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Cover Page Footnote

I would like to deeply thank Professor Anshelevich for all of his guidance, as this project would not have existed without him. I would also like to thank our reviewer for all of the helpful advice.

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A Characterization of Complex-Valued Random Variables With Rotationally-Invariant Moments

By *Michael Maiello*

Abstract. A complex-valued random variable Z has rotationally-invariant moments if $\mathbb{E}[Z^n\bar{Z}^m] = \mathbb{E}[e^{i\theta}Z^n\bar{Z}^m]$ for all $\theta \in [0, 2\pi)$. In the first part of the article, we characterize such random variables, in terms of "vanishing unbalanced moments," moment- and cumulant-generating functions, and polar decomposition. In the second part, we consider random variables whose moments are not necessarily finite, but which have a density. In this setting, we prove two characterizations that are equivalent to rotational invariance, one involving polar decomposition, and the other involving entropy. If a random variable has both a density and moments which determine it, all of these characterizations are equivalent.

1 Introduction

A complex random variable has the form $Z = X + iY$, where X and Y are real-valued random variables. The moments of a complex random variable are numbers of the form $\mathbb{E}[Z^n\bar{Z}^m]$, where $\mathbb{E}$ is the expectation and n, m are integers with $n, m \geq 0$. Unless Z is the zero random variable, its *balanced* moments $\mathbb{E}[Z^n\bar{Z}^n] = \mathbb{E}[|Z|^{2n}]$ are strictly positive. It is, however, possible for other moments of Z to be zero. For example, if X, Y are uncorrelated, with zero mean and equal variance, then it is easy to verify that $\mathbb{E}[Z^2] = 0$. In fact, it is possible for *all* the unbalanced moments of Z (those for $n \neq m$) to vanish. Two key examples are

- Z complex Gaussian where X and Y independent, normally distributed, with mean zero and equal variances.
- Z uniformly distributed on the unit circle in the complex plane.

In this article, we will describe all complex random variables all of whose unbalanced moments vanish. It turns out that this somewhat unusual condition is equivalent to a very familiar one, namely that the random variable have rotationally invariant moments. That is, such Z have the same moments as $e^{i\theta}Z$ for any real θ. Most of the article is dedicated to describing other conditions equivalent to rotational invariance and

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vanishing of unbalanced moments. For example, such Z has the same distribution as VZ, where V is uniformly distributed on the unit circle and independent from Z (see **Theorem** 2.3(4) for a more general statement). Other characterizations involve moment and cumulant generating functions. We also show that the sum of two independent random variables with rotationally invariant moments, or the product of a random variable with rotationally invariant moments and any random variable independent from it, again have rotationally invariant moments.

In the second part of the paper, we consider random variables whose moments may not be finite, but which have a density. Recall that neither of these properties implies the other. Thus we consider random variables with a rotationally invariant density, rather than just rotationally invariant moments. In this setting, we prove two additional characterizations. The first one states that in the polar decomposition $Z = Re^{iU}$ of Z, the radial part R and the polar part U are independent, and U is uniformly distributed. The second one states that among all random variables with the same (distribution of) radial part, the one with the rotationally invariant distribution has the largest entropy.

In the case when the random variables have both a density and moments which determine the density (for example, when they are bounded), all the characterizations above are equivalent.

We finish the introduction with some additional motivation for the questions considered in this article. The reader can safely skip the rest of the introduction, or (preferably) return to it after reading the article.

Free probability deals with random variable-like objects which do not necessarily commute, such as linear operators or random matrices. Since such an object a does not even necessarily commute with its adjoint a^*, the moments become more complicated, for example $\mathbb{E}[aa^*aa^*]$ is not necessarily the same as $\mathbb{E}[a^2(a^*)^2]$. Nevertheless, with appropriate definitions, one has the notions of "free" independence, "free" cumulants, "free" entropy, etc. The reader is encouraged to consult for example [5]. In particular, free cumulants also depend on the order, so that the joint free cumulant of (a, a^*, a, a^*) is in general different from that of (a, a, a^*, a^*).

One of the important notions which arose in free probability is that of R-diagonal elements: operators such that only free cumulants of the alternating form $(a, a^*, \ldots, a, a^*)$ and $(a^*, a, \ldots, a^*, a)$ are allowed to be non-zero, while all the other free cumulants have to vanish. Note that this is a strengthening of the "balanced" assumption, which would only require the total numbers of a's and a^*'s to be the same. Numerous characterizations and properties of R-diagonal elements have since been proved. The reader can consult Lecture 15 of [5] as well as [2] to see that most properties of random variables in this article are parallel to those of R-diagonal elements. Thus one can summarize the article as

> A commutative version of an R-diagonal element is a random variable with rotationally invariant moments.

We want to emphasize that there are additional properties of R-diagonal elements which do not seem to have commutative analogs.

2 Random Variables With Finite Moments

Throughout this section, we only consider random variables all of whose moments exist and are finite. Throughout the paper, unless stated otherwise, we let k, m, and n be non-negative integers.

2.1 Prerequisites I

Definition 2.1. Let $Z = X + iY$ be a complex random variable.

- If $n, m \geq 0$, the corresponding moment of Z is the expectation $\mathbb{E}[Z^n\bar{Z}^m]$.
- Z has v.u.m. (vanishing unbalanced moments) if for all $n \neq m$, $\mathbb{E}[Z^n\bar{Z}^m] = 0$.
- The moment-generating function of a complex random variable Z with $z \in \mathbb{C}$ is the formal power series
$$M_Z(z) = \sum_{n,m=0}^{\infty} \frac{\mathbb{E}[Z^n\bar{Z}^m]}{n!m!} z^n \bar{z}^m.$$
- The cumulant generating function of Z is the formal power series $C_Z(z) = \log M_Z(z)$. The coefficients in the expansion
$$C_Z(z) = \sum_{n,m=0}^{\infty} \frac{c_{n,m}(Z)}{n!m!} z^n \bar{z}^m$$
with $c_{0,0} = 0$ are called the cumulants of Z.
- The moments of a complex random variable Z are rotationally invariant if for any $\theta \in [0, 2\pi)$ the moments of Z are the same as the moments of $W = e^{i\theta}Z$.

If the reader wishes to learn more about the definition of a complex random variable and the moments of such a random variable, refer to the second chapter of [6]. If instead one wants information regarding Moment Generating functions, refer to [3].

Proposition 2.2. *If a random variable* U *with values in* $[0, 2\pi)$ *is uniformly distributed (that is, if it has constant density), then for any integer n,*
$$\mathbb{E}[e^{iUn}] = \frac{1}{2\pi}\int_0^{2\pi} e^{iun}\, du = \delta_{n,0}.$$

We omit the proof of the previous statement.

2.2 Main Results I

Theorem 2.3. *For a complex random variable* Z *all of whose moments are finite, the following statements are equivalent.*

1. *Z has v.u.m.*
2. *The moments of* Z *are rotationally invariant.*
3. Z *has the same moments as* $|Z|\,e^{iU}$*, with* $|Z|$ *and* e^{iU} *independent, and* U *uniformly distributed.*
4. *Let* V *be a random variable with values in* $[0,2\pi)$*, such that* e^{iV} *is independent from* Z *and no power of* e^{iV} *is equal to* 1 *almost everywhere. Then* Z *and* Ze^{iV} *have the same moments.*
5. *The moment generating function depends only on* $|z|$*, in the sense that* $M_Z(z) = h(|z|)$ *for some function* h*. Equivalently,* $M_Z(z) = M_Z(e^{i\theta}z)$ *for any* $\theta \in [0,2\pi)$.
6. *The cumulant generating function of* Z *depends only on* $|z|$.
7. *The cumulants of* Z *are zero for all* $n \neq m$.

The proof of 2.3 will come as a result of the lemmas in the next section.

Example 2.4. Let U be uniformly distributed on $[0,2\pi)$, and $Z = e^{iU}$ be uniformly distributed on the unit circle. Then

$$\mathbb{E}[Z^n\overline{Z}^m] = \mathbb{E}[e^{iU(n-m)}] = 0$$

for $n \neq m$ and 1 for $n = m$. Therefore Z satisfies condition (1) in the theorem.

Example 2.5. Let X, Y be independent random variables whose distributions are normal with mean 0 and equal variance σ^2. Define the complex Gaussian random variable $Z = X + iY$. Then using independence, and the well-known form of the moment generating function for a normal random variable,

$$\begin{aligned} M_Z(z) &= \mathbb{E}\left[e^{(X+iY)z}e^{(X-iY)\bar{z}}\right] \\ &= \mathbb{E}\left[e^{X(z+\bar{z})}e^{Yi(z-\bar{z})}\right] \\ &= \mathbb{E}\left[e^{X(z+\bar{z})}\right]\mathbb{E}\left[e^{Yi(z-\bar{z})}\right] \\ &= e^{\sigma^2(z+\bar{z})^2/2}e^{-\sigma^2(z-\bar{z})^2/2} \\ &= e^{2\sigma^2|z|^2}. \end{aligned}$$

Therefore Z satisfies condition (5) in **Theorem** 2.3.

The following proposition summarizes additional properties satisfied by random variables from **Theorem** 2.3.

Proposition 2.6. *Let* A *be a random variable with v.u.m. (or satisfying any of the equivalent conditions in **Theorem** 2.3).*

(a) If B *is another random variable, independent from* A*, which has v.u.m., then so does* $\mathrm{A}+\mathrm{B}$.

(b) If B *is any random variable independent from* A*, then* AB *has v.u.m.*

(c) For any $k \neq 0$*,* A^k *has v.u.m.*

Proof. For (a), we calculate

$$\begin{aligned}\mathbb{E}[(\mathrm{A}+\mathrm{B})^n(\bar{\mathrm{A}}+\bar{\mathrm{B}})^m] &= \mathbb{E}\left[\sum_{p=0}^{n}\sum_{q=0}^{m}\binom{n}{p}\binom{m}{q}\mathrm{A}^{n-p}\cdot\bar{\mathrm{A}}^{m-q}\mathrm{B}^p\bar{\mathrm{B}}^q\right] \\ &= \sum_{p=0}^{n}\sum_{q=0}^{m}\binom{n}{p}\binom{m}{q}\mathbb{E}[\mathrm{A}^{n-p}\cdot\bar{\mathrm{A}}^{m-q}]\mathbb{E}[\mathrm{B}^p\bar{\mathrm{B}}^q]\end{aligned}$$

Since $n \neq m$, we cannot have both $n-p = m-q$ and $p = q$. Therefore, $\mathbb{E}[\mathrm{A}^{n-p}\cdot\bar{\mathrm{A}}^{m-q}] = 0$ or $\mathbb{E}[\mathrm{B}^p\bar{\mathrm{B}}^q] = 0$. So, for $n \neq m$,

$$\mathbb{E}[(\mathrm{A}+\mathrm{B})^n(\bar{\mathrm{A}}+\bar{\mathrm{B}})^m] = \sum_{p=0}^{n}\sum_{q=0}^{m}\binom{n}{p}\binom{m}{q}\mathbb{E}[\mathrm{A}^{n-p}\cdot\bar{\mathrm{A}}^{m-q}]\mathbb{E}[\mathrm{B}^p\bar{\mathrm{B}}^q] = 0.$$

For (b), we note that for $n \neq m$,

$$\begin{aligned}\mathbb{E}[(\mathrm{AB})^n\overline{\mathrm{AB}}^m] &= \mathbb{E}[\mathrm{A}^n\bar{\mathrm{A}}^m]\mathbb{E}[\mathrm{B}^n\bar{\mathrm{B}}^m] \\ &= 0\end{aligned}$$

For (c), let $n \neq m$ be integers. Then,

$$\mathbb{E}[(\mathrm{A}^k)^n(\bar{\mathrm{A}}^k)^m] = \mathbb{E}[\mathrm{A}^{kn}\bar{\mathrm{A}}^{km}].$$

Since A has v.u.m., it follows that the above is zero. □

2.3 Proof of 2.3

Lemma 2.7. *Z has v.u.m. if and only if the moments of Z are rotationally invariant.*

Proof. (⇒) Given a complex random variable Z with $\mathbb{E}[Z^n\bar{Z}^m] = 0$ for $n \neq m$, let $\theta \in [0, 2\pi)$ be arbitrary. Then, since $e^{i\theta(n-m)}$ is a constant, it follows that for $n \neq m$,

$$\begin{aligned}\mathbb{E}[(e^{i\theta}Z)^n(\overline{e^{i\theta}Z})^m] &= \mathbb{E}[Z^n\bar{Z}^m e^{i\theta(n-m)}] \\ &= e^{i\theta(n-m)}\mathbb{E}[Z^n\bar{Z}^m] \\ &= e^{i\theta(n-m)} \cdot 0 \\ &= \mathbb{E}[Z^n\bar{Z}^m].\end{aligned}$$

For $n = m$, note that both sides become $\mathbb{E}[|Z|^{2n}]$.

(⇐) If the moments of Z are rotationally invariant, for $\theta = 1$ and for any $n, m \geq 0$,

$$e^{i(n-m)}\mathbb{E}[Z^n\bar{Z}^m] = \mathbb{E}[Z^n\bar{Z}^m].$$

Let $n \neq m$. Then $e^{i(n-m)} \neq 1$, and so $\mathbb{E}[Z^n\bar{Z}^m] = 0$. □

Lemma 2.8. *The distribution of Z has v.u.m. if and only if Z has the same moments as $|Z|\, e^{iU}$ with $|Z|$, e^{iU} independent, and U uniformly distributed.*

Proof. (⇒) Suppose that the distribution of Z has v.u.m. Let U be a uniformly distributed random variable such that e^{iU} is independent from $|Z|$. Consider the random variable $W = |Z|\, e^{iU}$. The moments of W are

$$\begin{aligned}\mathbb{E}[W^n\overline{W}^m] &= \mathbb{E}[|Z|^{n+m}\, e^{iU(n-m)}] \\ &= \mathbb{E}[|Z|^{n+m}]\ \mathbb{E}[e^{iU(n-m)}] \\ &= 0\end{aligned}$$

for $n \neq m$. Therefore, for $n \neq m$, it follows that

$$\begin{aligned}\mathbb{E}[Z^n\bar{Z}^m] &= 0 \\ &= \mathbb{E}[W^n\overline{W}^m].\end{aligned}$$

On the other hand, if $n = m$,

$$\begin{aligned}\mathbb{E}[W^n\overline{W}^n] &= \mathbb{E}[|Z|^{2n}] \\ &= \mathbb{E}[Z^n\bar{Z}^n].\end{aligned}$$

So Z has the same moments as $|Z|e^{iU}$.

($\Leftarrow$) Given a random variable Z satisfying the assumption, for $n \neq m$,

$$\mathbb{E}[Z^n \bar{Z}^m] = \mathbb{E}[|Z|^{n+m} e^{iU(n-m)}] = \mathbb{E}[|Z|^{n+m}]\,\mathbb{E}[e^{iU(n-m)}] = 0.$$

□

Lemma 2.9. *If Z has v.u.m., then the moment generating function depends only on* $|z|$.

Proof. Given Z with v.u.m., its moment generating function collapses to

$$\begin{aligned} M_Z(z) &= \sum_{n,m=0}^{\infty} \frac{\mathbb{E}[Z^n \bar{Z}^m]}{n!m!} z^n \bar{z}^m \\ &= \sum_{n=1}^{\infty} \frac{\mathbb{E}[|Z|^2]}{(n!)^2} |z|^2 \end{aligned}$$

Hence the moment generating function depends only on $|z|$. □

Lemma 2.10. *If Z is a random variable such that the moment generating function depends only on* $|z|$*, then the moments of Z are rotationally invariant.*

Proof. Suppose that the moment-generating function depends only on $|z|$. Then, note that since $|e^{i\theta}| = 1$ for any $\theta \in [0, 2\pi)$, $|z| = |e^{i\theta}| \cdot |z| = |e^{i\theta} z|$. Therefore, $h(|z|) = h(|e^{i\theta} z|)$, which implies $M_Z(z) = M_Z(e^{i\theta}(z))$ as $h(|z|) = M_Z(z)$ and $h(|e^{i\theta} z|) = M_Z(e^{i\theta} z)$.

So, suppose that for any $\theta \in [0, 2\pi)$, $M_Z(z) = M_Z(e^{i\theta} z)$. Then

$$\begin{aligned} M_{e^{i\theta}Z}(z) &= \sum_{n,m=0}^{\infty} \frac{\mathbb{E}[(e^{i\theta}Z)^n \overline{(e^{i\theta}Z)}^m]}{n!m!} z^n \bar{z}^m \\ &= M_Z(e^{i\theta} z) \\ &= M_Z(z). \end{aligned}$$

This immediately implies that the moments of $e^{i\theta}Z$ equal those of Z, and the moments of this random variable are rotationally invariant. □

Lemma 2.11. *The moment generating function depends only on* $|z|$ *if and only if the cumulant generating function depends only on* $|z|$.

Proof. This follows directly from the definition of the cumulant generating function. □

Lemma 2.12. *The cumulant generating function depends only on* $|z|$ *if and only if the cumulants of Z are zero unless* $n = m$.

Proof. (⇒) Given our assumptions, we may write that for $\theta = 1$,

$$\sum_{n,m=0}^{\infty} \frac{c_{n,m}(Z)z^n\bar{z}^m}{n!m!} = \sum_{n,m=0}^{\infty} \frac{e^{i(n-m)}c_{n,m}(Z)z^n\bar{z}^m}{n!m!},$$

that is,

$$0 = \sum_{n,m=0}^{\infty} \frac{c_{n,m}(Z)z^n\bar{z}^m(e^{i(n-m)}-1)}{n!m!}.$$

We now observe that $(e^{i(n-m)}-1) \neq 0$ for any choice of $n \neq m$. It follows that the coefficients $c_{n,m}(Z)$ of this formal power series are zero for each $n \neq m$.

(⇐) If the cumulants of Z are zero for $n \neq m$, then as above,

$$C_Z(z) = \sum_{n,m=0}^{\infty} \frac{c_{n,m}(Z)z^n\bar{z}^m}{n!m!} = \sum_{n=0}^{\infty} \frac{c_{n,n}(Z)|z|^{2n}}{n!n!}.$$ □

Lemma 2.13. *Let* V *be a random variable with values in* $[0,2\pi)$*, such that* e^{iV} *is independent from* Z *and no power of* e^{iV} *is equal to* 1 *almost everywhere.* Z *has v.u.m. if and only if* Z *and* Ze^{iV} *have the same moments.*

Proof. (⇒) If Z has v.u.m., by **Proposition** 2.6, so does Ze^{iV}. For the balanced moments, $\mathbb{E}[|Z|^n] = \mathbb{E}[\left|Ze^{iV}\right|^n]$.

(⇐) Suppose Z and Ze^{iV} have the same moments. Then in particular

$$\mathbb{E}[Z^n\bar{Z}^m] = \mathbb{E}[Z^n\bar{Z}^m]\mathbb{E}[(e^{iV})^{n-m}]$$

since Z and e^{iV} are independent. It follows from the assumption that for $n \neq m$, $\mathbb{E}[(e^{iV})^{n-m}] \neq 1$, and so $\mathbb{E}[Z^n\bar{Z}^m] = 0$. □

3 Random variables with a density

In this section we consider random variables whose moments are not necessarily finite, but which have a density. Some properties from **Theorem** 2.3, notably the "vanishing unbalanced moments" condition, no longer make sense. But we prove two new characterizations that are equivalent to rotational invariance in this setting, one in terms of polar decomposition, and the other in terms of entropy.

3.1 Prerequisites II

Definition 3.1. Let $Z = X + iY$ be a complex random variable. If the real-valued random variables X and Y have a joint density $f(x, y)$, we will also refer to it as the density of Z.

Definition 3.2. Let $Z = X + iY$ be a complex random variable.

- We may write the random variable Z in its polar form as $Z = |Z|\, e^{iU}$ where U is a random variable with values in $[0, 2\pi)$. This decomposition is unique on the set where $Z \neq 0$. For a random variable with a density, this is the case almost everywhere.

- The density of Z is rotationally invariant if for any θ, $e^{i\theta}Z$ has the same density as Z. equivalently,

$$f(x, y) = f(x\cos(\theta) - y\sin(\theta), x\sin(\theta) + y\cos(\theta)).$$

- If Z has a density $f(x, y)$, the entropy of Z is defined to be

$$H(Z) = \mathbb{E}[-\log f(X, Y)] = -\int_{\mathbb{R}^2} f(x, y)\log f(x, y)\, dx\, dy,$$

  where by convention $0\log 0 = 0$. Note that entropy of Z may be infinite.

For more information regarding Shannon Entropy, please refer to the tenth chapter of [3].

Definition 3.3. Random variables R and U with densities are independent if their joint density $h(r, u)$ decomposes as a product of their marginal densities. That is, $h(r, u) = h_R(r)h_U(u)$. If one simply sets $h_R(r) = f(r)$ and $h_U(u) = g(u)$, we can write $h(r, u) = f(r)g(u)$.

The following two results appear in many probability textbooks.

Proposition 3.4 (Jensen's inequality)**.** *Let $p(x)$ be a probability density and q a function. If ϕ is a convex function on an interval I, then*

$$\int_I \phi(q(x))\, p(x)\, dx \geq \phi\left(\int_I q(x)p(x)\, dx\right).$$

For a convex function ϕ which is not affine on any interval, the equality holds if and only if q is constant.

If the reader is interested in a thorough walk-through of Jensen's Inequality, please refer to [1].

Proposition 3.5 (Change of variables)**.** *Let A be a random variable, and $\alpha : \mathbb{R}^2 \to \mathbb{R}^2$ an invertible, differentiable function. If A has density f, then the random variable $\alpha^{-1}(A)$ has the density $(f \circ \alpha)\,|J_\alpha|$, where J_α is the Jacobian of α.*

For more information regarding the above proposition, refer to the fifth chapter of [3].

3.2 Main results II

Remark. Throughout this section, let $Z = X + iY$ be a complex-valued random variable with density $f(x, y)$ which is positive almost everywhere. Denote (R, U) the components in the polar decomposition $Z = Re^{iU}$, so that $X = R\cos U$ and $Y = R\sin U$. For $r \geq 0$ and $u \in [0, 2\pi)$, denote

$$g(r, u) = f(r\cos u, r\sin u),$$

and

$$g(r) = \frac{1}{2\pi}\int_0^{2\pi} g(r, u)\, du.$$

Corollary 3.6. *The joint density of* R *and* U *is* $h(r, u) = r\, g(r, u)$.

Proof. By assumption, almost surely R takes values in $(0, \infty)$. Denote $\alpha : (0, \infty) \times [0, 2\pi) \to \mathbb{R}^2 \setminus \{0\}$ the function defined by $\alpha(r, u) = (r\cos u, r\sin u)$. So, J_α is given by taking the determinant of the matrix $\begin{vmatrix} \cos u & -r\sin u \\ \sin u & r\cos u \end{vmatrix}$ which is of course r. The density of $f(x, y)$ may be turned into $f(r\cos u, r\sin u) = g(r, u)$, and the change of variables proposition implies

$$h(r, u) = (f(x, y) \circ \alpha)J_\alpha = r\, g(r, u).$$ □

The connection between the theorem below and the results in Section 2 of the article is provided by the following.

Proposition 3.7. *If the density of a complex random variable is determined by its moments, then it has rotationally invariant moments if and only if it has a rotationally invariant density. Thus under this assumption, all the conditions in* ***Theorem*** *2.3 and* ***Theorem*** *3.8 are equivalent.*

Proof. When the moments and the density of a random variable $Z = X + iY$ both exist, they are related by

$$\mathbb{E}[Z^n \bar{Z}^m] = \int_{\mathbb{R}^2} (x + iy)^n (x - iy)^m f(x, y)\, dx\, dy.$$

So if the densities of Z and $e^{i\theta}Z$ are the same, so are their moments. Conversely, if the moments of Z and $e^{i\theta}Z$ are the same, and they determine the densities, then the densities have to be the same as well. □

Theorem 3.8. *Let* $Z = Re^{iU}$ *be a complex random variable with a density. The following statements are equivalent.*

1. *The density of* Z *is rotationally invariant.*
2. $|Z|$ *and* U *are independent, and* U *is uniformly distributed.*

3. *Among all random variables with a fixed distribution of* $|Z|$, Z *has the largest entropy.*

Remark 3.9. The equivalence between (1) and (2) of **Theorem** 3.8 holds for general random variables which are non-zero almost surely. The proof is basically the same as the one below.

Proof. (1) ⇔ (2) Let $h(r,u)$ be the density of $Z = Re^{iU}$ and throughout let $\theta \in [0,2\pi)$. The density of Z is rotationally invariant if and only if $Z = Re^{iU}$ has the same density as $e^{i\theta}Z = Re^{i(U+\theta)}$. Using the Change of Variables proposition again, this states that Z is rotationally invariant if and only if $h(r,u) = h(r,u+\theta)$. In order for this to be the case, we must have $h(r,u) = \frac{h(r)}{2\pi}$. Since $h(r,u) = h(r)\ell(u)$ with $\ell(u) = 1/2\pi$, it follows that $R = |Z|$ and U are independent with U uniformly distributed. The converse is clear.

(2) ⇔ (3) The entropy of $Z = X + iY = |Z|\,e^{iU}$ is

$$-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\log f(x,y)\,dx\,dy = -\int_0^{\infty}\int_0^{2\pi} r\,g(r,u)\log g(r,u)\,du\,dr$$
$$= -\int_0^{\infty} 2\pi r\cdot\int_0^{2\pi}\frac{1}{2\pi}\phi(g(r,u))\,du\,dr,$$

where $\phi(a) = a\log a$. $\phi''(a) = 1/a$, so ϕ is a convex function. Therefore it satisfies Jensen's inequality that for every r and the uniform density $\frac{1}{2\pi}$ on $[0,2\pi)$:

$$\int_0^{2\pi}\phi(g(r,u))\frac{1}{2\pi}\,du \geq \phi\left(\int_0^{2\pi} g(r,u)\,\frac{1}{2\pi}\,du\right).$$

It follows that

$$\begin{aligned}
-\int_0^{\infty} 2\pi r\cdot\int_0^{2\pi}\frac{1}{2\pi}\phi(g(r,u))\,du\,dr &\leq -\int_0^{\infty} 2\pi r\cdot\phi\left(\int_0^{2\pi}\frac{1}{2\pi}g(r,u)\,du\right)dr\\
&= -\int_0^{\infty} 2\pi r\cdot\phi\big(g(r)\big)\,dr\\
&= -\int_0^{\infty} 2\pi r\,g(r)\log g(r)\,dr\\
&= -\int_0^{2\pi}\int_0^{\infty} r\,g(r,u)\log g(r)\,dr\,du\\
&= -\mathbb{E}[\log g(|Z|)].
\end{aligned}$$

Note that the right-hand side depends only on the distribution of $|Z|$.

The equality holds if and only if for almost every $r \neq 0$, $g(r,u)$ is a constant function of u, in other words if $g(r,u) = g(r)$ almost everywhere, in which case $|Z|$ and U are independent with U uniform. □

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