concept of the Laplace Transform in which we may transform a function of any real variable into a function of a complex variable. This idea will be expanded upon in section 3.

2 Existence and Uniqueness of Solutions

We begin our discussion of uniqueness by defining a few terms. To fully explain each definition and result would unreasonably elongate this paper due to the complexity of the topic, so much of this section will be left unexplained. For a fuller explanation of L^p spaces, Sobolev spaces, and results regarding the existence and uniqueness of weak solutions of a PDE, please see [2] and [4]. We have the following five definitions from [2].

Definition 2.1. The ***Sobolev space*** $W^{k,p}(U)$ consists of all locally summable functions $u : U \to \mathbb{R}$ such that for each multiindex α with $|\alpha| \leq k$, $D^\alpha u$ exists in the weak sense and belongs to $L^p(U)$. If $p = 2$, then we may write

$$H^k(U) = W^{k,2}(U) \quad (k = 0, 1, 2, \ldots).$$

Definition 2.2. We define

$$W_0^{k,p}(U)$$

to be the closure of $C_c^\infty(U)$ in $W^{k,p}(U)$, where $C_c^\infty(U)$ is the space of infinitely differentiable functions with compact support on U as described in [2].

We commonly use the notation

$$\mathrm{H}_0^k(\mathrm{U}) = \mathrm{W}_0^{k,2}(\mathrm{U}).$$

Definition 2.3. If $u \in \mathrm{W}^{k,p}(\mathrm{U})$, we define its norm to be

$$\|u\|_{\mathrm{W}^{k,p}(\mathrm{U})} := \begin{cases} \left(\sum_{|\alpha|\le k} \int_{\mathrm{U}} |\mathrm{D}^\alpha u|^p \, dx\right)^{\frac{1}{p}} & (1 \le p < \infty) \\ \sum_{|\alpha|\le k} \operatorname{ess\,sup}_{\mathrm{U}} |\mathrm{D}^\alpha u| & (p = \infty), \end{cases}$$

where $x = (x_1, x_2, \ldots, x_n)$.

Now, consider the following BVP:

$$\begin{cases} \mathrm{L}u = f & \text{in U} \\ u = 0 & \text{on } \partial\mathrm{U}, \end{cases} \tag{2.1}$$

where U is an open subset of $\mathbb{R}^n$. We have that $f : \mathrm{U} \to \mathbb{R}$ is a given function and L denotes a first order partial differential operator, the definition of which immediately follows.

Definition 2.4. The bilinear form $\mathrm{B}[\cdot,\cdot]$ associated with the first order partial differential operator L defined by

$$\mathrm{L}[u] = \sum_{i=1}^{n} a^i(x) u_{x_i} + b(x) u$$

is described by

$$\mathrm{B}[u, v] := \int_{\mathrm{U}} \sum_{i=1}^{n} a^i u_{x_i} v + buv \, \mathrm{d}x$$

for $u,\ v \in \mathrm{H}^1(\mathrm{U})$.

Definition 2.5. We say that $u \in \mathrm{H}_0^1(\mathrm{U})$ is a ***weak solution*** of the boundary value problem given in eq. (2.1) if

$$\mathrm{B}[u, v] = (f, v)$$

for all $v \in \mathrm{H}_0^1(\mathrm{U})$, where $(\cdot,\cdot)$ denotes the inner product in the space $\mathrm{L}^2(\mathrm{U})$.

Now, let us assume that H is a real Hilbert space equipped with norm $\|\cdot\|$ and inner product $(\cdot,\cdot)$. Let $\langle\cdot,\cdot\rangle$ denote the pairing of H with its dual space, as described in [2].

Theorem 2.6 (Lax-Milgram Theorem)**.** *Assume that*

$$\mathrm{B} : \mathrm{H} \times \mathrm{H} \to \mathbb{R}$$

is a bilinear mapping, for which there exists constants $\alpha, \beta > 0$ *such that*

$$|\mathrm{B}[u, v]| \le \alpha \|u\| \|v\| \quad (u, v \in \mathrm{H})$$

and

$$\beta\|u\|^2 \le B[u,u] \quad (u \in H).$$

Finally, let $f : H \to \mathbb{R}$ *be a bounded linear functional on* H.

Then there exists a unique element $u \in H$ *such that*

$$B[u,v] = \langle f, v\rangle$$

for all $v \in H$.

For the proof, please refer to Evans [2]. The first inequality in the hypothesis of **theorem 2.6** implies that our bilinear operator is *bounded*. The second inequality, if satisfied, implies that our bilinear operator is *coercive*. We now show that unique weak solutions of the PDE of the form given in eq. (1.14) do indeed exist. First, we construct two lemmas in order to do so.

Lemma 2.7. *Suppose that* $p(x,t)$, $q(x,t)$, *and* $r(x,t)$ *are all bounded on* $\mathbb{R}^2$. *Define* $L : H^1(U) \to L^2(U)$ *by* $L := p(x,t)\frac{\partial}{\partial t} + q(x,t)\frac{\partial}{\partial x} + r(x,t)$, *where* $U = \mathbb{R}\times(0,\infty)$. *Then if* B *is the form associated with* L, B *is a bounded bilinear operator.*

Proof. Suppose that α and β are constants. We have that

$$\begin{aligned} L[\alpha u + \beta v] &= p\frac{\partial(\alpha u + \beta v)}{\partial t} + q\frac{\partial(\alpha u + \beta v)}{\partial x} + r(\alpha u + \beta v) \\ &= p\left(\alpha\frac{\partial u}{\partial t} + \beta\frac{\partial v}{\partial t}\right) + q\left(\alpha\frac{\partial u}{\partial x} + \beta\frac{\partial v}{\partial x}\right) + r\alpha u + r\beta v \\ &= \alpha\left(p\frac{\partial u}{\partial t} + q\frac{\partial u}{\partial x} + ru\right) + \beta\left(p\frac{\partial v}{\partial t} + q\frac{\partial v}{\partial x} + rv\right) \\ &= \alpha L[u] + \beta L[v], \end{aligned}$$

so L is a linear operator. Consequently, B is bilinear as well. By definition of the bilinear form $B[u,v]$ and the use of Hölder's Inequality, we have that

$$\begin{aligned} |B[u,v]| &= \left|\int_U p\frac{\partial u}{\partial t}v + q\frac{\partial u}{\partial x}v + ruv \, dx\right| \\ &\le \left|\int_U c_1|u_t||v| + c_2|u_x||v| + c_3|u||v| \, dx\right| \\ &\le \left|c_1\left(\int_U |u_t^2| dx\right)^{\frac{1}{2}}\left(\int_U |v^2| dx\right)^{\frac{1}{2}}\right| + \left|c_2\left(\int_U |u_x^2| dx\right)^{\frac{1}{2}}\left(\int_U |v^2| dx\right)^{\frac{1}{2}}\right| \\ &\quad + \left|c_3\left(\int_U |u^2| dx\right)^{\frac{1}{2}}\left(\int_U |v^2| dx\right)^{\frac{1}{2}}\right|, \end{aligned}$$

where c_1, c_2, and c_3 are positive constants determined by the bounds of $p(x,t)$, $q(x,t)$, and $r(x,t)$ respectively. Since $u \in \mathrm{H}^1(\mathrm{U})$, it follows that $u_t \in \mathrm{L}^2(\mathrm{U})$ and so the L^2-norm of u_t is bounded. Thus,

$$\left|\left(\int_{\mathrm{U}} |u_t^2| \mathrm{d}x\right)^{\frac{1}{2}} \left(\int_{\mathrm{U}} |v^2| \mathrm{d}x\right)^{\frac{1}{2}}\right| \leq \mathrm{M}$$

for some positive constant M, and we have

$$\begin{aligned} |\mathrm{B}[u,v]| &\leq \mathrm{M} + \left| c_2 \left(\int_{\mathrm{U}} |u_x^2| \mathrm{d}x\right)^{\frac{1}{2}} \left(\int_{\mathrm{U}} |v^2| \mathrm{d}x\right)^{\frac{1}{2}} \right| + \left| c_3 \left(\int_{\mathrm{U}} |u^2| \mathrm{d}x\right)^{\frac{1}{2}} \left(\int_{\mathrm{U}} |v^2| \mathrm{d}x\right)^{\frac{1}{2}} \right| \\ &\leq \mathrm{M} + \left(\int_{\mathrm{U}} |v^2| \mathrm{d}x\right)^{\frac{1}{2}} \left[\gamma \left(\int_{\mathrm{U}} |u_x^2| + |u^2| \mathrm{d}x\right)^{\frac{1}{2}} \right] \leq \tilde{\gamma} \|u\|_{\mathrm{H}^1} \|v\|_{\mathrm{H}^1}, \end{aligned}$$

where γ is some positive constant dependent on c_2 and c_3 while $\tilde{\gamma}$ is determined by M and γ. So $\mathrm{B}[u,v]$ is bounded. □

Lemma 2.8. *Suppose that $y(x,t)$ is bounded on $\mathbb{R}^2$ and defined at (x_0,t_0), and suppose that for any continuous $u(x,t)$, the functional $\mathrm{J}[u]$ is defined by*

$$\mathrm{J}[u](x_0,t_0) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u(x,t)y(x,t)\delta(x-x_0,t-t_0)\mathrm{d}x\mathrm{d}t.$$

Then $\mathrm{J}[u]$ is a bounded, linear functional. Consequently, since any function in H^1 may be written as a limit of functions in C^∞, J is also bounded and linear in H^1.

Proof. By our definition in eq. (1.10), it immediately follows that

$$|\mathrm{J}[u](x_0,t_0)| = |u(x_0,t_0)y(x_0,t_0)| \leq \mathrm{M}|u(x_0,t_0)|$$

for some constant M by the boundedness of $y(x,t)$ and $u(x,t)$. If α and β are constants, then

$$\begin{aligned} \mathrm{J}[\alpha u + \beta v](x_0,t_0) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (\alpha u + \beta v)y(x,t)\delta(x-x_0,t-t_0)\,\mathrm{d}x\mathrm{d}t \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \alpha u y(x,t)\delta(x-x_0,t-t_0) + \beta v y(x,t)\delta(x-x_0,t-t_0)\,\mathrm{d}x\mathrm{d}t \\ &= \alpha \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u y(x,t)\delta(x-x_0,t-t_0)\,\mathrm{d}x\mathrm{d}t + \beta \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} v y(x,t)\delta(x-x_0,t-t_0)\,\mathrm{d}x\mathrm{d}t \\ &= \alpha\mathrm{J}[u](x_0,t_0) + \beta\mathrm{J}[v](x_0,t_0). \end{aligned}$$

Therefore, $\mathrm{J}[u]$ is a bounded, linear functional. □

Theorem 2.9. *Let $p(x,t)$, $q(x,t)$, $r(x,t)$, and $y(x,t)$ be bounded on $\mathbb{R}^2$ and let $u \equiv u(x,t)$. Suppose that the bilinear form* B *corresponding to*

$$\mathrm{L}[u] := p(x,t)u_t + q(x,t)u_x + r(x,t)u \tag{2.2}$$

as in ***lemma 2.7*** *satisfies the coercivity condition $\beta\|u\|^2 \le \mathrm{B}[u,u]$. Then a solution of the PDE*

$$p(x,t)u_t + q(x,t)u_x + r(x,t)u = y(x,t)\delta(x-x_0, t-t_0)$$

exists. If u is prescribed to be zero on the boundary, the solution is also guaranteed to be unique.

Proof. We show in **lemma 2.7** and **lemma 2.8** that B is a bounded bilinear operator and that $\mathrm{J}[u]$ is a bounded linear functional. Due to the generality of the coefficients, we assume coercivity, which is a convention often taken in PDE. If $u = 0$ on the boundary, then by a simple application of the Lax-Milgram Theorem, it follows that there exists a unique element $u \in \mathrm{H}_0^1$ such that

$$\mathrm{B}[u,v] = \langle f, v\rangle$$

for all $v \in \mathrm{H}_0^1$. Else, if u is prescribed a nontrivial boundary condition, then there exists at least one element $u \in \mathrm{H}^1$ which satisfies the PDE on the domain U. □

3 A General Theory for the Linear Problem

In this section, we derive weak solutions of eq. (1.14) when it is coupled with a boundary condition and an initial condition. The Laplace Transform is especially useful in this scenario, so we use it to derive a general solution of eq. (1.14) when $u = u(x,t)$. Worked examples of boundary value problems of this type, along with graphical representations of their weak solutions, are reserved for the next section.

3.1 The Constant Coefficients BVP

We begin our study of eq. (1.14) and it's prescribed conditions below by assuming that $p(x,t) = a$, $q(x,t) = b$, $r(x,t) = c$, and $y(x,t) = d$ are all constant coefficients. We also assume the condition that $a \neq 0$ and $b \neq 0$. Upon dividing both sides by the nonzero coefficient of the u_t term, we write our BVP as

$$u_t + \mathrm{A}u_x + \mathrm{B}u = \mathrm{C}\delta(x-x_0)\delta(t-t_0), \quad u(x,0) = f(x), \quad u(\alpha,t) = k(t), \tag{3.1}$$
$$t > 0, \quad t_0 > 0, \quad x > \alpha, \quad x_0 > \alpha$$

where $\mathrm{A} = \frac{b}{a}$, $\mathrm{B} = \frac{c}{a}$, $\mathrm{C} = \frac{d}{a}$, and α is a real constant.

Applying the 1D-LT to eq. (3.1) with respect to the time variable, we have

$$\begin{aligned}
&\mathscr{L}\{u_t + \mathrm{A}u_x + \mathrm{B}u\}(s) = \mathscr{L}\{\mathrm{C}\delta(x-x_0)\delta(t-t_0)\}(s) \\
\Rightarrow\quad & s\bar{u} - u(x,0) + \mathrm{A}\bar{u}_x + \mathrm{B}\bar{u} = \mathrm{C}\delta(x-x_0)e^{-st_0} \\
\Rightarrow\quad & \mathrm{A}\bar{u}_x + (\mathrm{B}+s)\bar{u} = \mathrm{C}\delta(x-x_0)e^{-st_0} + u(x,0) = \mathrm{C}\delta(x-x_0)e^{-st_0} + f(x) \\
\Rightarrow\quad & \bar{u}_x + \bar{\mathrm{B}}\bar{u} = \bar{\mathrm{C}}\delta(x-x_0)e^{-st_0} + \bar{f}(x), \qquad (3.2)
\end{aligned}$$

where $\bar{u} \equiv \bar{u}(x,s)$ is the transformation of $u(x,t)$, $\bar{\mathrm{B}} = \frac{\mathrm{B}+s}{\mathrm{A}}$, $\bar{\mathrm{C}} = \frac{\mathrm{C}}{\mathrm{A}}$, and $\bar{f}(x) = \frac{f(x)}{\mathrm{A}}$. We now find an integrating factor μ for this result.

$$\mu = e^{\int \bar{\mathrm{B}}\mathrm{d}x} = e^{\bar{\mathrm{B}}x}$$

Multiplying both sides of eq. (3.2) by μ, we have that

$$\begin{aligned}
& e^{\bar{\mathrm{B}}x}\bar{u}_x + e^{\bar{\mathrm{B}}x}\bar{\mathrm{B}}\bar{u} = \bar{\mathrm{C}}\delta(x-x_0)e^{\bar{\mathrm{B}}x-st_0} + e^{\bar{\mathrm{B}}x}\bar{f}(x) \\
\Rightarrow\ & \frac{\partial}{\partial x}\left[e^{\bar{\mathrm{B}}x}\bar{u}\right] = \bar{\mathrm{C}}\delta(x-x_0)e^{\bar{\mathrm{B}}x-st_0} + e^{\bar{\mathrm{B}}x}\bar{f}(x).
\end{aligned}$$

Integrating both sides with respect to x results in

$$\begin{aligned}
e^{\bar{\mathrm{B}}x}\bar{u} &= \int \bar{\mathrm{C}}\delta(x-x_0)e^{\bar{\mathrm{B}}x-st_0}\,dx + \int \bar{f}(x)e^{\bar{\mathrm{B}}x}\,dx + g(s) \\
&= \bar{\mathrm{C}}\mathrm{H}(x-x_0)e^{\bar{\mathrm{B}}x_0-st_0} + \int \bar{f}(x)e^{\bar{\mathrm{B}}x}\,dx + g(s) \\
\Rightarrow\quad \bar{u} &= \bar{\mathrm{C}}\mathrm{H}(x-x_0)e^{\bar{\mathrm{B}}x_0-st_0-\bar{\mathrm{B}}x} + e^{-\bar{\mathrm{B}}x}\int \bar{f}(x)e^{\bar{\mathrm{B}}x}\,dx + e^{-\bar{\mathrm{B}}x}g(s) \\
&= \bar{\mathrm{C}}\mathrm{H}(x-x_0)e^{-(\bar{\mathrm{B}}(x-x_0)+st_0)} + e^{-\bar{\mathrm{B}}x}\int \bar{f}(x)e^{\bar{\mathrm{B}}x}\,dx + e^{-\bar{\mathrm{B}}x}g(s), \qquad (3.3)
\end{aligned}$$

where $\mathrm{H}(x-x_0)$ is the Heaviside step function centered at x_0 and $g(s)$ is an arbitrary function of s. We now find an expression that helps us to eliminate the arbitrary $g(s)$ so that we may take the inverse Laplace Transform of our newly transformed function $\bar{u}$. Denoting the transform of $k(t)$ with respect to the time variable by $\bar{k}(s)$ and recalling that $\alpha < x_0$, we have that

$$\begin{aligned}
\bar{u}(\alpha,s) := \bar{k}(s) &= \bar{\mathrm{C}}\mathrm{H}(\alpha-x_0)e^{-st_0}e^{\bar{\mathrm{B}}(x_0-\alpha)} + e^{-\bar{\mathrm{B}}\alpha}\left[\int \bar{f}(x)e^{\bar{\mathrm{B}}x}\mathrm{d}x\right]_{x=\alpha} + g(s)e^{-\bar{\mathrm{B}}\alpha} \\
\Rightarrow g(s) &= e^{\bar{\mathrm{B}}\alpha}\left(\bar{k}(s) - e^{-\bar{\mathrm{B}}\alpha}\left[\int \bar{f}(x)e^{\bar{\mathrm{B}}x}\mathrm{d}x\right]_{x=\alpha}\right) = \bar{k}(s)e^{\bar{\mathrm{B}}\alpha} - \left[\int \bar{f}(x)e^{\bar{\mathrm{B}}x}\mathrm{d}x\right]_{x=\alpha.}
\end{aligned}$$

Substituting $g(s)$ back into eq. (3.3), our already complicated result becomes

$$\begin{aligned}\bar{u} &= \bar{\mathrm{C}}\mathrm{H}(x-x_0)e^{-st_0}e^{\bar{\mathrm{B}}(x_0-x)} + e^{-\bar{\mathrm{B}}x}\int \bar{f}(x)e^{\bar{\mathrm{B}}x}\mathrm{d}x + e^{-\bar{\mathrm{B}}x}\left(\bar{k}(s)e^{\bar{\mathrm{B}}\alpha} - \left[\int \bar{f}(x)e^{\bar{\mathrm{B}}x}\mathrm{d}x\right]_{x=\alpha}\right)\\
&= \bar{\mathrm{C}}\mathrm{H}(x-x_0)e^{-st_0}e^{(\frac{\mathrm{B}+s}{\mathrm{A}})(x_0-x)} + e^{-(\frac{\mathrm{B}+s}{\mathrm{A}})x}\int \bar{f}(x)e^{\bar{\mathrm{B}}x}\mathrm{d}x\\
&\quad + \bar{k}(s)e^{(\frac{\mathrm{B}+s}{\mathrm{A}})(\alpha-x)} - e^{-(\frac{\mathrm{B}+s}{\mathrm{A}})x}\left[\int \bar{f}(x)e^{\bar{\mathrm{B}}x}\mathrm{d}x\right]_{x=\alpha}\\
&= \bar{\mathrm{C}}\mathrm{H}(x-x_0)e^{\frac{\mathrm{B}}{\mathrm{A}}(x_0-x)}e^{-s\left(t_0-\frac{x_0-x}{\mathrm{A}}\right)} + e^{\frac{-\mathrm{B}x}{\mathrm{A}}}e^{\frac{-sx}{\mathrm{A}}}\int \bar{f}(x)e^{\bar{\mathrm{B}}x}\mathrm{d}x\\
&\quad + e^{\frac{\mathrm{B}}{\mathrm{A}}(\alpha-x)}\bar{k}(s)e^{-s\left(\frac{x-\alpha}{\mathrm{A}}\right)} - e^{-\frac{\mathrm{B}x}{\mathrm{A}}}e^{-\frac{sx}{\mathrm{A}}}\left[\int \bar{f}(x)e^{\bar{\mathrm{B}}x}\mathrm{d}x\right]_{x=\alpha.}\end{aligned}$$

While this appears extraordinarily daunting to deal with, our expression is now written in such a way that all functions involving an s are now easily identifiable. This will aid us as we take the inverse Laplace Transform, showing us which pieces of this puzzle are functions of x only and may therefore be treated as constants in our next step.

Applying the 1D-ILT with respect to the variable s, we arrive at

$$\begin{aligned}u(x,t) &= \mathscr{L}^{-1}\{\bar{u}(x,s)\}(t)\\
&= \bar{\mathrm{C}}e^{\frac{\mathrm{B}}{\mathrm{A}}(x_0-x)}\mathrm{H}(x-x_0)\delta\left(t-t_0+\frac{x_0-x}{\mathrm{A}}\right) + e^{-\frac{\mathrm{B}x}{\mathrm{A}}}\mathscr{L}^{-1}\left\{e^{-\frac{sx}{\mathrm{A}}}\int \bar{f}(x)e^{\bar{\mathrm{B}}x}\mathrm{d}x\right\}(t)\\
&\quad + e^{\frac{\mathrm{B}}{\mathrm{A}}(\alpha-x)}\mathrm{H}\left(t-\frac{x-\alpha}{\mathrm{A}}\right)k\left(t-\frac{x-\alpha}{\mathrm{A}}\right) - e^{-\frac{\mathrm{B}x}{\mathrm{A}}}\mathscr{L}^{-1}\left\{e^{-\frac{sx}{\mathrm{A}}}\left[\int \bar{f}(x)e^{\bar{\mathrm{B}}x}\mathrm{d}x\right]_{x=\alpha}\right\}(t),\end{aligned}$$

which is further simplified to

$$\begin{aligned}u(x,t) &= \bar{\mathrm{C}}e^{\frac{\mathrm{B}}{\mathrm{A}}(x_0-x)}\delta\left(t-t_0+\frac{x_0-x}{\mathrm{A}}\right)\mathrm{H}(x-x_0) + e^{\frac{\mathrm{B}}{\mathrm{A}}(\alpha-x)}\mathrm{H}\left(t-\frac{x-\alpha}{\mathrm{A}}\right)k\left(t-\frac{x-\alpha}{\mathrm{A}}\right)\\
&\quad + e^{-\frac{\mathrm{B}x}{\mathrm{A}}}\mathscr{L}^{-1}\left\{e^{-\frac{sx}{\mathrm{A}}}\left(\int \bar{f}(x)e^{\bar{\mathrm{B}}x}\mathrm{d}x - \left[\int \bar{f}(x)e^{\bar{\mathrm{B}}x}\mathrm{d}x\right]_{x=\alpha}\right)\right\}(t). \qquad (3.4)\end{aligned}$$

Without any further information regarding $f(x)$, we are unable to simplify the integrals appearing in our final solution. As such, we are forced to leave our result in terms of an inverse Laplace Transform.

3.2 The Coefficients in Terms of x BVP

We now assume that the coefficients in eq. (1.14) are reduced to functions of x only, so we have that $p(x,t) = p(x)$, $q(x,t) = q(x)$, $r(x,t) = r(x)$, and $y(x,t) = y(x)$. We couple this with the assumption that $p(x) \neq 0$ and $q(x) \neq 0$ for any $x \in \mathbb{R}$. The following process is largely similar to the previous constant coefficients problem. Dividing both sides of eq. (1.14) by $p(x)$ and letting $\mathrm{P}(x) = \frac{q(x)}{p(x)}$, $\mathrm{Q}(x) = \frac{r(x)}{p(x)}$, and $\mathrm{R}(x) = \frac{y(x)}{p(x)}$, we have

$$u_t + \mathrm{P}(x)u_x + \mathrm{Q}(x)u = \mathrm{R}(x)\delta(x-x_0)\delta(t-t_0), \quad u(x,0) = f(x), \quad u(\alpha,t) = k(t), \tag{3.5}$$
$$t > 0,\ \ t_0 > 0,\ \ x > \alpha,\ \ x_0 > \alpha$$

where α once again represents any real constant. Applying the 1D-LT to eq. (3.5) with respect to the time variable results in

$$\begin{aligned}
&\mathscr{L}\{u_t + \mathrm{P}(x)u_x + \mathrm{Q}(x)u\}(s) = \mathscr{L}\{\mathrm{R}(x)\delta(x-x_0)\delta(t-t_0)\}(s)\\
\Rightarrow\ & s\bar{u} - u(x,0) + \mathrm{P}(x)\bar{u}_x + \mathrm{Q}(x)\bar{u} = \mathrm{R}(x)\delta(x-x_0)e^{-st_0}\\
\Rightarrow\ & \mathrm{P}(x)\bar{u}_x + (\mathrm{Q}(x)+s)\bar{u} = \mathrm{R}(x)\delta(x-x_0)e^{-st_0} + u(x,0) = \mathrm{R}(x)\delta(x-x_0)e^{-st_0} + f(x)\\
\Rightarrow\ & \bar{u}_x + \bar{\mathrm{Q}}(x,s)\bar{u} = \bar{\mathrm{R}}(x)\delta(x-x_0)e^{-st_0} + \bar{f}(x),
\end{aligned} \tag{3.6}$$

where $\bar{\mathrm{Q}}(x,s) = \frac{\mathrm{Q}(x)+s}{\mathrm{P}(x)}$, $\bar{\mathrm{R}}(x) = \frac{\mathrm{R}(x)}{\mathrm{P}(x)}$, $\bar{f}(x) = \frac{f(x)}{\mathrm{P}(x)}$, and $\bar{u} \equiv \bar{u}(x,s)$ is the transform of $u(x,t)$. Because we have a linear PDE whose only derivative is with respect to x, we have reason to find an integrating factor μ.

$$\mu = e^{\int \bar{\mathrm{Q}}(x,s)\ \mathrm{d}x} = e^{\bar{\bar{\mathrm{Q}}}(x,s)}$$

Multiplying both sides of eq. (3.6) by μ, we have

$$\begin{aligned}
&\bar{u}_x e^{\bar{\bar{\mathrm{Q}}}(x,s)} + \bar{\mathrm{Q}}(x,s)\bar{u}e^{\bar{\bar{\mathrm{Q}}}(x,s)} = \bar{\mathrm{R}}(x)\delta(x-x_0)e^{-st_0}e^{\bar{\bar{\mathrm{Q}}}(x,s)} + \bar{f}(x)e^{\bar{\bar{\mathrm{Q}}}(x,s)}\\
\Rightarrow\ & \frac{\partial}{\partial x}\left[e^{\bar{\bar{\mathrm{Q}}}(x,s)}\bar{u}\right] = \bar{\mathrm{R}}(x)\delta(x-x_0)e^{\bar{\bar{\mathrm{Q}}}(x,s)-st_0} + \bar{f}(x)e^{\bar{\bar{\mathrm{Q}}}(x,s)}.
\end{aligned}$$

Just as with the constant coefficients problem, we integrate both sides with respect to x:

$$\begin{aligned}
e^{\bar{\bar{\mathrm{Q}}}(x,s)}\bar{u} &= \int \bar{\mathrm{R}}(x)\delta(x-x_0)e^{\bar{\bar{\mathrm{Q}}}(x,s)-st_0}\mathrm{d}x + \int \bar{f}(x)e^{\bar{\bar{\mathrm{Q}}}(x,s)}\mathrm{d}x + g(s)\\
&= \int \bar{\mathrm{R}}(x_0)\delta(x-x_0)e^{\bar{\bar{\mathrm{Q}}}(x_0,s)-st_0}\mathrm{d}x + \int \bar{f}(x)e^{\bar{\bar{\mathrm{Q}}}(x,s)}\mathrm{d}x + g(s)\\
&= \bar{\mathrm{R}}(x_0)\mathrm{H}(x-x_0)e^{\bar{\bar{\mathrm{Q}}}(x_0,s)-st_0} + \int \bar{f}(x)e^{\bar{\bar{\mathrm{Q}}}(x,s)}\mathrm{d}x + g(s)\\
\Rightarrow\quad \bar{u} &= \bar{\mathrm{R}}(x_0)\mathrm{H}(x-x_0)e^{\bar{\bar{\mathrm{Q}}}(x_0,s)-\bar{\bar{\mathrm{Q}}}(x,s)-st_0}\\
&\quad + e^{-\bar{\bar{\mathrm{Q}}}(x,s)}\int \bar{f}(x)e^{\bar{\bar{\mathrm{Q}}}(x,s)}\mathrm{d}x + g(s)e^{-\bar{\bar{\mathrm{Q}}}(x,s)},
\end{aligned} \tag{3.7}$$

where $g(s)$ is an arbitrary function of s. Once again denoting $\bar{k}(s)$ as the Laplace Transform of $k(t)$ with respect to the time variable, we now use our boundary condition in order to determine $g(s)$. Substituting $x = \alpha$ into eq. (3.7) and once again recalling that $\alpha < x_0$, we have

$$\bar{u}(\alpha,s) := \bar{k}(s) = \bar{\text{R}}(x_0)\text{H}(\alpha-x_0)e^{\bar{\bar{\text{Q}}}(x_0,s)-\bar{\bar{\text{Q}}}(\alpha,s)-st_0} + e^{-\bar{\bar{\text{Q}}}(\alpha,s)}\left[\int \bar{f}(x)e^{\bar{\bar{\text{Q}}}(x,s)}\text{d}x\right]_{x=\alpha} + g(s)e^{-\bar{\bar{\text{Q}}}(\alpha,s)}$$
$$\Rightarrow g(s) = e^{\bar{\bar{\text{Q}}}(\alpha,s)}\left(\bar{k}(s) - e^{-\bar{\bar{\text{Q}}}(\alpha,s)}\left[\int \bar{f}(x)e^{\bar{\bar{\text{Q}}}(x,s)}\text{d}x\right]_{x=\alpha}\right)$$
$$= \bar{k}(s)e^{\bar{\bar{\text{Q}}}(\alpha,s)} - \left[\int \bar{f}(x)e^{\bar{\bar{\text{Q}}}(x,s)}\text{d}x\right]_{x=\alpha.}$$

Plugging this result back into eq. (3.7) shows us that

$$\begin{aligned}\bar{u}(x,s) &= \bar{\text{R}}(x_0)\text{H}(x-x_0)e^{\bar{\bar{\text{Q}}}(x_0,s)-\bar{\bar{\text{Q}}}(x,s)-st_0} + e^{-\bar{\bar{\text{Q}}}(x,s)}\int \bar{f}(x)e^{\bar{\bar{\text{Q}}}(x,s)}\text{d}x \\ &\quad + e^{-\bar{\bar{\text{Q}}}(x,s)}\left(\bar{k}(s)e^{\bar{\bar{\text{Q}}}(\alpha,s)} - \left[\int \bar{f}(x)e^{\bar{\bar{\text{Q}}}(x,s)}\text{d}x\right]_{x=\alpha}\right) \\ &= \bar{\text{R}}(x_0)\text{H}(x-x_0)e^{\bar{\bar{\text{Q}}}(x_0,s)-\bar{\bar{\text{Q}}}(x,s)-st_0} + e^{-\bar{\bar{\text{Q}}}(x,s)}\int \bar{f}(x)e^{\bar{\bar{\text{Q}}}(x,s)}\text{d}x \\ &\quad + \bar{k}(s)e^{\bar{\bar{\text{Q}}}(\alpha,s)-\bar{\bar{\text{Q}}}(x,s)} - e^{-\bar{\bar{\text{Q}}}(x,s)}\left[\int \bar{f}(x)e^{\bar{\bar{\text{Q}}}(x,s)}\text{d}x\right]_{x=\alpha.} \end{aligned} \tag{3.8}$$

Unfortunately, this result is very complicated. In the constant coefficients problem, it is possible to break up the term $\bar{\text{B}}$ into $\frac{\text{B}+s}{\text{A}}$ in order to directly use the inverse Laplace Transform later on. Due to the generality of $\bar{\bar{\text{Q}}}(x,s)$, there is no way for us to apply the same treatment here. We are therefore forced to write our weak solution as

$$u(x,t) = \mathscr{L}^{-1}\{\bar{u}(x,s)\}, \tag{3.9}$$

where $\bar{u}(x,s)$ is as represented in eq. (3.8). It is regrettable that we cannot write our solution in any other way, but for sufficiently simple choices of our coefficients, as well as our initial and boundary conditions, we may still calculate the solution by hand.

3.3 The Coefficients in Terms of t BVP

In an almost identical fashion to the previous section, we now assume that the coefficients in eq. (1.14) are reduced to functions of t only, so we have that $p(x,t) = p(t)$, $q(x,t) = q(t)$, $r(x,t) = r(t)$, and $y(x,t) = y(t)$. For the sake of consistency, however, let us rewrite the problem as

$$p(x,t)u_x + q(x,t)u_t + r(x,t)u = y(x,t)\delta(x-x_0)\delta(t-t_0).$$

We couple this with the assumption that $p(t) \neq 0$ and $q(t) \neq 0$ for any $t > 0$. Dividing both sides of section 3.3 by $p(t)$ and letting $\text{P}(t) = \frac{q(t)}{p(t)}$, $\text{Q}(t) = \frac{r(t)}{p(t)}$, and $\text{R}(t) = \frac{y(t)}{p(t)}$, we have

$$u_x + \mathrm{P}(t)u_t + \mathrm{Q}(t)u = \mathrm{R}(t)\delta(x-x_0)\delta(t-t_0), \quad u(x,\alpha) = f(x), \quad u(0,t) = k(t), \tag{3.10}$$
$$x>0, \ x_0>0, \quad t>\alpha, \ t_0>\alpha,$$

where $\alpha \geq 0$. From here, we apply the unilateral Laplace Transform with respect to the variable x to eq. (3.10), resulting in

$$s\bar{u}(s,t) - k(t) + \mathrm{P}(t)\bar{u}_t(s,t) + \mathrm{Q}(t)\bar{u}(s,t) = \mathrm{R}(t)\delta(t-t_0)e^{-sx_0}.$$

By following the exact same process as in the previous section, with the roles of $f(x)$ and $k(t)$ now effectively reversed, we arrive at a very similar weak solution of this BVP:

$$\bar{u}(s,t) = \bar{\mathrm{R}}(t_0)\mathrm{H}(t-t_0)e^{\bar{\bar{\mathrm{Q}}}(s,t_0)-\bar{\bar{\mathrm{Q}}}(s,t)-sx_0} + e^{-\bar{\bar{\mathrm{Q}}}(s,t)}\int \bar{k}(t)e^{\bar{\bar{\mathrm{Q}}}(s,t)}\mathrm{d}t$$
$$+ \bar{f}(s)e^{\bar{\bar{\mathrm{Q}}}(s,\alpha)-\bar{\bar{\mathrm{Q}}}(s,t)} - e^{-\bar{\bar{\mathrm{Q}}}(s,t)}\left[\int \bar{k}(t)e^{\bar{\bar{\mathrm{Q}}}(s,t)}\mathrm{d}t\right]_{t=\alpha} \tag{3.11}$$
$$\Rightarrow u(x,t) = \mathscr{L}^{-1}\{\bar{u}(s,t)\}, \tag{3.12}$$

where $\bar{\bar{\mathrm{Q}}}(s,t) = \int \frac{\mathrm{Q}(t)+s}{\mathrm{P}(t)}dt$, $\bar{\mathrm{R}}(t) = \frac{\mathrm{R}(t)}{\mathrm{P}(t)}$, $\bar{k}(t) = \frac{k(t)}{\mathrm{P}(t)}$, and $\bar{f}(s)$ is the unilateral Laplace Transform of $f(x)$ with respect to x.

It is apparent that the solutions for both the preceding problem as well as this one are very similar in nature. The main difference is that in the previous section, we use the initial condition $f(x)$ at the outset of the problem upon applying the Laplace Transform with respect to the time variable. Here, the initial condition is used to eliminate an arbitrary function, just as $k(t)$ is used to eliminate $g(s)$ in the previous section.

4 Examples

We begin with a basic example where all of the coefficients are constant. Consider the following BVP:

Example 4.1.

$$u_t + 2u_x + 4u = 5\delta(x-1,\ t-2), \quad u(x,0) = 4, \quad u(0,t) = t^3,$$
$$t>0, \ x>0.$$

Using our results from section 3.1, we quickly find that the weak solution must be

$$
\begin{aligned}
u(x,t) &= \frac{5}{2}e^{\frac{4}{2}(1-x)}\delta\left(t-2+\frac{1-x}{2}\right)\mathrm{H}(x-1)+e^{\frac{4}{2}(0-x)}\mathrm{H}\left(t-\frac{x-0}{2}\right)\left(t-\frac{x-0}{2}\right)^3 \\
&\quad + e^{-\frac{4}{2}x}\mathscr{L}^{-1}\left\{e^{-\frac{sx}{2}}\left(\int\frac{4}{2}e^{\frac{(4+s)x}{2}}\mathrm{d}x-\left[\int\frac{4}{2}e^{\frac{(4+s)x}{2}}\mathrm{d}x\right]_{x=0}\right)\right\}(t) \\
&= \frac{5}{2}e^{2-2x}\delta\left(t-\frac{x+3}{2}\right)\mathrm{H}(x-1)+e^{-2x}\left(t-\frac{x}{2}\right)^3\mathrm{H}\left(t-\frac{x}{2}\right) \\
&\quad + e^{-2x}\mathscr{L}^{-1}\left\{e^{-\frac{sx}{2}}\left(\frac{4}{s+4}e^{2x+\frac{sx}{2}}-\frac{4}{s+4}\right)\right\}(t) \\
&= \frac{5}{2}e^{2-2x}\delta\left(t-\frac{x+3}{2}\right)\mathrm{H}(x-1)+e^{-2x}\left(t-\frac{x}{2}\right)^3\mathrm{H}\left(t-\frac{x}{2}\right) \\
&\quad + e^{-2x}\left(e^{2x}\mathscr{L}^{-1}\left\{\frac{4}{s+4}\right\}(t)-\mathscr{L}^{-1}\left\{\frac{4}{s+4}e^{-\frac{sx}{2}}\right\}(t)\right) \\
&= \frac{5}{2}e^{2-2x}\delta\left(t-\frac{x+3}{2}\right)\mathrm{H}(x-1)+e^{-2x}\left(t-\frac{x}{2}\right)^3\mathrm{H}\left(t-\frac{x}{2}\right) \\
&\quad + e^{-2x}\left(4e^{2x}e^{-4t}-4e^{-4\left(t-\frac{x}{2}\right)}\mathrm{H}\left(t-\frac{x}{2}\right)\right) \\
&= \frac{5}{2}e^{2-2x}\delta\left(t-\frac{x+3}{2}\right)\mathrm{H}(x-1)+e^{-2x}\left(t-\frac{x}{2}\right)^3\mathrm{H}\left(t-\frac{x}{2}\right)+4e^{-4t}-4e^{-4t}\mathrm{H}\left(t-\frac{x}{2}\right).
\end{aligned}
$$

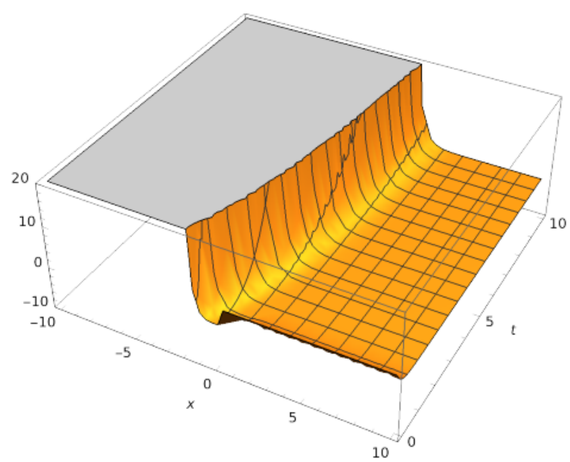

Figure 1: Solution of Example 1

We should note several things here. First, the computer program used to generate this image, Wolfram Mathematica, has no value associated with $\delta(0)$. As such, the plot above is not representative of the weak solution at any point in the xt-plane along the line $2t = x+3$. These points constitute a set of discontinuities that the computer cannot plot, so our graph instead looks like a classical solution of the homogeneous version of this problem. Second, the especially keen observer may have noticed that the initial condition is not satisfied for $x \leq 0$. This is why we restrict our domain to include only positive values of x. In general, the initial condition is only satisfied for $x > \alpha$, implying that in this example, the solution is truly only valid for positive x and positive t. Additionally, we have a discontinuity of boundary conditions at the origin, leading to a sharp change in the solution at that point. There are BVP in which the initial condition is satisfied for $x = \alpha$, however. For a more concrete example of this phenomenon, please view **example 4.3**. One may also algebraically show that the boundary condition is satisfied.

Let us now consider an example where the coefficients of the problem are given to be functions of x only. For the sake of brevity, this problem is shorter than the previous one.

Example 4.2.

$$u_t + u_x - \left(\frac{1}{1+x^2}\right)u = (1-x^2)\delta(x-1,\, t-1), \quad u(x,0) = 0, \quad u(0,t) = 10\sin(2t),$$
$$t > 0, \;\; x > 0.$$

Before proceeding, we note that

$$\bar{\bar{Q}}(x,s) = \int \left(-\frac{1}{1+x^2} + s\right)\,\mathrm{d}x = -\arctan(x) + sx.$$

It is also given that $f(x) = 0$, so we neglect all terms involving $\bar{f}(x)$. Based on our work in section 3.2, we have that

$$\begin{aligned}\bar{u}(x,s) &= (1-1^2)\mathrm{H}(x-1)e^{-\arctan(1)+s-(-\arctan(x)+sx)-st_0} + \frac{20}{s^2+4}e^{-\arctan(0)+0(s)-(-\arctan(x)+sx)} \\ &= \frac{20}{s^2+4}e^{\arctan(x)-sx} = \frac{20}{s^2+4}e^{\arctan(x)}e^{-sx}\end{aligned}$$

$$\Rightarrow u(x,t) = \mathscr{L}^{-1}\left\{\frac{20}{s^2+4}e^{\arctan(x)}e^{-sx}\right\}(t) = 10e^{\arctan(x)}\sin(2(t-x))\mathrm{H}(t-x)$$

This example is much simpler than the last for several reasons. The coefficients are chosen in such a way that calculations are very short and the choice of x_0 and t_0 results in a trivial impulse as $1-1^2 = 0$. Such a short solution is rare with these types of problems and only appears when provided with well-chosen coefficients, initial and boundary conditions, and points of impulse.

Clearly, we may verify that the initial condition holds for $x \geq 0$ by substituting $t = 0$ into our solution. Similarly, we may show that the boundary condition is satisfied by substituting $x = 0$. Because there is no discontinuity of boundary conditions at the origin, we have a smoother graph in this example than that of **example 4.1**.

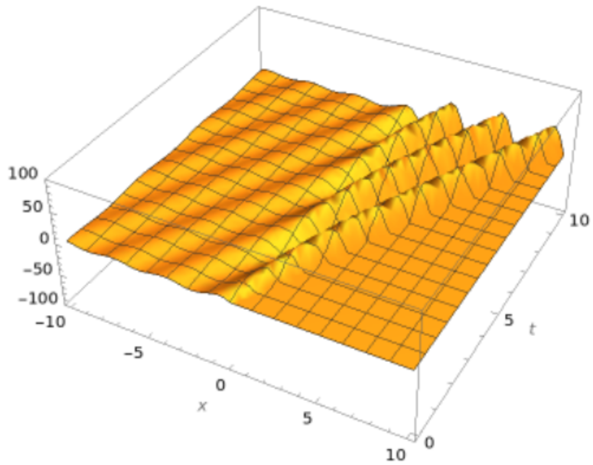

Figure 2: Solution of Example 2

Example 4.3.

$$u_t + u_x - \left(\frac{1}{1+x^2}\right)u = (1-x^2)\delta(x-1,\, t-1), \quad u(x,0) = 0, \quad u(-8,t) = 10\sin(2t),$$
$$t > 0, \;\; x > -8.$$

The only difference between this problem and the last is that $\alpha = -8$ here. The calculation follows in the exact same manner, leaving us with only one term that does not evaluate to zero. Our weak solution is then:

$$\bar{u}(x,s) = \frac{20}{s^2+4} e^{-\arctan(-8)-8s+\arctan(x)-sx} = \frac{20}{s^2+4} e^{\arctan(x)-\arctan(-8)} e^{-sx-8s}$$

$$\Rightarrow u(x,t) = \mathscr{L}^{-1}\left\{\frac{20}{s^2+4} e^{\arctan(x)-\arctan(-8)} e^{-sx-8s}\right\}(t)$$
$$= 10e^{\arctan(x)-\arctan(-8)} \sin(2(t-x-8))\mathrm{H}(t-x-8).$$

It is clear to see that the initial condition $u(x,0) = 0$ is only satisfied when $x \geq -8$. In this case, it works out nicely to where the initial condition is satisfied at $x = \alpha$ because there is no discontinuity of boundary conditions at (-8,0), but this is generally not the case.

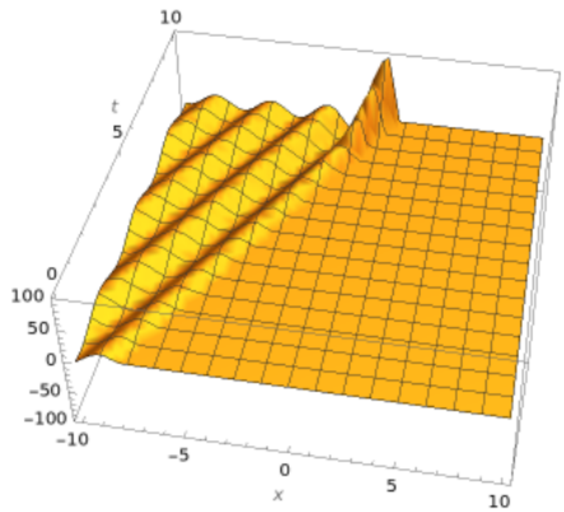

Figure 3: Solution of Example 3

Example 4.4.

$$u_x + \frac{1}{t+1} u_t = (t^2-4)\delta(x-4,\ t-1), \quad u(x,0) = e^x, \quad u(0,t) = 0,$$
$$t > 0, \quad x > 0.$$

We follow the procedure from section 3.3. It follows that $\bar{\mathrm{R}}(t_0) = (1^2-4)(1+1) = -6$ and any term involving $k(t) = 0$ vanishes. We should note that

$$\bar{\bar{\mathrm{Q}}}(s,t) = \int s(t+1)\mathrm{d}t = s\left(\frac{t^2}{2}+t\right),$$

so by utilizing the inverse Laplace Transform in order to convert our complex variable s back into our spatial variable x, we have that

$$\bar{u}(s,t) = -6\mathrm{H}(t-1)e^{\frac{3s}{2}-s\left(\frac{t^2}{2}+t\right)-4s} + \frac{1}{s-1}e^{-s\left(\frac{t^2}{2}+t\right)} = -6\mathrm{H}(t-1)e^{-s\left(\frac{5}{2}+t+\frac{t^2}{2}\right)} + \frac{1}{s-1}e^{-s\left(\frac{t^2}{2}+t\right)}$$

$$\Rightarrow u(x,t) = \mathscr{L}^{-1}\{\bar{u}(s,t)\}(x)$$
$$= e^{x-t-\frac{t^2}{2}}\mathrm{H}\left(x-t-\frac{t^2}{2}\right) - 6\mathrm{H}(t-1)\mathrm{H}\left(x-\frac{t^2}{2}-t-\frac{5}{2}\right)\delta\left(x-\frac{t^2}{2}-t-\frac{5}{2}\right)$$

Because Mathematica does not have any value associated with $\delta(0)$, the below figure is only representative of the first term of our solution and as such looks like a classical solution of the homogeneous version of this problem. For all points (x,t) such that $2x - t^2 - 2t - 5 = 0$, the figure is an inaccurate representation of the weak solution given by $u(x,t)$ above.

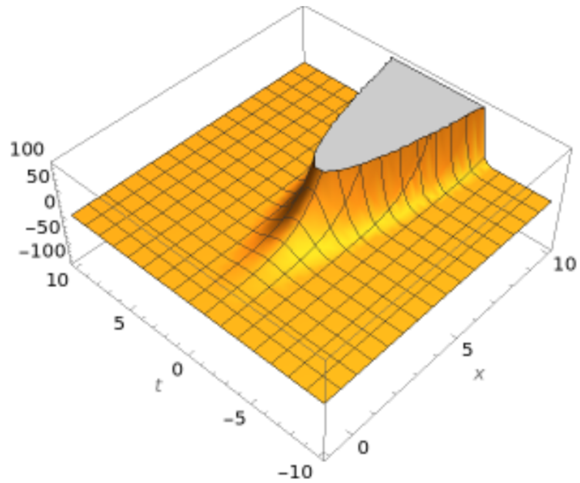

Figure 4: Solution of Example 4

References

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Appendix

Common Laplace Transforms in the Time Variable	
$f(t)$	$\mathrm{F}(s)$
1	$\frac{1}{s}$
$t^n, n \in \mathbb{N}$	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}$
$\cos(at)$	$\frac{s}{s^2+a^2}$
$\sin(at)$	$\frac{a}{s^2+a^2}$
$\mathrm{H}(t-a)$	$\frac{e^{-as}}{s}$
$\delta(t-a)$	e^{-as}
$\mathrm{H}(t-a)f(t-a)$	$e^{-as}\mathrm{F}(s)$
$\frac{\partial f(x,t)}{\partial x}$	$\frac{\partial \mathrm{F}(x,s)}{\partial x}$
$\frac{\partial f(x,t)}{\partial t}$	$s\mathrm{F}(x,s) - f(x,0)$

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