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Cover Page Footnote

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The Existence of Solutions to a System of Nonhomogeneous Difference Equations

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Abstract. In this article we use a fixed point theorem to determine the existence of multiple positive solutions for a type of system of nonhomogeneous even ordered boundary value problems on a discrete domain. We first reconstruct the problem by transforming the system so that it satisfies homogeneous boundary conditions. We then create a cone and an operator sufficient to apply the Guo-Krasnosel'skii Fixed Point Theorem. The majority of the work involves developing the constraints needed to utilize the theorem. The theorem is then applied three times, guaranteeing the existence of at least three distinct solutions. Thus, solutions to this class of boundary value problems exist and are not unique.

1 Introduction

When first beginning the study of boundary value problems, the student usually deals with standard differential equations such as the wave or heat equations. However, the study of the existence and uniqueness of solutions for boundary value problems is an amazingly rich field, branching into areas such as quantum mechanics and electrostatics. This paper contributes to this highly applicable and widely researched field.

In [8], Marcos, Lorca, and Ubilla determined the existence of multiple solutions for the fourth order differential equation $u^{(4)}(t) = \lambda h(t, u(t), u''(t))$ where $t \in (0,1)$. Their process involves a transformation technique that replaces the fourth order equation with a system of second order equations. This technique served as motivation for Henderson and Hopkins in [6], who utilized a fixed point theorem, namely the Guo-Krasnosel'skii Fixed Point Theorem, to show the multiplicity of solutions to the fourth order difference equation $\Delta^4 u(t-2) = \lambda h(t, u(t), \Delta^2 u(t-1))$ for $t \in (0, \mathrm{N}+2)_{\mathbb{Z}}$. We have adopted the process used by Henderson and Hopkins as a model for our work wherein we establish the existence of multiple positive solutions to a particular type of system of nonhomogeneous even ordered boundary value problems on a discrete domain. Other works utilizing similar processes to different problems include [1], [2], and [7].

While we will explicitly state the Guo-Krasnosel'skii Fixed Point Theorem in Section 2, we informally describe a version of it here, as its result is central to our work. Conceptually, the Krasnosel'skii Fixed Point Theorem begins with a closed convex non-empty

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subset C of a Banach space $(X, ||\cdot||)$. If we then have maps A and B from C into X which satisfy

(i) $Ax + By \in C$ for each $x, y \in C$,

(ii) A is continuous and AC is contained in some compact set,

(iii) B is a contraction with constant $\alpha < 1$,

then there exists some $z \in C$ such that $Az + Bz = z$. In other words, there is a fixed point for the map $A + B$. This version of the theorem is a bit more intuitive than the form we will present below where many of the above conditions will be replaced by the notion of a cone. The version of the theorem we employ multiple times in our work is also used for existence results on other types of problems. For instance, in [5] it is used to determine the existence of positive solutions under various conditions of nonlinearity in terms of parameter intervals, and in [4] it is used to determine the existence of positive solutions of nonlinear fractional differential equations with integral boundary value conditions.

In the following sections, we will determine the existence of at least three positive solutions to the system of second order discrete nonhomogeneous boundary value problems having the form:

$$-\Delta^2 u_n = \lambda f(t, u_1, \dots, u_n), \ t \in (0, N+2)_{\mathbb{Z}}, \tag{1}$$

$$-\Delta^2 u_k = g_k(t, u_1, \dots, u_n), \ k = 1, \dots, n-1, \ t \in (0, N+2)_{\mathbb{Z}}, \tag{2}$$

$$u_k(0) = 0, \ u_k(N+2) = a_k, \ k = 1, \dots, n, \tag{3}$$

where $a_1, \dots, a_n, \lambda \geq 0$, $\sum_{i=1}^{n} a_i > 0$ and $f, g_k : [0, N+2]_{\mathbb{Z}} \times [0, \infty)^n \to [0, \infty)$ for $k = 1, \dots, n-1$. Our process begins in Section 2 with a transformation of the system of second order equations (1)–(3) into a system that satisfies homogeneous boundary conditions. We then provide, given that solutions exist, the general form of the solutions to the homogeneous system. As previously mentioned, this is also where we introduce the Guo-Krasnosel'skii Fixed Point Theorem, construct a cone and operator that meet the criteria for the theorem, and provide other necessary preliminary information. In Section 3, we state and prove several lemmas that establish the inequalities necessary for the application of the Guo-Krasnosel'skii Fixed Point Theorem. Finally, in Section 4, we combine this information which allows us to apply the Guo-Krasnosel'skii Fixed Point Theorem three times to yield our main result which establishes the existence of multiple positive solutions.

2 Preliminaries

To begin, note that given any set $S \subseteq \mathbb{R}$, $S_{\mathbb{Z}}$ denotes the intersection of the set S with the set of integers; that is

$$S_{\mathbb{Z}} = S \cap \mathbb{Z}.$$

Recall our original system,

$$\begin{aligned}
&-\Delta^2 u_n = \lambda f(t, u_1, \dots, u_n),\ t \in (0, \mathrm{N}+2)_{\mathbb{Z}},\\
&-\Delta^2 u_k = g_k(t, u_1, \dots, u_n),\ t \in (0, \mathrm{N}+2)_{\mathbb{Z}},\\
&u_k(0) = 0,\ u_k(\mathrm{N}+2) = a_k,\ k = 1, \dots, n
\end{aligned}$$

where $a_1, \dots, a_n, \lambda \geq 0$, $\sum_{i=1}^{n} a_i > 0$ and $f, g_k : [0, \mathrm{N}+2]_{\mathbb{Z}} \times [0, \infty)^n \to [0, \infty)$ for $k = 1, \dots, n-1$. To achieve our main result, we need to place the following requirements on the functions f and g_k for $k = 1, \dots, n-1$:

(H0) $f, g_k : [0, \mathrm{N}+2]_{\mathbb{Z}} \times [0, \infty)^n \to [0, \infty)$ for $k = 1, \dots, n-1$ are continuous functions that are nondecreasing in the last n variables.

(H1) Suppose there is an $\alpha, \beta \in (0, \mathrm{N}+2)_{\mathbb{Z}}$, where $\alpha < \beta$, such that given $(x_1, \dots, x_n) \in [0, \infty)^n$, there is a $k > 0$ such that $f(t, x_1, \dots, x_n) > k$ for $t \in [\alpha, \beta]_{\mathbb{Z}}$.

(H2) Let $z = \sum_{i=1}^{n} x_i$. Then $\lim_{z \to 0^+} \dfrac{f(t, x_1, \dots, x_n)}{x_1 + \cdots + x_n} = 0$ uniformly for $t \in [0, \mathrm{N}+2]_{\mathbb{Z}}$.

(H3) Let $z = \sum_{i=1}^{n} x_i$. Then $\lim_{z \to \infty} \dfrac{f(t, x_1, \dots, x_n)}{x_1 + \cdots + x_n} = 0$ uniformly for $t \in [0, \mathrm{N}+2]_{\mathbb{Z}}$.

(H4) There exists $\gamma_k, q > 0$ where $\sum_{k=1}^{n-1} \gamma_k < \dfrac{8}{(\mathrm{N}+2)^2}$, such that, for

$(x_1, \dots, x_n) \in [0, \infty)^n$ with $\sum_{i=1}^{n} x_i < q$, we have

$g_k(t, x_1, \dots, x_n) \leq \gamma_k \cdot \sum_{i+1}^{n} x_i$ for $k = 1, \dots, n-1$ and $t \in [0, \mathrm{N}+2]_{\mathbb{Z}}$.

(H5) There exists both a $0 < \eta_k < \dfrac{8}{(\mathrm{N}+2)^2}$ and a $p_1 > 0$ such that, for

$(x_1, \dots, x_n) \in [0, \infty)^n$ with $\sum_{i=1}^{n} x_i > p_1$, we have

$g_k(t, x_1, \dots, x_n) \leq \eta_k \cdot \sum_{i=1}^{n} x_i$ for $k = 1, \dots, n-1$ and $t \in [0, \mathrm{N}+2]_{\mathbb{Z}}$.

In order for (1)–(3) to satisfy homogeneous boundary conditions, we apply the following transformation. For $t \in [0, \mathrm{N}+2]_{\mathbb{Z}}$ and $k = 1, \dots, n$, let $\overline{u_k}(t) = u_k(t) - \mathrm{A}_k t$ where $\mathrm{A}_k = \dfrac{a_k}{\mathrm{N}+2}$, which yields

$$-\Delta^2 u_n = \lambda f(t, \overline{u_1}(t) + \mathrm{A}_1 t, \dots, \overline{u_n}(t) + \mathrm{A}_n t),\ t \in (0, \mathrm{N}+2)_{\mathbb{Z}} \tag{4}$$

$$-\Delta^2 u_k = g_k(t, \overline{u_1}(t) + \mathrm{A}_1 t, \dots, \overline{u_n}(t) + \mathrm{A}_n t),\ k = 1, \dots, n-1,\ t \in (0, \mathrm{N}+2)_{\mathbb{Z}} \tag{5}$$

$$u_k(0) = 0,\ u_k(\mathrm{N}+2) = 0,\ k = 1, \dots, n. \tag{6}$$

If we can show (4)–(6) has solutions, then (1)–(3) has solutions as well. Now that we have a system subject to homogeneous boundary conditions (4)–(6), we know the solutions are of the form

$$u_n(t) = \lambda \sum_{s=1}^{N+1} G(t,s) f(s, \overline{u_1}(s) + A_1 s, \ldots, \overline{u_n}(s) + A_n s),$$
$$u_k(t) = \sum_{s=1}^{N+1} G(t,s) g_k(s, \overline{u_1}(s) + A_1 s, \ldots, \overline{u_n}(s) + A_n s),\ k = 1, \ldots, n-1$$

where $G(t,s)$ denotes the Green's function,

$$G(t,s) = \frac{1}{N+2} \begin{cases} t(N+2-s), & 0 \le t \le s \le N+1, \\ s(N+2-t), & 1 \le s \le t \le N+2. \end{cases}$$

Now that we know what solutions should look like, the question is whether they exist. To show this, we will need some more information. We will start by showing that the Green's function, defined above, has an upper bound. Notice $\sum_{s=1}^{N+1} G(t,s) = \frac{N+2}{2}t - \frac{1}{2}t^2$. Consider $\phi(x) = \frac{N+2}{2}x - \frac{1}{2}x^2$, where $x \in \mathbb{R}$. Then $\phi'(x) = \frac{N+2}{2} - x$, and $\phi'(x) = 0$ when $x = \frac{N+2}{2}$. Thus $x^* = \frac{N+2}{2}$ maximizes $\phi(x)$, since Q is quadratic and $Q'' < 0$. In fact,

$$\phi\left(\frac{N+2}{2}\right) = \left(\frac{N+2}{2}\right)^2 - \frac{1}{2}\left(\frac{N+2}{2}\right)^2 = \frac{(N+2)^2}{8}.$$

Let $\hat{t} = \left[\left[\frac{N+2}{2}\right]\right]$ where $[[\cdot]]$ denotes the greatest integer function. Therefore $\sum_{s=1}^{N+1} G(\hat{t},s) \le \frac{(N+2)^2}{8}$, which will be useful in the following section. Furthermore, $G(t,s)$ is clearly nonnegative, and for $k = 1, \ldots, n$, u_k must be positive since f and g_k are assumed to be nonnegative.

Set $Y = \{u(t) \mid u : [0, N+2]_{\mathbb{Z}} \to \mathbb{R}\}$ and let $(X, ||\cdot||)$ denote the Banach space $X = \prod_{i=1}^{n} Y$, endowed with the norm $||(u_1, \ldots, u_n)|| = \sum_{i=1}^{n} ||u_i||_\infty$, where $||u||_\infty = \max_{t \in [0,N+2]_{\mathbb{Z}}} |u(t)|$. Also, let Ω_r be the set $\Omega_r = \{(u_1, \ldots, u_n) \in X \mid ||(u_1, \ldots, u_n)|| < r\}$, which is open. Next define $C \subset X$ by

$$C = \{(u_1, \ldots, u_n) \in X \mid (u_1, \ldots, u_n)(0) = (u_1, \ldots, u_n)(N+2) = (0, \ldots, 0) \text{ and } u_i \text{ is concave for } i = 1, \ldots, n\}.$$

Note C is a cone as it is a nonempty, closed, convex subset of X that satisfies the following properties:

1. If $x \in \mathrm{C}$ and $\lambda > 0$, then $\lambda x \in \mathrm{C}$.

2. If $x, -x \in \mathrm{C}$, then $x = 0$.

Lastly, define $\mathrm{T} : \mathrm{X} \to \mathrm{X}$ as the operator $\mathrm{T}(u_1, \ldots, u_n) = (\mathrm{T}_1(u_1, \ldots, u_n), \ldots, \mathrm{T}_n(u_1, \ldots, u_n))$, where

$$\mathrm{T}_n = \lambda \sum_{s=1}^{\mathrm{N}+1} \mathrm{G}(t,s)\, f(s, \overline{u_1}(s) + \mathrm{A}_1 s, \ldots, \overline{u_n}(s) + \mathrm{A}_n s),$$

$$\mathrm{T}_k = \sum_{s=1}^{\mathrm{N}+1} \mathrm{G}(t,s)\, g_k(s, \overline{u_1}(s) + \mathrm{A}_1 s, \ldots, \overline{u_n}(s) + \mathrm{A}_n s),$$

for $i = 1, \ldots, n-1$. Thus, T satisfies the following lemma.

Lemma 2.1. *T is a completely continuous cone-preserving operator.*

A standard Arzela-Ascoli argument can be used to show that T is completely continuous, and the proof that T is cone-preserving is straight forward.

Finally, the following theorem will be used to obtain our main result:

Theorem 2.2 (Guo-Krasnosel'skii Fixed Point Theorem)**.** *Let* $(\mathrm{X}, ||\cdot||)$ *be a Banach space and* $\mathrm{C} \subset \mathrm{X}$ *be a cone. Suppose* Ω_1 *and* Ω_2 *are open subsets of* X *satisfying* $0 \in \Omega_1 \subset \overline{\Omega_1} \subset \Omega_2$. *If* $\mathrm{T} : \mathrm{C} \cap (\overline{\Omega_2} \setminus \Omega_1) \to \mathrm{C}$ *is a completely continuous operator such that either*

1. $||\mathrm{T}u|| \leq ||u||$ *for* $u \in \mathrm{C} \cap \partial\Omega_1$ *and* $||\mathrm{T}u|| \geq ||u||$ *for* $u \in \mathrm{C} \cap \partial\Omega_2$*, or*

2. $||\mathrm{T}u|| \geq ||u||$ *for* $u \in \mathrm{C} \cap \partial\Omega_1$ *and* $||\mathrm{T}u|| \leq ||u||$ *for* $u \in \mathrm{C} \cap \partial\Omega_2$*,*

then T *has a fixed point in* $\mathrm{C} \cap (\overline{\Omega_2} \setminus \Omega_1)$.

3 Technical Results

In this section, we present four lemmas that, when combined, show T satisfies the required inequalities of the Guo-Kasnosel'skii Fixed Point Theorem a total of three times.

Lemma 3.1. *Suppose* (H0) *and* (H1) *hold and let* $\rho^* > 0$. *Then there exists a* $\Lambda > 0$ *such that, for every* $\lambda \geq \Lambda$ *and* $(a_1, \ldots, a_n) \in [0, \infty)^n$,

$$||\mathrm{T}(u_1, \ldots, u_n)|| \geq ||(u_1, \ldots, u_n)||,$$

where $(u_1, \ldots, u_n) \in \mathrm{C} \cap \partial\Omega_{\rho^*}$.

Proof. Let $\rho^* > 0$ and take $(u_1,\dots,u_n) \in C \cap \partial\Omega_{\rho^*}$. Let $\alpha,\beta \in (0,N+2)_{\mathbb{Z}}$ with $\alpha < \beta$ and set

$$r_i = \frac{\min\limits_{t\in[\alpha,\beta]_{\mathbb{Z}}} u_i(t)}{||u_i||_\infty},$$

for $i = 1,\dots,n$. Also, let $r = \min\{r_i \mid i = 1,\dots,n\}$. Note that, for $i = 1,\dots,n$, $\min\limits_{t\in[\alpha,\beta]_{\mathbb{Z}}} u_i(t)$ exists and $\min\limits_{t\in[\alpha,\beta]_{\mathbb{Z}}} u_i(t), ||u_i||_\infty > 0$ as u_i is concave, $u_i(0) = u_i(N+2) = 0$, and $\alpha,\beta \in (0,N+2)_{\mathbb{Z}}$. This gives that r_i exists and $r_i > 0$ for $i = 1,\dots,n$. Thus r exists, since i is finite, and $r > 0$. Recall $\max\limits_{t\in(0,N+2)_{\mathbb{Z}}} \sum_{s=1}^{N+1} G(t,s) \geq \sum_{s=1}^{N+1} G(\hat{t},s) \geq \sum_{s=1}^{N+1} G(t,s)$ for all $t \in (0,N+2)_{\mathbb{Z}}$. Let

$$M = \inf\left\{\frac{f(t,rz_1,\dots,rz_n)}{r\cdot(z_1+\dots+z_n)} : t \in [\alpha,\beta]_{\mathbb{Z}},\, (z_1,\dots,z_n) \in (0,\infty)^n,\, \sum_{i=1}^{n} z_i = \rho^*\right\}$$

and

$$\Lambda = \left[rM\sum_{s=\alpha}^{\beta} G\left(\hat{t},s\right)\right]^{-1}.$$

Under (H1), since $t \in [\alpha,\beta]_{\mathbb{Z}}$, $(z_1,\dots,z_n) \in (0,\infty)^n$, $\sum_{i=1}^{n} z_i = \rho^* \geq 0$ and $r > 0$, we know $\exists k > 0$ such that

$$f(t,rz_1,\dots,rz_n) > k > 0.$$

Furthermore, given $\sum_{i=1}^{n} z_i = \rho^*$, notice

$$r\cdot(z_1+\dots+z_n) = r\cdot\rho^* > 0$$

as r, $\rho^* > 0$. Since $f(t,rz_1,\dots,rz_n) > k > 0$, and $r\cdot(z_1+\dots+z_n) > 0$, $M > 0$. Recall

$$G(t,s) = \frac{1}{N+2}\begin{cases} t(N+2-s), & 0 \leq t \leq s \leq N+1, \\ s(N+2-t), & 1 \leq s \leq t \leq N+2. \end{cases}$$

Note $G(t,s) > 0$ for any $t,s \in (0,N+2)_{\mathbb{Z}}$, so $\sum_{s=\alpha}^{\beta} G(t,s) > 0$ for any $t \in (0,N+2)_{\mathbb{Z}}$. Now, since $r, M, \sum_{s=\alpha}^{\beta} G(\hat{t},s) > 0$, $\Lambda > 0$. Let $\lambda \geq \Lambda$. Then

$$
\begin{aligned}
||T(u_1,\ldots,u_n)|| &= ||T_1(u_1,\ldots,u_n)||_\infty + \ldots + ||T_n(u_1,\ldots,u_n)||_\infty \\
&\geq \sup_{t\in[0,N+2]_{\mathbb{Z}}} |T_n(u_1,\ldots,u_n)(t)| \\
&= \sup_{t\in[0,N+2]_{\mathbb{Z}}} \left|\lambda \sum_{s=1}^{N+1} G(t,s)\, f(s,u_1(s)+A_1 s,\ldots,u_n(s)+A_n s)\right| \\
&\geq \left|\lambda \sum_{s=1}^{N+1} G(\hat{t},s)\, f(s,u_1(s)+A_1 s,\ldots,u_n(s)+A_n s)\right| \\
&\geq \lambda \sum_{s=\alpha}^{\beta} G(\hat{t},s)\, f(s,u_1(s)+A_1 s,\ldots,u_n(s)+A_n s) \\
&\geq \lambda \sum_{s=\alpha}^{\beta} G(\hat{t},s)\, f(s,r||u_1||_\infty,\ldots,r||u_n||_\infty) \\
&= \lambda\, r\rho^* \sum_{s=\alpha}^{\beta} G(\hat{t},s)\, \frac{f(s,r||u_1||_\infty,\ldots,r||u_n||_\infty)}{r\cdot\rho^*} \\
&\geq \lambda\, r||(u_1,\ldots,u_n)|| \sum_{s=\alpha}^{\beta} G(\hat{t},s)\cdot M \\
&\geq \Lambda\, rM||(u_1,\ldots,u_n)|| \sum_{s=\alpha}^{\beta} G(\hat{t},s) \\
&= \left[rM \sum_{s=\alpha}^{\beta} G\left(\hat{t},s\right)\right]^{-1} rM||(u_1,\ldots,u_n)|| \sum_{s=\alpha}^{\beta} G(\hat{t},s) \\
&= ||(u_1,\ldots,u_n)||.
\end{aligned}
$$

□

Lemma 3.2. *Fix* $\Lambda > 0$ *and suppose* (H0) *and* (H1) *hold. Then,* $\forall \lambda \geq \Lambda$ *and* $(a_1,\ldots,a_n) \in [0,\infty)^n$ *with* $\sum_{i=1}^{n} a_i > 0$*, there is a* $\rho_1 = \rho_1(\Lambda, a_1,\ldots,a_n)$ *such that,* $\forall \rho \leq \rho_1$*, we have*

$$
||T(u_1,\ldots,u_n)|| \geq ||(u_1,\ldots,u_n)||,
$$

for $(u_1,\ldots,u_n) \in C \cap \partial\Omega_\rho$.

Proof. Suppose (H0) and (H1) hold, $\Lambda > 0$, and $(u_1,\ldots,u_n) \in C$. Let $(a_1,\ldots,a_n) \in [0,\infty)^n$ with $\sum_{i=1}^{n} a_i > 0$. Given $\overline{\alpha} > 0$, $\left(\frac{\overline{\alpha}}{N+2}\, a_1,\ldots,\frac{\overline{\alpha}}{N+2}\, a_n\right) = (\overline{\alpha}A_1,\ldots,\overline{\alpha}A_n) \in [0,\infty)^n$, since $(a_1,\ldots,a_n) \in$

$[0,\infty)^n$. Then by (H1), $\exists\, k>0$ such that

$$f(t,\alpha A_1,\ldots,\alpha A_n)>k$$

for $t\in[\alpha,\beta]_{\mathbb{Z}}$ where $\alpha,\beta\in(0,N+2)_{\mathbb{Z}}$. Let $t\in[\alpha,\beta]$. Under (H0), since f is nondecreasing,

$$\begin{aligned} f(t,u_1(t)+A_1t,\ldots,u_n(t)+A_nt) &\geq f(t,A_1t,\ldots,A_nt)\\ &\geq f(t,\alpha A_1,\ldots,\alpha A_n). \end{aligned}$$

That is $f(t,\overline{u_1}(t)+A_1t,\ldots,\overline{u_n}(t)+A_nt)>k$. Take $\rho_1=\Lambda k\sum_{s=\alpha}^{\beta}G(\hat{t},s)$. Note $\rho_1>0$ as $\Lambda,k>0$ and $G(t,s)>0$ for all $t,s\in(0,N+2)_{\mathbb{Z}}$, so $\sum_{s=\alpha}^{\beta}G(\hat{t},s)>0$. Let $\rho\leq\rho_1$ and $\lambda\geq\Lambda$. Then $\forall(u_1,\ldots,u_n)\in C\cap\Omega_\rho$,

$$\begin{aligned} ||T(u_1,\ldots,u_n)|| &\geq \lambda\sum_{s=\alpha}^{\beta}G(\hat{t},s)f(s,u_1(s)+A_1s,\ldots,u_n(s)+A_ns)\\ &\geq \lambda\sum_{s=\alpha}^{\beta}G(\hat{t},s)f(s,\alpha A_1,\ldots,\alpha A_n)\\ &\geq \Lambda k\sum_{s=\alpha}^{\beta}G(\hat{t},s)\\ &\geq \Lambda k\frac{\rho}{\rho_1}\sum_{s=\alpha}^{\beta}G(\hat{t},s)\\ &> \Lambda k\frac{||(u_1,\ldots,u_n)||}{\rho_1}\sum_{s=\alpha}^{\beta}G(\hat{t},s)\\ &= ||(u_1,\ldots,u_n)||. \end{aligned}$$

□

Lemma 3.3. *Suppose* (H0), (H2), (H4) *hold and fix* $\rho^*>0$. *Then, given* $\lambda>0$, *there is a* $\rho_2\in(0,\rho^*)$ *and a* $\delta>0$, *such that for every* $(a_1,\ldots,a_n)\in[0,\infty)^n$, *with* $0<\sum_{i=1}^{n}a_i<\delta$,

$$||T(u_1,\ldots,u_n)||\leq||(u_1,\ldots,u_n)||,$$

for $(u_1,\ldots,u_n)\in C\cap\partial\Omega_{\rho_2}$.

Proof. Suppose (H0), (H2), (H4) hold and fix $\rho^* > 0$. Given $\lambda > 0$, pick $\epsilon > 0$ so that $\lambda\epsilon < \frac{4}{(N+2)^2}$. Then there is a $\rho_2 \in (0, \rho^*)$ such that, for $\sum_{i=1}^{n} x_i = \rho_2$ with $(x_1, \ldots, x_n) \in [0, \infty)^n$ and $\sum_{i=1}^{n} R_i \leq \rho_2$, we have that

$$\frac{f(t, x_1 + R_1, \ldots, x_n + R_n)}{(x_1 + R_1) + \ldots + (x_n + R_n)} < \epsilon$$

as $\frac{f(t, x_1 + R_1, \ldots, x_n + R_n)}{(x_1 + R_1) + \ldots + (x_n + R_n)}$ converges uniformly by (H2) for $t \in [0, N+2]_{\mathbb{Z}}$. It follows that

$$f(t, x_1 + R_1, \ldots, x_n + R_n) < \epsilon \cdot \sum_{i+1}^{n} (x_i + R_i)$$

for $t \in [0, N+2]_{\mathbb{Z}}$. Take $(u_1, \ldots, u_n) \in C \cap \partial\Omega_{\rho_2}$ and suppose $\sum_{i=1}^{n} a_i < \rho_2$. Then, for $t \in [0, N+2]_{\mathbb{Z}}$,

$$\begin{aligned}
T_n(u_1, \ldots, u_n)(t) &= \lambda \sum_{s=1}^{N+1} G(t,s) f(s, u_1(s) + A_1 s, \ldots, u_n(s) + A_n s) \\
&\leq \lambda \sum_{s=1}^{N+1} G(t,s) f(s, ||u_1||_\infty + a_1, \ldots, ||u_n||_\infty + a_n) \\
&< \lambda \sum_{s=1}^{N+1} G(t,s) \cdot \epsilon \cdot \sum_{i=1}^{n} (||u_i||_\infty + a_i) \\
&= \lambda\epsilon \left[||(u_1, \ldots, u_n)|| + \sum_{i=1}^{n} a_i \right] \sum_{s=1}^{N+1} G(t,s) \\
&< \lambda\epsilon \left[||(u_1, \ldots, u_n)|| + \rho_2 \right] \sum_{s=1}^{N+1} G(t,s) \\
&= 2\lambda\epsilon ||(u_1, \ldots, u_n)|| \sum_{s=1}^{N+1} G(t,s) \\
&\leq 2\lambda\epsilon ||(u_1, \ldots, u_n)|| \frac{(N+2)^2}{8} \\
&= \frac{\lambda\epsilon (N+2)^2}{4} ||(u_1, \ldots, u_n)||.
\end{aligned}$$

Thus,

$$||T_n(u_1, \ldots, u_n)||_\infty = \sup_{t \in [0, N+2]_{\mathbb{Z}}} |T_n(u_1, \ldots, u_n)| \leq \frac{\lambda\epsilon (N+2)^2}{4} ||(u_1, \ldots, u_n)||.$$

Now consider the remaining T_k's. By (H0), the g_k's are nondecreasing in the last n variables. Furthermore, by (H4), there is a γ_k for $k = 1, \ldots, n-1$ such that

$$0 < \sum_{k=1}^{n-1} \gamma_k < \frac{8}{(N+2)^2}$$

and a q such that, for $(x_1 + R_1, \ldots, x_n + R_n) \in [0, \infty)^n$ with $\sum_{i=1}^{n} (x_i + R_i) < q$, we have

$$g_k(t, x_1 + R_1, \ldots, x_n + R_n) \leq \gamma_k \cdot \sum_{i=1}^{n} (x_i + R_i),$$

for $k = 1, \ldots, n-1$ and $t \in [0, N+2]_{\mathbb{Z}}$. Pick ρ_2 such that $\rho_2 < \frac{q}{2}$. Notice

$$\sum_{i=1}^{n} (||u_i||_\infty + a_i) = ||(u_1, \ldots, u_n)|| + (a_1 + \cdots + a_n) < 2\rho_2 = q.$$

for $t \in [0, N+2]_{\mathbb{Z}}$. Let $\delta' < 1$ and set $\delta = \delta' \rho_2$. Take $(u_1, \ldots, u_n) \in C \cap \partial\Omega_{\rho_2}$ and suppose $\sum_{i=1}^{n} a_i < \delta$. Then, for $k = 1, \ldots, n-1$ and $t \in [0, N+2]_{\mathbb{Z}}$, we have

$$
\begin{aligned}
T_k(u_1,\dots,u_n)(t) &= \sum_{s=1}^{N+1} G(t,s)\, g_k(s, u_1(s)+A_1 s,\dots,u_n(s)+A_n s) \\
&\le \sum_{s=1}^{N+1} G(t,s)\, g_k(s, \|u_1\|_\infty + a_1,\dots,\|u_n\|_\infty + a_n) \\
&\le \sum_{s=1}^{N+1} G(t,s)\, \gamma_k \cdot \sum_{i=1}^{n} (\|u_i\|_\infty + a_i) \\
&= \gamma_k \left[\|(u_1,\dots,u_n)\| + \sum_{i=1}^{n} a_i \right] \sum_{s=1}^{N+1} G(t,s) \\
&< \gamma_k \left[\|(u_1,\dots,u_n)\| + \delta \right] \sum_{s=1}^{N+1} G(t,s) \\
&= \gamma_k \left[\|(u_1,\dots,u_n)\| + \delta' \rho_2 \right] \sum_{s=1}^{N+1} G(t,s) \\
&= \gamma_k \left[\|(u_1,\dots,u_n)\| + \delta' \|(u_1,\dots,u_n)\| \right] \sum_{s=1}^{N+1} G(t,s) \\
&= \gamma_k \left(1+\delta'\right) \|(u_1,\dots,u_n)\| \sum_{s=1}^{N+1} G(t,s) \\
&\le \frac{\gamma_k \left(1+\delta'\right)(N+2)^2}{8} \|(u_1,\dots,u_n)\|.
\end{aligned}
$$

Thus, for $k = 1,\dots,n-1$,

$$
\|T_k(u_1,\dots,u_n)\|_\infty = \sup_{t\in[0,N+2]_{\mathbb{Z}}} T_k(u_1,\dots,u_n) \le \frac{\gamma_k \left(1+\delta'\right)(N+2)^2}{8} \|(u_1,\dots,u_n)\|.
$$

So, taking $\sum_{i=1}^{n} a_i < \delta$ and $(u_1,\dots,u_n) \in C \cap \partial\Omega_{\rho_2}$ gives

$$
\begin{aligned}
\|T(u_1,\dots,u_n)\| &= \|T_1(u_1,\dots,u_n)\|_\infty + \cdots + \|T_n(u_1,\dots,u_n)\|_\infty \\
&\le \frac{\gamma_1 \left(1+\delta'\right)(N+2)^2}{8} \|(u_1,\dots,u_n)\| + \cdots + \frac{\gamma_{n-1} \left(1+\delta'\right)(N+2)^2}{8} \|(u_1,\dots,u_n)\| \\
&\quad + \frac{\lambda\epsilon (N+2)^2}{4} \|(u_1,\dots,u_n)\| \\
&= \left(\frac{\gamma_1 \left(1+\delta'\right)(N+2)^2}{8} + \cdots + \frac{\gamma_{n-1} \left(1+\delta'\right)(N+2)^2}{8} + \frac{\lambda\epsilon (N+2)^2}{4} \right) \|(u_1,\dots,u_n)\| \\
&= \frac{(N+2)^2}{8} \left[\left(1+\delta'\right) \sum_{k=1}^{n-1} \gamma_k + 2\lambda\epsilon \right] \|(u_1,\dots,u_n)\|.
\end{aligned}
$$

Thus, choosing δ' and ϵ small enough so that $\left(1+\delta'\right)\sum_{k=1}^{n-1}\gamma_k + 2\lambda\epsilon \leq \frac{8}{(\mathrm{N}+2)^2}$, we will have

$$\begin{aligned}
||\mathrm{T}(u_1,\ldots,u_n)|| &\leq \frac{(\mathrm{N}+2)^2}{8}\left[\left(1+\delta'\right)\sum_{k=1}^{n-1}\gamma_k + 2\lambda\epsilon\right]||(u_1,\ldots,u_n)|| \\
&= ||(u_1,\ldots,u_n)||.
\end{aligned}$$

□

Lemma 3.4. *Suppose* (H0), (H3), (H5) *hold and let* $(a_1,\ldots,a_n)\in[0,\infty)^n$ *satisfy* $0<\sum_{i=1}^{n}a_i<\delta$*, where* $\delta>0$ *is given. Then, for every* $\lambda>0$*, there is a* $\rho_3=\rho_3(\delta,\lambda)$ *such that* $\forall\rho\geq\rho_3$,

$$||\mathrm{T}(u_1,\ldots,u_n)|| \leq ||(u_1,\ldots,u_n)||,$$

where $(u_1,\ldots,u_n)\in\mathrm{C}\cap\partial\Omega_\rho$.

Proof. Suppose (H0), (H3), (H5) hold and let $(a_1,\ldots,a_n)\in[0,\infty)^n$ satisfy $0<\sum_{i=1}^{n}a_i<\delta$, where $\delta>0$ is given. Let $\lambda>0$. By (H0), g_k is nondecreasing in the last n variables for $k=1,\ldots,n$, and by (H5), there are

$$0<\eta_k<\frac{8}{(\mathrm{N}+2)^2}$$

for $k=1,\ldots,n-1$ and a $p_1>0$ such that for $(x_1+\mathrm{R}_1,\ldots,x_n+\mathrm{R}_n)\in[0,\infty)^n$ with $\sum_{i=1}^{n}(x_i+\mathrm{R}_i)>p_1$, we have

$$g_k(t,x_1+\mathrm{R}_1,\ldots,x_n+\mathrm{R}_n)\leq\eta_k\cdot\sum_{i=1}^{n}(x_i+\mathrm{R}_i),$$

for $t\in[0,\mathrm{N}+2]_{\mathbb{Z}}$. Let $\eta=\max\{\eta_k\mid k=1,\ldots,n-1\}$. Let $\epsilon>0$ and choose q_1 large enough so that $q_1 + \sum_{i=1}^{n} \mathrm{R}_i > p$ and $\epsilon > \frac{\eta\delta}{q_1}$, which gives $\eta < \frac{q_1\epsilon}{\delta}$. Then

$$\begin{aligned} g_k(t, x_1 + a_1, \dots, x_n + a_n) &\le \eta_k \cdot \sum_{i=1}^{n} (x_i + a_i) \\ &\le \eta \cdot \sum_{i=1}^{n} (x_i + a_i) \\ &< \eta \cdot \sum_{i=1}^{n} x_i + \frac{q_1\epsilon}{\delta} \cdot \sum_{i=1}^{n} a_i \\ &< \eta \cdot \sum_{i=1}^{n} x_i + \frac{q_1\epsilon}{\delta} \cdot \delta \\ &= \eta \cdot \sum_{i=1}^{n} x_i + q_1\epsilon \\ &\le \eta \cdot \sum_{i=1}^{n} x_i + \epsilon \cdot \sum_{i=1}^{n} x_i \\ &= (\eta + \epsilon) \sum_{i=1}^{n} x_i, \end{aligned}$$

for $t \in [0, \mathrm{N}+2]_{\mathbb{Z}}$. Therefore, for any $(u_1, \dots, u_n) \in \mathrm{C} \cap \partial\Omega_{q_1}$,

$$\begin{aligned} \mathrm{T}_k(u_1, \dots, u_n)(t) &= \sum_{s=1}^{\mathrm{N}+1} \mathrm{G}(t,s)\, g_k(s, u_1(s) + \mathrm{A}_1 s, \dots, u_n(s) + \mathrm{A}_n s) \\ &\le \sum_{s=1}^{\mathrm{N}+1} \mathrm{G}(t,s)\, g_k(s, ||u_1||_\infty + a_1, \dots, ||u_n||_\infty + a_n) \\ &< \sum_{s=1}^{\mathrm{N}+1} \mathrm{G}(t,s)\, (\eta + \epsilon) \cdot (||u_1||_\infty + \dots + ||u_n||_\infty) \\ &= (\eta + \epsilon) ||(u_1, \dots, u_n)|| \sum_{s=1}^{\mathrm{N}+1} \mathrm{G}(t,s) \\ &\le \frac{(\eta + \epsilon)(\mathrm{N}+2)^2}{8} ||(u_1, \dots, u_n)||, \end{aligned}$$

for $t \in 0, \mathrm{N}+2]_{\mathbb{Z}}$ and $k = 1, \dots, n-1$. Thus, for $k = 1, \dots, n-1$,

$$||\mathrm{T}_k(u_1, \dots, u_n)||_\infty \le \frac{(\eta + \epsilon)(\mathrm{N}+2)^2}{8} ||(u_1, \dots, u_n)||.$$

Next, consider $\mathrm{T}_n(u_1, \dots, u_n)$. Let $\delta' > 0$. Then, by (H3) there is a $p_2, q_2 > 0$ such that, for $(x_1+R_1,\ldots,x_n+R_n)\in[0,\infty)^n$ with $\sum_{i=1}^{n}(x_i+R_i)\ge q_2+\sum_{i=1}^{n}R_i\ge p_2$, we have that

$$\frac{f(t,x_1+R_1,\ldots,x_n+R_n)}{(x_1+R_1)+\ldots+(x_n+R_n)}<\delta'$$

as $\dfrac{f(t,x_1+R_1,\ldots,x_n+R_n)}{(x_1+R_1)+\ldots+(x_n+R_n)}$ converges uniformly by (H3) for $t\in[0,N+2]_{\mathbb{Z}}$. It follows that

$$f(t,x_1+R_1,\ldots,x_n+R_n)<\delta'\cdot\sum_{i=1}^{n}(x_i+R_i)$$

for $t\in[0,N+2]_{\mathbb{Z}}$. Let $q_3=\max\{q_2,\delta\}$. Then, for any $(x_1,\ldots,x_n)\in[0,\infty)^n$ with $\sum_{i=1}^{n}x_i\ge q_3$,

$$\begin{aligned}
f(t,x_1+R_1,\ldots,x_n+R_n) &< \delta'\cdot\sum_{i=1}^{n}(x_i+R_i)\\
&< \delta'\cdot\sum_{i=1}^{n}x_i+\delta'\cdot\delta\\
&\le \delta'\cdot\sum_{i=1}^{n}x_i+\delta'\cdot q_3\\
&= 2\delta'\sum_{i=1}^{n}x_i,
\end{aligned}$$

for $t\in[0,N+2]_{\mathbb{Z}}$. It follows that

$$\begin{aligned}
T_n(u_1,\ldots,u_n)(t) &= \lambda\sum_{s=1}^{N+1}G(t,s)\,f(s,u_1(s)+A_1s,\ldots,u_n(s)+A_ns)\\
&\le \lambda\sum_{s=1}^{N+1}G(t,s)\,f(s,||u_1||_\infty+a_1,\ldots,||u_n||_\infty+a_n)\\
&< \lambda\sum_{s=1}^{N+1}G(t,s)\cdot 2\delta'(||u_1||_\infty+\ldots+||u_n||_\infty)\\
&= 2\delta'\lambda||(u_1,\ldots,u_n)||\sum_{s=1}^{N+1}G(t,s)\\
&\le \frac{\delta'\lambda(N+2)^2}{4}||(u_1,\ldots,u_n)||,
\end{aligned}$$

for $t\in[0,N+2]_{\mathbb{Z}}$, giving

$$||T_n(u_1,\ldots,u_n)||_\infty\le\frac{\delta'\lambda(N+2)^2}{4}||(u_1,\ldots,u_n)||.$$

Therefore,

$$\begin{aligned}||T(u_1,\dots,u_n)|| &\le \frac{(n-1)(\eta+\epsilon)(N+2)^2}{8}||(u_1,\dots,u_n)|| + \frac{\delta'\lambda(N+2)^2}{4}||(u_1,\dots,u_n)|| \\ &= \frac{((n-1)(\eta+\epsilon)+2\delta'\lambda)(N+2)^2}{8}||(u_1,\dots,u_n)||.\end{aligned}$$

Thus, choosing δ' and ϵ small enough so that $(n-1)\epsilon + 2\delta'\lambda \le \frac{8}{(N+2)^2} - (n-1)\eta$, we will have

$$\begin{aligned}||T(u_1,\dots,u_n)|| &\le \frac{((n-1)(\eta+\epsilon)+2\delta'\lambda)(N+2)^2}{8}||(u_1,\dots,u_n)|| \\ &\le \frac{(N+2)^2}{8}\left[(n-1)\eta + \frac{8}{(N+2)^2} - (n-1)\eta\right]||(u_1,\dots,u_n)|| \\ &= ||(u_1,\dots,u_n)||.\end{aligned}$$

□

4 Main Result

Theorem 4.1. *Let f and g_k satisfy* (H0)–(H5) *for $k = 1,\dots,n-1$. Then, there exists a $\Lambda > 0$ such that, given any $\lambda \ge \Lambda$, there is a $\delta > 0$ such that, for every $a_1,\dots,a_n \ge 0$ satisfying $0 < \sum_{i=1}^{n} a_i < \delta$, the system* (4)–(6) *has at least three positive solutions.*

Proof. Suppose f and $g_1,\dots g_{n-1}$ satisfy (H0)–(H5) . Fix $\rho^* > 0$. By **lemma 3.1**, there is a $\Lambda > 0$ such that for every $\lambda \ge \Lambda$ and $a_1,\dots,a_n \ge 0$,

$$\|T(u_1,\dots,u_n)\| \ge \|(u_1,\dots,u_n)\|, \text{ for } (u_1,\dots,u_n) \in C \cap \partial\Omega_{\rho^*}.$$

Now, fix $\lambda \ge \Lambda$. **Lemma 3.2**, **lemma 3.3**, and **lemma 3.4** give that there is a $\delta > 0$ and $\rho_1, \rho_2, \rho_3 > 0$, with $\rho_1 < \rho_2 < \rho^* < \rho_3$, such that for $(a_1,\dots,a_n) \in [0,\infty)^n$, satisfying $0 < \sum_{i=1}^{n} a_i < \delta$, we have

$$\begin{aligned}\|T(u_1,\dots,u_n)\| &\ge \|(u_1,\dots,u_n)\|, \text{ for } (u_1,\dots,u_n) \in C \cap \partial\Omega_{\rho_1}, \\ \|T(u_1,\dots,u_n)\| &\le \|(u_1,\dots,u_n)\|, \text{ for } (u_1,\dots,u_n) \in C \cap \partial\Omega_{\rho_2}, \\ \|T(u_1,\dots,u_n)\| &\le \|(u_1,\dots,u_n)\|, \text{ for } (u_1,\dots,u_n) \in C \cap \partial\Omega_{\rho_3}.\end{aligned}$$

Therefore, by appealing to the Guo-Krasnosel'skii Fixed Point Theorem, there exist three positive solutions, $(x_1,\dots,x_n), (y_1,\dots,y_n), (z_1,\dots,z_n) \in C$ of (4)–(6) such that,

$$\rho_1 < \|(x_1,\dots,x_n)\| < \rho_2 < \|(y_1,\dots,y_n)\| < \rho^* < \|(z_1,\dots,z_n)\| < \rho_3.$$

□

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