Rose-Hulman Undergraduate Mathematics Journal

Volume 23
Issue 2

Article 2

On Cantor Sets Defined by Generalized Continued Fractions

Danielle Hedvig
College of the Canyons, danihedvig@gmail.com

Masha Gorodetski
masha.gorodetski.2005@gmail.com

Cover Page Footnote

We are grateful to Professor Vladimir Baranovsky, the University of California Irvine Department of Mathematics, for his mentorship. We also like to thank Sasha Volokh, a Ph.D. Student at USC Viterbi Department of Computer Science, for helping us optimize our Python computer programs.

ℝose-Hulman Undergraduate Mathematics Journal
Volume 23, Issue 2, 2022

On Cantor Sets Defined by Generalized Continued Fractions

By *Masha Gorodetski* and *Danielle Hedvig*

Abstract. We study a special class of generalized continued fractions, both in real and complex settings, and show that, in many cases, the set of numbers that can be represented by a continued fraction for that class forms a Cantor set. Also, we notice that in some regimes those sets form a peculiar fractal and formulate some questions and conjectures on its properties.

1 Introduction

Any irrational number ξ can be represented as an infinite continued fraction, i.e. in the form

$$\xi = a_0 + \cfrac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \ldots}}} \tag{1}$$

where $a_0 \in \mathbb{Z}$ and $a_i \in \mathbb{N}$ for all $i \geq 1$, and any rational number can be represented as a similar but finite continued fraction.

In the literature, one can find the notation $\xi = [a_0; a_1, a_2, a_3, \ldots]$ or Gauss notation

$$\xi = a_0 + \mathbf{K}_{i=1}^{\infty} \frac{1}{a_i},$$

but here we will use an explicit form (1).

Continued fractions are useful to study approximations of irrational numbers by rational approximations (so called Diophantine approximations, see [8] and references there) to understand rotations of the circle [10] and in number theory, as a tool to prove irrationality of some explicit numbers (e.g. irrationality of π, see [7]). For an introductory text in continued fractions, see [6].

What is the set of numbers that can be represented as a continued fraction of some specific form? For example, it was shown by Euler that a continued fraction with a periodic sequence of coefficients must be a quadratic irrational (i.e. a root of a quadratic equation with integer coefficients), and Lagrange proved the converse of Euler's theorem:

Mathematics Subject Classification. 1A55,37F99
Keywords. Cantor Sets, Fractals, Continued Fractions

for any quadratic irrational the continued fraction expansion must have eventually periodic (i.e. must have a periodic "tail").

As another example, continued fractions of "bounded" type were considered in [1], see also [4] for the motivation. Namely, one can consider a set of numbers on the interval $[0,1]$ that can be represented by a continued fraction with coefficients no greater than some given $n \geq 2$. It is known that this set must be a Cantor set for each fixed $n \geq 2$. Moreover, for any finite set of coefficients $\mathrm{M} \subset \mathbb{N}$ let us denote by $\mathrm{C_M}$ the set of points on $[0,1]$ that can be represented as a continued fraction with coefficients from M. For example, if $\mathrm{M} = \{2,3\}$, the set $\mathrm{C_M}$ is a set of numbers whose continued fractions only have coefficients 2 and 3. One can show that if M is finite and contains more than two numbers, then $\mathrm{C_M}$ is a Cantor set.

Here we would like to address several similar questions in the case of generalized continued fractions. A generalized continued fraction is an expression of the following form:

$$a_0 + \cfrac{w_1}{a_1 + \cfrac{w_2}{a_2 + \cfrac{w_3}{a_3 + \dots}}},$$

where we allow the coefficients a_i and ω_i to take not only integer, but some real or even complex values depending on a problem. Considering generalized continued fractions turned out to be useful, particularly with number theory. For example, in 1761, Lambert gave the first proof that π is irrational [7] by using the following generalized continued fraction for $\tan x$:

$$\tan(x) = \cfrac{x}{1 + \cfrac{-x^2}{3 + \cfrac{-x^2}{5 + \cfrac{-x^2}{7 + \ddots}}}}$$

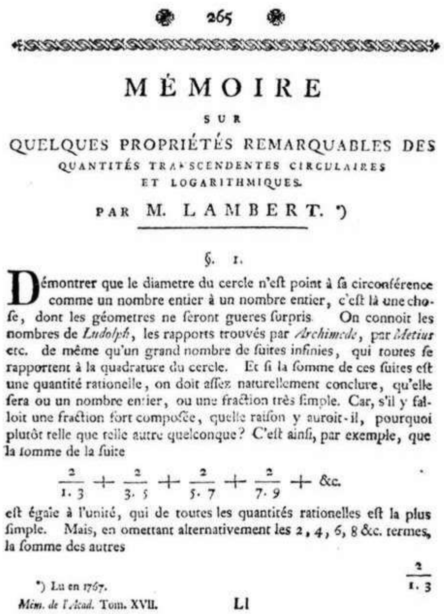
Below we reproduce two pages from the original Lambert's manuscript, see Fig. 1 and Fig. 2.

Let us also give two examples (from [5]) of explicit representations of π via generalized continued fractions:

$$\pi = \cfrac{4}{1 + \cfrac{1^2}{3 + \cfrac{2^2}{5 + \cfrac{3^2}{7 + \ddots}}}} = \cfrac{4}{1 + \cfrac{1^2}{2 + \cfrac{3^2}{2 + \cfrac{5^2}{2 + \ddots}}}},$$

Notice that, in contrast with the regular continued fractions, the representation of a number as a generalized continued fraction is not unique.

For our first problem, suppose that one has a calculator that can do one of the three operations: operation A to change a sign of the number, operation B to add 2, and



265

MÉMOIRE

SUR

QUELQUES PROPRIÉTÉS REMARQUABLES DES QUANTITÉS TRANSCENDENTES CIRCULAIRES ET LOGARITHMIQUES.

PAR M. LAMBERT. *)

§. 1.

Démontrer que le diametre du cercle n'eſt point à ſa circonférence comme un nombre entier à un nombre entier, c'eſt là une choſe, dont les géometres ne ſeront gueres ſurpris. On connoit les nombres de *Ludolph*, les rapports trouvés par *Archimede*, par *Metius* etc. de même qu'un grand nombre de ſuites infinies, qui toutes ſe rapportent à la quadrature du cercle. Et ſi la ſomme de ces ſuites eſt une quantité rationelle, on doit aſſez naturellement conclure, qu'elle ſera ou un nombre entier, ou une fraction très ſimple. Car, s'il y falloit une fraction fort compoſée, quelle raiſon y auroit-il, pourquoi plutôt telle que telle autre quelconque? C'eſt ainſi, par exemple, que la ſomme de la ſuite

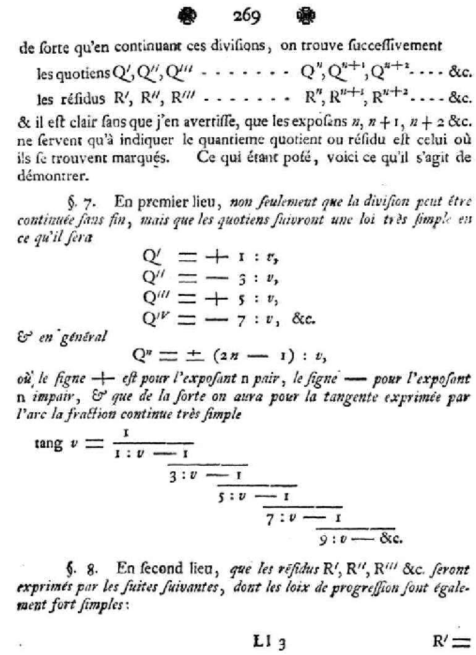
$$\frac{2}{1\cdot 3} + \frac{2}{3\cdot 5} + \frac{2}{5\cdot 7} + \frac{2}{7\cdot 9} + \&c.$$

eſt égale à l'unité, qui de toutes les quantités rationelles eſt la plus ſimple. Mais, en omettant alternativement les 2, 4, 6, 8 &c. termes, la ſomme des autres

*) Lu en 1767.

Mem. de l'Acad. Tom. XVII. Ll $\frac{2}{1\cdot 3}$

Figure 1: The first page from [7].



269

de ſorte qu'en continuant ces diviſions, on trouve ſucceſſivement

les quotiens $Q', Q'', Q''' \; - \; - \; - \; - \; - \; - \; - \; Q^n, Q^{n+1}, Q^{n+2} \; - \; - \; - \; - \; \&c.$

les réſidus $R', R'', R''' \; - \; - \; - \; - \; - \; - \; - \; R^n, R^{n+1}, R^{n+2} \; - \; - \; - \; - \; \&c.$

& il eſt clair ſans que j'en avertiſſe, que les expoſans n, $n+1$, $n+2$ &c. ne ſervent qu'à indiquer le quantieme quotient ou réſidu eſt celui où ils ſe trouvent marqués. Ce qui étant poſé, voici ce qu'il s'agit de démontrer.

§. 7. En premier lieu, *non ſeulement que la diviſion peut être continuée ſans fin, mais que les quotiens ſuivront une loi très ſimple en ce qu'il ſera*

$$Q' = +1 : v,$$
$$Q'' = -3 : v,$$
$$Q''' = +5 : v,$$
$$Q^{IV} = -7 : v, \quad \&c.$$

& en général

$$Q^n = \pm (2n - 1) : v,$$

où le ſigne + eſt pour l'expoſant n *pair, le ſigne — pour l'expoſant* n *impair, & que de la ſorte on aura pour la tangente exprimée par l'arc la fraction continue très ſimple*

$$\text{tang } v = \cfrac{1}{1:v - \cfrac{1}{3:v - \cfrac{1}{5:v - \cfrac{1}{7:v - \cfrac{1}{9:v - \&c.}}}}}$$

§. 8. En ſecond lieu, *que les réſidus* R', R'', R''' &c. *ſeront exprimés par les ſuites ſuivantes, dont les loix de progreſſion ſont également fort ſimples:*

Ll 3 $R' =$

Figure 2: Formula for $\tan x$ from [7].

operation C to take a reciprocal of a number. Start with any number, say zero, and start applying these operations. Suppose we start applying them in periodic order, i.e. ABCABCABCA.... In the limit, if we consider the results we get after each application of the operation C, we get

$$\cfrac{1}{2-\cfrac{1}{2-\cfrac{1}{2-\cfrac{1}{2-\dots}}}} = 1.$$

What if the button responsible for the operation A is broken, and pushing that button does not change the sign of a number? Then, in the limit, the calculator actually gives

$$\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\dots}}}} = \sqrt{2}-1,$$

which is the only positive root of the equation $\frac{1}{2+x} = x$.

Suppose now that the button A is broken in such a way that sometimes it works and sometimes it doesn't. What numbers can we see on a screen after many iterates? In other words, let us consider the generalized continued fractions given by an expression of the form

$$\frac{1}{2 \pm \frac{1}{2 \pm \frac{1}{2 \pm \frac{1}{2 \pm \dots}}}},$$

Which real numbers can be represented in this way?

Theorem 1.1. *A set that consists of real numbers that can be represented as*

$$\frac{1}{2 \pm \frac{1}{2 \pm \frac{1}{2 \pm \frac{1}{2 \pm \dots}}}}$$

for some choices of signs is a Cantor set.

By the Cantor set here, we mean a subset of the real line that is bounded, is closed, has no isolated points, and contains no intervals.

In fact, the choice of the number 2 in this expression is somewhat arbitrary, so the following more general statement holds:

Theorem 1.2. *For any $a \geq 2$ the set that consists of real numbers that can be represented as*

$$\frac{1}{a \pm \frac{1}{a \pm \frac{1}{a \pm \frac{1}{a \pm \dots}}}}$$

for some choices of signs is a Cantor set.

A Cantor set is the simplest example of a fractal set. But the most impressive fractal, such as Julia sets, are two dimensional. So, let us try to make our "broken calculator" more interesting and introduce another operation D of multiplication by an imaginary unit i, $i^2 = -1$, such that a corresponding button is also broken, so sometimes it works and sometimes it doesn't. What is the set of complex numbers we can get in the limit by pushing ABCDABCDABCDABCDA...? In other words, what is the set of complex numbers that can be represented as

$$\frac{1}{2 + \frac{\omega_1}{2 + \frac{\omega_2}{2 + \frac{\omega_3}{2 + \dots}}}},$$

where ω_i can be $1, -1, i$, or $-i$? As above, 2 can be replaced by any larger number, and we can prove the following:

Theorem 1.3. *For any real $a \geq 2$, the set of complex numbers that can be represented as*

$$\frac{1}{a + \frac{\omega_1}{a + \frac{\omega_2}{a + \frac{\omega_3}{a + \dots}}}},$$

where ω_i can be $1, -1, i$, or $-i$, is a Cantor set on the complex plane.

By a Cantor set on the complex plane we mean a set that is bounded, closed, totally disconnected (i.e. each connected component consists of one point), and has no isolated points. We will use the following sufficient condition for a set to be totally disconnected: a set is totally disconnected if that for any point from that set one can find a small disc of a radius that is as small as one wants that contains that point inside and does not contain any other points of the set on its boundary.

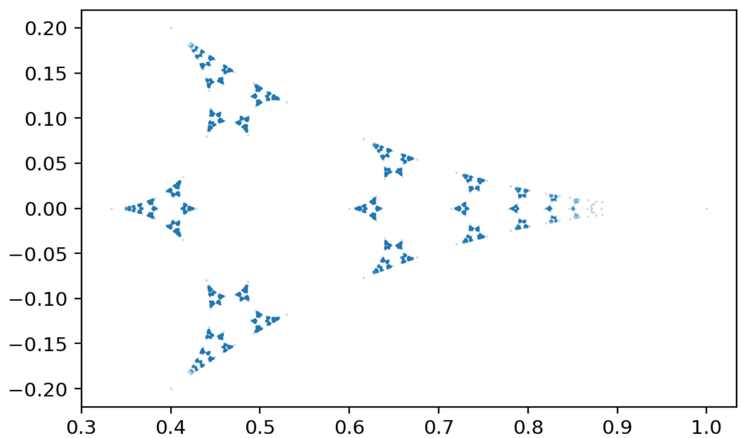


Figure 3: Cantor set on the complex plane as in Theorem 1.3, $a = 2$.

We were also experimenting with smaller values of a in Theorem 1.3, and it seems that the corresponding set of points, if a is essentially smaller than 2, is another type of fractal, not a Cantor set. We provide the results of our numerical experiments below as well as several open questions and other directions that could be explored.

2 Preliminaries

Many fractals can be generated using Iterated Function Systems (IFS). An IFS is defined by a finite collection of contractions

$$\{F_i : X \to X |\ i = 1, 2, 3, ..., N\}, N \in \mathbb{N} \tag{2}$$

Here, X is a bounded closed subset of a line, plane, or a higher dimensional Euclidean space, and each of N contractions maps X to itself. Now one can consider the following sequence of sets defined subsequently:

$$X_0 = X,\ X_{n+1} = F_1(X_n) \cup \ldots \cup F_N(X_n).$$

Finally, one can consider an intersection $C = \cap_{n=0}^{\infty} X_n$. In many cases this intersection forms a fractal set.

The first example is the standard Cantor set. The standard Cantor set $\mathscr{C}$ is formed by the following IFS:

$$\mathscr{C}_0 = [0,1] \tag{3}$$

$$\mathscr{C}_{n+1} = \frac{1}{3}\mathscr{C}_n \bigcup \left(\frac{2}{3} + \frac{1}{3}\mathscr{C}_n\right) \tag{4}$$

$$\mathscr{C} := \bigcap_{n=1}^{\infty} \mathscr{C}_n \tag{5}$$

The IFS images are commonly made of one shape that is continuously translated, rotated, and scaled to create complicated fractals. For example, the Sierpinski's Triangle, which is the most commonly known fractal, consists of equilateral triangles that are subdivided repeatedly into smaller triangles.

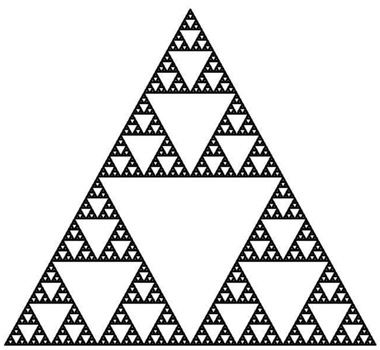


Figure 4: Serpinski's Triangle

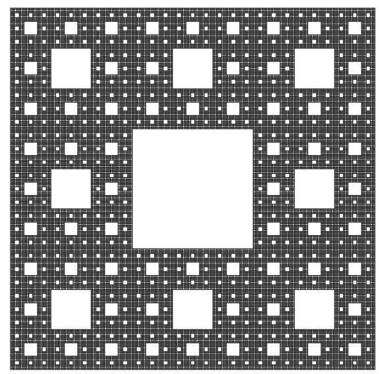


Figure 5: Serpinski's Square (aka Serpinski's Carpet)

Sierpinki's Triangle can be represented by an IFS with three contractions, each of them with contracts the distances with coefficient $1/2$. The vertices of a Sierpinski's Triangle are the centers of those contractions.

Other examples of fractals generated by iterated function systems include Serpinski's square (generated by 8 contractions, each one with coefficient of contraction $\frac{1}{3}$) and Pythagoras Tree (generated by two contractions, each one with coefficient $\frac{1}{\sqrt{2}}$).

Here we provide the pictures that were generated by the programs the we wrote but just for illustration, as those fractals were known for a long time. Indeed, Serpinski's Carpet was described first by Waclaw Sierpinski in 1916, and the Pythagoras Tree was discovered by Albert E. Bosman in 1942 (see [9], and [2] for some modern variations). The history of the discovery of Pythagoras Tree is quite interesting. Albert Bosman was a Dutch electrical engineer and mathematics teacher. He was employed by the Germans to design parts for submarines during World War II, but his drawing of increasingly smaller squares and triangles had nothing to do with that. It was a form of a silent sabotage.

Here we provide two pictures of Pythagoras Tree - one that was created by a computer program that we wrote and another one is an original hand-drawn by Albert Bosman.

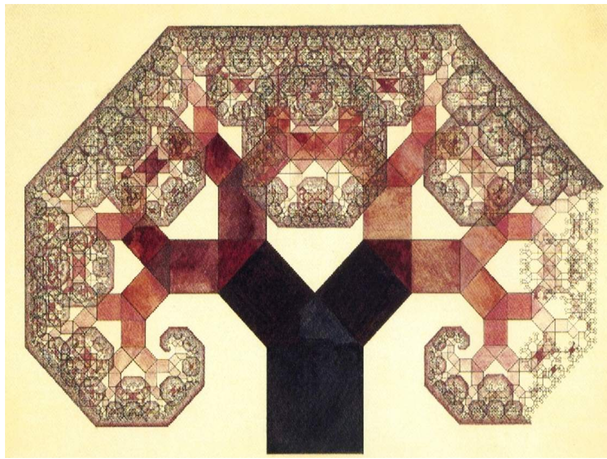


Figure 6: Pythagoras Tree, original hand drawing by Albert E. Bosman, 1942

Its important to note that while the fractals resulting from IFS are often self-similar and the functions that transform the fractals are usually contractive, it is not required for Cantor sets to be generated by IFS.

3 Proof of Theorem 1

Lets consider $g_0(x) = \frac{1}{2-x}$ and $g_1(x) = \frac{1}{2+x}$.

$$I_0 = [0, 1] \tag{6}$$

$$I_{n+1} = g_0(I_n) \bigcup g_1(I_n) \tag{7}$$

Therefore,

$$I_1 = [\frac{1}{3}, 1] \tag{8}$$

$$I_2 = [\frac{1}{3}, \frac{3}{7}] \bigcup [\frac{3}{5}, 1] \tag{9}$$

$$I_3 = [\frac{1}{3}, \frac{5}{13}] \bigcup [\frac{7}{17}, \frac{3}{7}] \bigcup [\frac{3}{5}, \frac{7}{11}] \bigcup [\frac{5}{7}, 1] \tag{10}$$

The maximal interval for each iteration is the interval that 1 lies within. For the maximal intervals, the left bound can be expressed as $L(n) = \frac{2n-1}{2n+1}$, and the right bound is always 1.

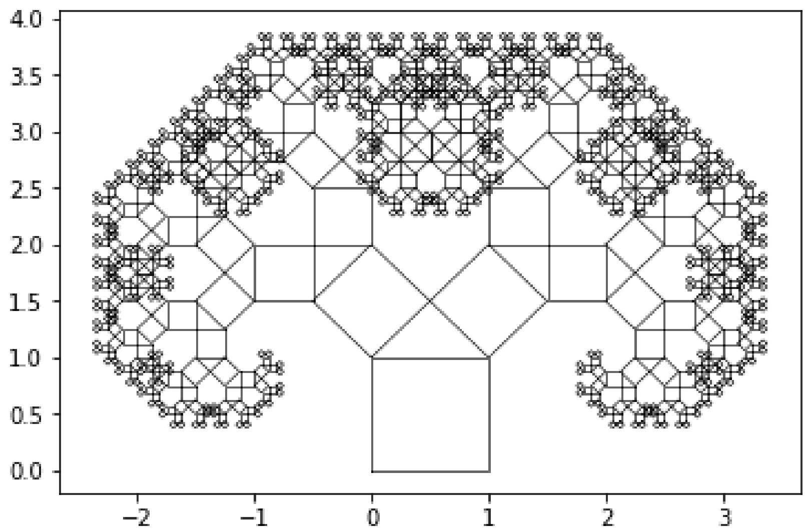


Figure 7: Pythagoras Tree, computer generated picture.

Since $\lim_{n\to\infty}(\frac{2n-1}{2n+1}) \to 1$, the length of the maximal interval in I_n as $n \to \infty$ tends to 0. Therefore, the set $\cap_{n=1}^{\infty}\mathrm{I}_n$ cannot contain any intervals.

Secondly, I_n is the union of a finite number of closed intervals and, hence, is a closed set. Since any intersection of closed sets is closed, the intersection $\cap_{n=1}^{\infty}\mathrm{I}_n$ is a closed set.

Lastly, each interval in I_n contains two intervals from I_{n+1}, and each of those two intervals contains two intervals of I_{n+2}, and so on. Pick a point x from the intersection $\cap_{n=1}^{\infty}\mathrm{I}_n$, and let us prove that there is some other point on the intersection that is very close to x. The point x must be from some small interval from the set I_n. So, it must be from one of the two intervals of I_{n+1} nearby. In the second interval, there are points from $\cap_{n=1}^{\infty}\mathrm{I}_n$. This shows that the intersection $\cap_{n=1}^{\infty}\mathrm{I}_n$ cannot have isolated points.

Therefore, the intersection $\cap_{n=1}^{\infty}\mathrm{I}_n$ contains no intervals, has no isolated pairs, and is closed, so Theorem 1 has been proven.

The proof of Theorem 2 is almost the same, but instead of the maps $g_0(x) = \frac{1}{2-x}$ and $g_1(x) = \frac{1}{2+x}$ we would need to take the maps $g_0(x) = \frac{1}{a-x}$ and $g_1(x) = \frac{1}{a+x}$.

4 Proof of Theorem 3

Let us prove Theorem 3 for $a = 2$. If $a > 2$ is another number, the proof is almost the same.

We need to take a unit disk of the complex plane. Denote

$$\mathrm{D} = \{z \in \mathbb{C} \mid |z| \leq 1\}.$$

Let us consider four maps,

$$g_0(z) = \frac{1}{2-z},\ g_1(z) = \frac{1}{2+z},\ g_2(z) = \frac{1}{2-iz},\ g_3(z) = \frac{1}{2+iz}.$$

We set

$$\mathrm{D}_0 = \mathrm{D},\ \mathrm{D}_{n+1} = g_0(\mathrm{D}_n) \cup g_1(\mathrm{D}_n) \cup g_2(\mathrm{D}_n) \cup g_3(\mathrm{D}_n),$$

and

$$\mathscr{M} = \cap_{n=0}^{\infty} \mathrm{D}_n.$$

We use $\mathscr{M}$ (i.e. "M beautiful") to denote this set since this procedure produces really beautiful pictures.

We need to prove that $\mathscr{M}$ is a Cantor set on a complex plane.

Notice that each map g_0, g_1, g_2, g_3 is a composition of a rotation or translation of a complex plane (e.g. the map $z \mapsto 2-z$ for g_0, or the map $z \mapsto 2+iz$ for g_4) and the map $\mathrm{T} : z \mapsto \frac{1}{z}$. Any rotation and translation maps circles to circles, and lines into lines. The map T is known to be a composition of a circle inversion and complex conjugacy $z \mapsto \bar{z}$. Complex conjugacy is a symmetry with respect to the real line and, therefore, also sends circles to circles, and lines into lines. As for a circle inversion, the following statement holds (see [3]):

Proposition 4.1. *A circle inversion, with center at the origin and of radius 1, maps any line or circle that contains an origin to a line and any line or circle that does not contain an origin to a circle.*

This implies that in order to find an image of the disc D_0 (or any other disc, circle, or line) under application of g_0, g_1, g_2, or g_3 it is enough to calculate the images of three points. Then the line or circle that contains those three images will determine the image.

This implies that D_1 (as one can establish by direct calculations using the properties described above) is a union of four discs, namely $g_0(\mathrm{D})$, $g_1(\mathrm{D})$, $g_2(\mathrm{D})$, and $g_3(\mathrm{D})$. It turns out that those four discs coincide, and each of them is a disc centered at a point with coordinates $(\frac{2}{3}, 0)$ with radius $\frac{1}{3}$.

Let us now analyze D_2. The image $g_0(\mathrm{D}_1)$ is a disc centered at a point with coordinates $(\frac{4}{5}, 0)$ with radius $\frac{1}{5}$. The image $g_1(\mathrm{D}_1)$ is a disc centered at a point with coordinates $(\frac{8}{21}, 0)$ and with radius $\frac{1}{21}$. Images $g_2(\mathrm{D}_1)$ and $g_3(\mathrm{D}_1)$ are harder to calculate explicitly, but both are disjoint discs inside of the larger disc D_1. As a result, we see that the set $\mathscr{M} = \cap_{n=0}^{\infty} \mathrm{D}_n$ is closed, for each point of $\mathscr{M}$ there are other points of $\mathscr{M}$ nearby, and each point of $\mathscr{M}$ is enclosed in a small circle (boundary of one of the discs that form D_n for large n) that does not intersect $\mathscr{M}$. Therefore, $\mathscr{M}$ is a Cantor set on the complex plane.

5 Further questions and Models

We proved that the set of points on $[0,1]$ that can be represented as

$$\frac{1}{a_1 \pm \frac{1}{a_2 \pm \frac{1}{a_3 \pm \frac{1}{a_4 \pm \dots}}}},$$

where all coefficients a_i are equal to 2 or to some real number $a > 2$ is a Cantor set. What if we allow the coefficients to take values 2 or 3 in any order? Or, in more general terms, from some finite set K, for example $\mathrm{K} = \{2,3,4,\dots,2022\}$? Let us denote such a set $\mathrm{C_K}$. We know that if we do not allow minus signs in the corresponding continued fractions and only allow coefficients from a finite subset of natural numbers, our construction would return a Cantor set, see [1]. However, for generalized continued fractions of our form, the answer seems to be more complicated. For example, the following conjecture seems to be reasonable:

Conjecture 5.1. There exists a finite set K of natural numbers such that the set $\mathrm{C_K}$ as defined above is not a Cantor set.

We do not know whether this conjecture actually holds. If it holds, it's unclear whether the corresponding set $\mathrm{C_K}$ will be an interval, be a union of intervals, or have a more complicated structure. For example, it could be a *Cantorval*. Cantorvals are closed subsets of $[0,1]$, such that they consist of an infinite number of distinct points and closed intervals where any point and closed interval has other points and closed intervals arbitrarily close. For example, if in the construction of the standard $1/3$-Cantor set one removes intervals from every other iteration, the set of points that will remain is a Cantorval.

Conjecture 5.2. There exists a finite set K of natural numbers such that the set $\mathrm{C_K}$ as defined above is a Cantorval.

Also, we can ask what one can say about the set of points on a complex plain that can be represented as

$$\frac{1}{a_1 + \frac{\omega_1}{a_2 + \frac{\omega_2}{a_3 + \frac{\omega_3}{a_4 + \dots}}}},$$

where ω_i can be $1, -1, i$, or $-i$ and coefficients a_i can take values from some finite set K. Let us denote this set $\mathrm{C^*_K}$. Similarly to the numerical experiments that we described, this set generally does not have to be a Cantor set. However, it seems to have some fractal structure. We do not know whether it can contain some discs on a complex plane, but we conjecture that it is possible.

Conjecture 5.3. There exists a finite set K of natural numbers such that the set $\mathrm{C^*_K}$ contains a disc.

Appendix A: Pictures

Here we provide the results of the numerical experiments related to Theorem 3. We demonstrate that for $a > 2$ the set described in Theorem 3 indeed looks like a Cantor set, and we provide the pictures that correspond to the values $a < 2$, which do seem to have some fractal structure but fail to be a Cantor set.

Let us remind that we denoted the set of complex numbers that can be represented as

$$\cfrac{1}{a+\cfrac{\omega_1}{a+\cfrac{\omega_2}{a+\cfrac{\omega_3}{a+\ldots}}}}$$

by $\mathscr{M}$ (the set $\mathscr{M} \subset \mathbb{C}$ depends on a real number a), where ω_i can be $1, -1, i$, or $-i$.

Let us start with a few pictures that correspond to $a \geq 2$ for which we can prove that they form a Cantor set on the complex plane.

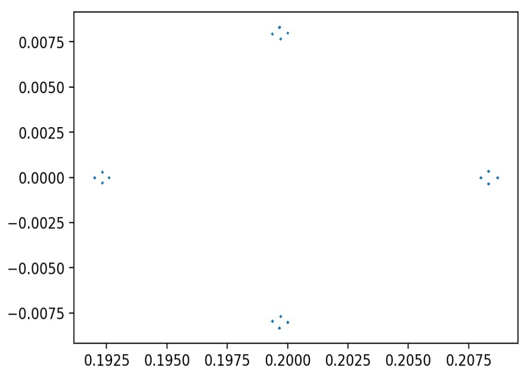


Figure 8: The set $\mathscr{M}$ for $a = 5$.

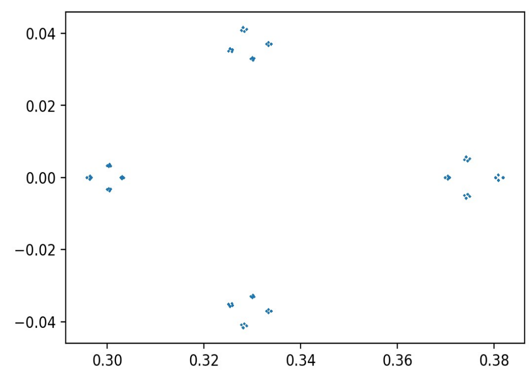


Figure 9: The set $\mathscr{M}$ for $a = 3$.

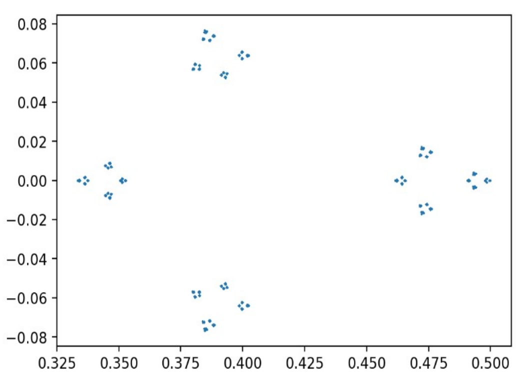


Figure 10: The set $\mathscr{M}$ for $a = 2.5$.

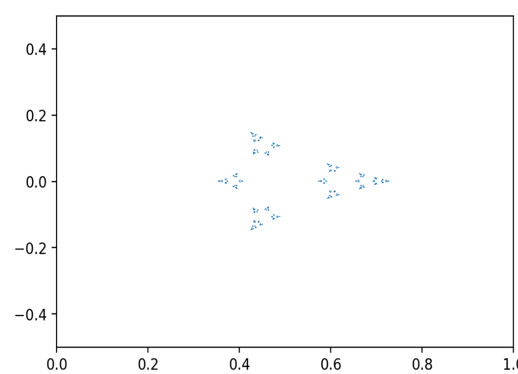


Figure 11: The set $\mathscr{M}$ for $a = 2.1$.

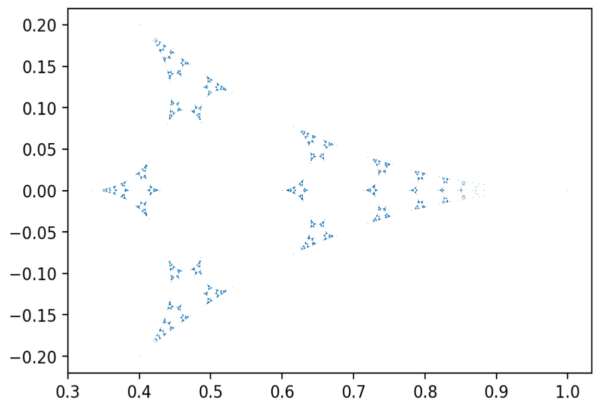


Figure 12: The set $\mathscr{M}$ for $a = 2$

Notice that a decreases and gets closer to 2. The set $\mathscr{M}$ becomes "bigger" and increases in diameter. One can probably show that its fractal dimension is increasing, but we did not elaborate on that in this project.

Let us now present the pictures for values $a < 2$. The sets have a very interesting structure, but it is not clear to us what are the properties of these sets. We believe that these pictures represent new type of fractals, and it would be interesting to understand their structure.

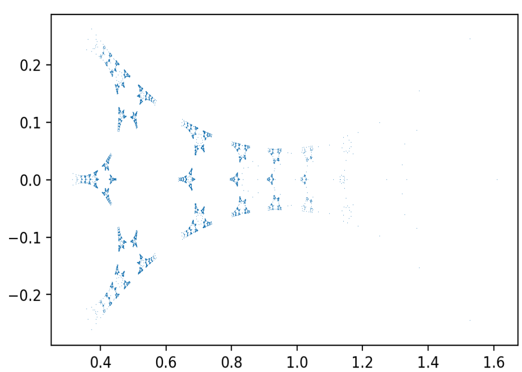


Figure 13: The set $\mathscr{M}$ for $a = 1.9$.

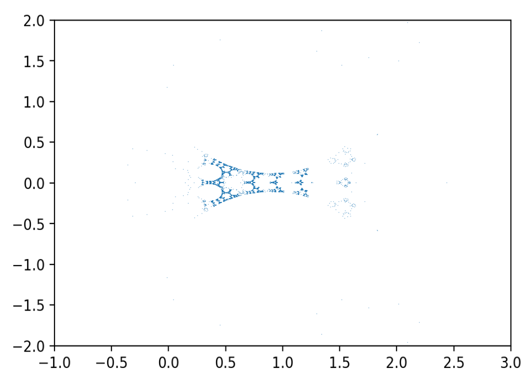


Figure 14: The set $\mathscr{M}$ for $a = 1.8$.

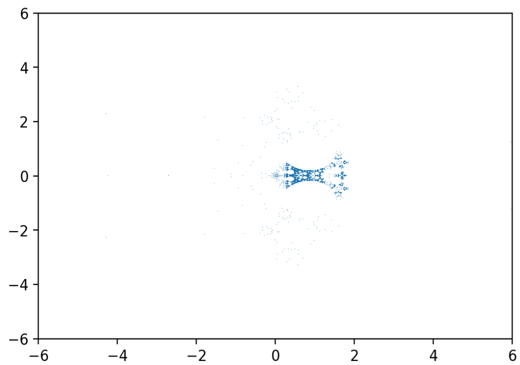


Figure 15: The set $\mathscr{M}$ for $a = 1.7$.

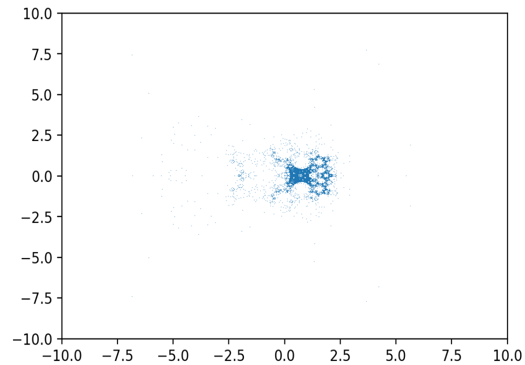


Figure 16: The set $\mathscr{M}$ for $a = 1.5$.

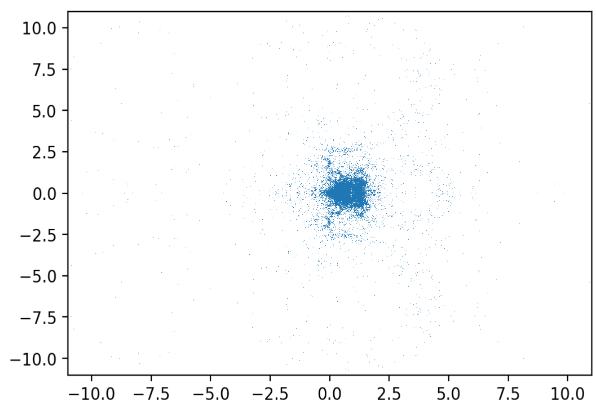


Figure 17: The set $\mathscr{M}$ for $a = 1.3$

Appendix B: Code

Here we provide the code in Python that we used to generate the pictures related to Theorem 3 (and that we provided in Appendix A).

```
import matplotlib.pyplot as plt
import numpy as np
import math
def sequences(m, n):
    seq = [1 for i in range(n)]
    while True:
```

```
        yield seq
        seq[n-1] += 1
        i = n-1
        while i > 0:
            if seq[i] > m:
                seq[i] = 1
                seq[i-1] += 1
            i -= 1
        if seq[0] > m:
            break
a = 2
def g1(z):
    return 1/(a + z)
def g2(z):
    return 1/(a - z)
def g3(z):
    return 1/complex(a, z)
def g4(z):
    return 1/complex(a, -z)
g = [g1, g2, g3, g4]
n = 8
m = 4
points = [ ]
for seq in sequences(m, n):
    i = n - 1
    x = 1
    while i > = 0:
        x = g[seq[i] - 1](x)
        i -= 1
    points.append(x)
fig, ax = plt.subplots()
fig.set_dpi(200)
x = list()
y = list()
for pt in points:
    x.append(pt.real)
    y.append(pt.imag)
ax.set_xlim([0, 1])
ax.set_ylim([-0.5, 0.5])
ax.plot(x, y, ',', markersize=0.1)
```

Acknowledgements

We are grateful to Professor Vladimir Baranovsky from UC Irvine Department of Mathematics for his mentorship. Also, we would like to thank Sasha Volokh, a Ph.D. Student at USC Vitebri Department of Computer Science, for helping us optimize our Python computer programs.

References

[1] S. Astels, Sums of numbers with small partial quotients. II., *J. Number Theory* **91** (2001), 187–205.

[2] T. Munz, M. Burch, T. van Benthem, Y. Poels, F. Beck, D. Weiskopf, Overlap-free drawing of generalized pythagoras trees for hierarchy visualization, *2019 IEEE Visualization Conference (VIS)*, pp. 251–255.

[3] H. Coxeter, "Inversion in a Circle" and "Inversion of Lines and Circles," §6.1 and 6.3 in *Introduction to Geometry*, 2nd ed. New York: Wiley, pp. 77–83, 1969.

[4] T. Cusick and M. Flahive, The Markoff and Lagrange Spectra, *American Mathematical Society*, 1989.

[5] J. Havil, The Irrationals: A Story of the Numbers You Can't Count On, *Princeton University Press*, 2012.

[6] A. Ya. Khinchin, Continued Fractions, (1964) (Originally published in Russian, 1935), *University of Chicago Press.*

[7] Johann Heinrich Lambert, Mémoire sur quelques propriét'es remarquables des quantit'es transcendantes et logarithmiques, Histoire de l'Acad'emie royale des sciences et belles-lettres (1761), Berlin.

[8] S. Lang, Introduction to Diophantine Approximations, *Springer-Verlag New York, Inc.*, 1995.

[9] H. Lauwerier, Fractals: Endlessly Repeated Geometric Figures. Princeton, NJ: Princeton University Press, pp. 67–77 and 111–113, 1991.

[10] J. Milnor, Appendix C. Rotations, Continued Fractions, and Rational Approximation, in "Dynamics in One Complex Variable", Third Edition, *Princeton: Princeton University Press*, 2011, pp. 234–245.

Masha Gorodetski
Academy of the Canyons
masha.gorodetski.2005@gmail.com

Danielle Hedvig
University High School
danihedvig@gmail.com