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# The Determining Number and Cost of 2-Distinguishing of Select Kneser Graphs

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### Cover Page Footnote

Thank you to Dr. Sarah Loeb, without whom this work would have been impossible

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# The Determining Number and Cost of 2-Distinguishing of Select Kneser Graphs

By *James Garrison*

**Abstract.** A graph G is said to be *$d$-distinguishable* if there exists a not-necessarily proper coloring with $d$ colors such that only the trivial automorphism preserves the color classes. For a 2-distinguishing labeling, the *cost of* 2*-distinguishing*, denoted $\rho(G)$, is defined as the minimum size of a color class over all 2-distinguishing colorings of G. Our work also utilizes *determining sets* of G, sets of vertices $S \subseteq G$ such that every automorphism of G is uniquely determined by its action on S. The *determining number* of a graph is the size of a smallest determining set. We investigate the cost of 2-distinguishing families of Kneser graphs $K_{n:k}$ by using optimal determining sets of those families. We show the determining number of $K_{n:2}$ is equal to $\lceil \frac{2n-2}{3} \rceil$ and give linear bounds on $\rho(K_{n:2})$ when $n$ is sufficiently sized.

## 1 Introduction

A graph G consists of a set of vertices V(G) and a set of edges E(G) where an edge is an unordered set of vertices of size 2. Among the many areas of interest in graph theory is that of the graph automorphism, a form of symmetry of a graph in which the graph is mapped onto itself while preserving the edge-vertex relationships. Formally, an *automorphism* of a graph G is a permutation $\sigma$ of the vertex set V(G) such that the pair of vertices $\{u, v\}$ form an edge if and only if the pair $\{\sigma(u), \sigma(v)\}$ also form an edge. The automorphism group of a graph G, denoted Aut(G), is the group of all possible automorphisms of a graph, each of which reveal symmetries in a graph's structure.

One can "add structure" to a graph G in the form of *labels* (or *colors*) on a graph's edges or vertices that help one better understand Aut(G). One such mode of graph labeling uses colors to make Aut(G) trivial: this style of labeling is called a *distinguishing labeling*. A $d$-distinguishing labeling colors vertices (or, for edges, a $d$-edge-distinguishing labeling) with $d$ colors in such a way that only the trivial automorphism—that is, the automorphism $\varphi : V(G) \to V(G)$ which maps each vertex $v \in V(G)$ to itself—preserves the color classes of the coloring. Because graphs vary so widely in structure, the number of colors required to distinguish a graph is a parameter of interest. We call a graph *$d$-distinguishable* if the graph can be distinguished with $d$ colors. For examples of distinguishing in action, see [1], [9], and [10].

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A natural question that falls out of the discussion of distinguishing labelings is this: given a $d$-distinguishable graph colored white, what is the fewest number of vertices one must color using $d-1$ colors to distinguish G? This is what Boutin calls the *paint cost of d-distinguishing* [4]. In this paper, we focus on the paint cost of 2-distinguishing G, also called simply the *cost of 2-distinguishing* G or the *cost*, denoted $\rho(G)$, which was first discussed by Boutin in [6].

An important concept for investigating the cost of 2-distinguishing is that of *determining sets* of a graph. Intuitively, a determining set is a subset of vertices of $S \subseteq V(G)$ such that if the effects of an automorphism $\varphi$ are known for S, the effects of $\varphi$ are known for *every* vertex of G. A determining set can also be thought of as a set of vertices that is fixed pointwise only by the trivial automorphism. The *determining number* of a graph G, denoted Det(G), is the minimum size of a determining set of G.

In this paper, we investigate the determining number and the cost of 2-distinguishing of a family of graphs known as Kneser graphs. It is shown in [2] that all Kneser graphs $K_{n:k}$ with $n \geq 6$ and $k \geq 2$ are 2-distinguishable. Moreover, there has been progress in identifying the cost and determining number of many subfamilies of Kneser graphs: in [5] it is shown that $\rho(K_{2^m-1:2^{m-1}-1}) = m+1$; all Kneser graphs with determining number 2, 3, or 4 are given in [3]; [8] improves bounds on $\mathrm{Det}(K_{n:k})$ given in [3]. An example of a simple Kneser graph and one of its determining sets can be seen in Figure 4. In this paper, we focus our results on the comparatively large subfamily $K_{n:2}$.

This paper is organized as follows. Formal definitions and examples of distinguishing labelings, determining sets, cost of 2-distinguishing, and Kneser graphs are given in Section 2. In Section 3, we establish $\mathrm{Det}(K_{n:2})$ when $n \geq 6$. In Section 4, we identify linear upper and lower bounds for $\rho(K_{n:2})$. Finally, in Section 5 we provide some future directions.

## 2 Definitions and Background

**Definition 1.** For a graph G, a labeling of vertices $f : V(G) \to \{1, \ldots, d\}$ is *d-distinguishing* if $\varphi \in \mathrm{Aut}(G)$ and $f(\varphi(x)) = f(x)$ for all $x \in V(G)$ implies that $\varphi = id$. The *distinguishing number* of G, denoted Dist(G), is the minimum $d$ such that G has a $d$-distinguishing labeling.

In Figure 1, one can see a graph G with a 3-distinguishing labeling; moreover, it can be shown that Dist(G) = 3. By choosing the black and gray vertices, one can ensure that the only automorphism that preserves the color classes of the labeling is the trivial automorphism. Every graph has a distinguishing labeling, since one can assign a different label to every vertex; the interesting aspect of this concept, then, is the distinguishing labeling of G which uses the fewest possible colors, or Dist(G).

**Definition 2.** Let G be a 2-distinguishable graph. The minimum size of a color class over

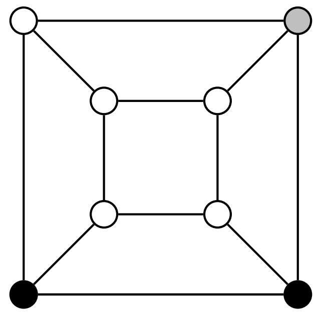

Figure 1: A 3-distinguishing coloring of the 3-dimensional hypercube.

all 2-distinguishing labelings of G is called the *cost of 2-distinguishing* G. The cost of 2-distinguishing a graph G is denoted $\rho(G)$.

Where the distinguishing number minimizes the number of colors needed to distinguish a graph G, the cost of 2-distinguishing minimizes the number of vertices needed to be colored in a 2-distinguishing labeling. To facilitate the investigation of $\rho(K_{n:2})$, one can use determining sets.

**Definition 3.** A subset $S \subseteq V(G)$ is a *determining set* if whenever $g, h \in \mathrm{Aut}(G)$ and $g(x) = h(x)$ for all $x \in S$, then $g = h$. The determining number of G, denoted Det(G), is the minimum $r$ such that G has a determining set of cardinality $r$.

An automorphism performs a permutation of the vertices of a graph, and a determining set can be thought of as a subset of vertices that tells one everything about what that permutation does to the graph. That is, every automorphism of G is uniquely determined by its action on the vertices of a determining set. Just as every graph is distinguishable, every graph has a determining set, since a set containing all but one vertex is a determining set. The determining set is of key importance, for by distinguishing any determining set of G, one distinguishes G. It can be seen that the minimum number of vertices required to $d$-distinguish a graph—that is, the paint cost of $d$ distinguishing—is equal to Det(G), since one cannot color a smaller set and still have a distinguishing labeling. Boutin tied determining sets and the cost of 2-distinguishing with the notion of the *distinguishing class*, a label class in a 2-distinguishing labeling,in the following lemma adapted from [6]:

**Lemma 2.1.** *A subset of vertices* S *is a distinguishing class if and only if* S *is a determining set for* G *with the property that every automorphism of* G *that fixes* S *setwise, also fixes it pointwise.*

What Lemma 2.1 makes formal is the notion that distinguishing classes, labels that "add structure" to the graph G so that only the trivial automorphism can preserve that

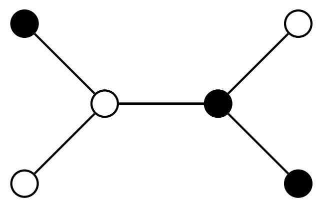

Figure 2: A 2-distinguishing labeling of $S_{2,2}$.

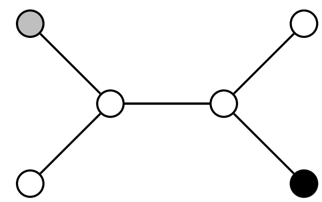

Figure 3: A 3-distinguishing labeling of $S_{2,2}$.

structure, must also be determining sets that have each of their vertices fixed pointwise. The distinguishing class tells us everything about the action of the identity automorphism —and thus the structure of the graph— for the identity is the only automorphism left in Aut(G) after all the labels have been added. Therefore, a distinguishing class is a determining set in its own right, the vertices of which are fixed pointwise by the trivial automorphism.

**Example 1.** An example is helpful in observing the relationships and behavior of distinguishing labelings, determining sets, and distinguishing classes as laid out above. A *double-star graph* $S_{n,m}$ is a graph with the following structure: one edge $\{x, y\}$ with $n$ pendant vertices $\{v_1, \ldots, v_n\}$ adjacent to $x$ and $m$ pendant vertices $\{u_1, \ldots, u_m\}$ adjacent to $y$ and no other edges. One can see that $\text{Det}(S_{n,m}) = n + m - 2$ and $\text{Dist}(S_{n,m}) = \max\{n, m\}$. Figures 2 and 3 portray two different distinguishing labelings of $S_{2,2}$. Figure 2 shows that $\text{Dist}(S_{2,2})$ is indeed 2, and the black labeled vertices are also a determining set that is fixed pointwise, *i.e.*, a distinguishing class of $S_{2,2}$. Consequently, $\rho(S_{2,2}) = 3$ as evidenced in Figure 2, because a 2-distinguishing labeling will require at least 3 vertices to be colored to break all of the symmetries in $S_{2,2}$. It is important to note that the distinguishing class in Figure 2, while a determining set, is not the minimally sized determining set. Distinguishing the graph by distinguishing the optimal determining set requires more than 2 colors, as seen in Figure 3. Though not addressed in this paper, a question inspired by [4] concerning distinguishing the optimal determining set is posed in Section 5.

**Definition 4.** A Kneser graph $K_{n:k}$ is a graph with vertices that are subsets of $[n] = \{1, \ldots, n\}$ of size $k$ and edges between only those subsets that are disjoint.

Unlike the double-star graph $S_{n,m}$ in Example 1, the structure of Kneser graphs $K_{n:k}$ becomes much more complicated as $n$ increases. Consequently, finding distinguishing classes and determining sets becomes more complicated, too. For example, in Figure 4, the determining set, labeled in black, need only account for two "types" of automorphism: exchanging vertices connected by a single edge and exchanging edges. However, in Figure 5, a distinguishing labeling must account for far more "types" of automorphism, because the graph is far more complicated: $\text{Aut}(K_{5:2})$ is isomorphic to the symmetric group on 5 vertices, which has 120 elements [11]. That means $\text{Aut}(K_{5:2})$ is larger than $\text{Aut}(K_{4:2})$ by a large multiplicative factor. The trend of increasing complexity in $\text{Aut}(K_{n:2})$

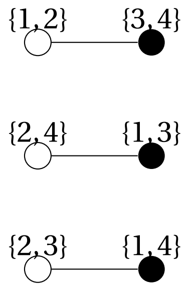


Figure 4: $K_{4:2}$ with a determining set in black.

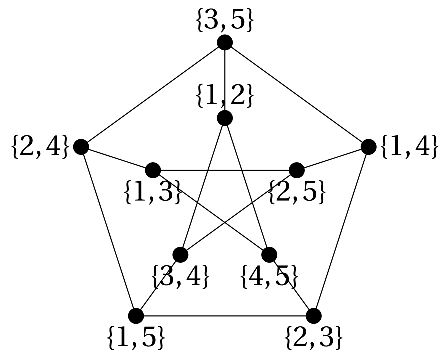


Figure 5: The Kneser graph $K_{5:2}$.

continues as $n$ increases. Consequently, we turn to the determining number of $K_{n:2}$ to find a starting point for investigating $\rho(K_{n:2})$ regardless of the graph's complexity.

## 3 Determining Number of $K_{n:2}$

In this section we prove our major result:

**Theorem 3.1.** For $n \geq 6$, $\text{Det}(K_{n:2}) = \left\lceil \frac{2n-2}{3} \right\rceil$

The proof has two parts: first, we proceed by induction to provide a lower bound on $\text{Det}(K_{n:2})$; second, we provide a determining set with size equal to that lower bound, yielding the result. Note also the ceiling function in Theorem 3.1 is equal to the piecewise function in Lemmas 3.4 and 3.5.

### 3.1 Characteristic Matrices & Lower Bound

Because the structure of Kneser graphs becomes more complex as $n$ increases, we use a concept from Boutin used on Kneser graphs in [3] called a *characteristic matrix* to find $\text{Det}(K_{n:2})$.

**Definition 5.** Let $S = \{V_1, \ldots, V_m\}$ be an ordered set of vertices of $K_{n:2}$, each written as a vector $\{v_1, \ldots, v_m\}$ of length $n$ with ones in the coordinate positions corresponding to the elements in $V_i$ and zeros in the coordinate positions corresponding to the elements not in $V_i$. Define M to be the $m \times n$ matrix whose $i^{th}$ row is $v_i$. Call M a *characteristic matrix of* S.

Characteristic matrices allow for easily manipulated and understood representations of determining sets in Kneser graphs. Two vertices $V_i$, $V_j \in V(K_{n:2})$ are adjacent only if there is no column of a characteristic matrix M where $v_i$ and $v_j$ both contain a 1. For

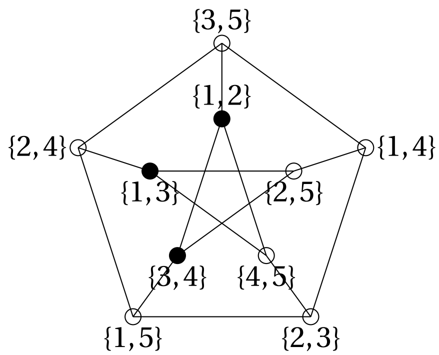


Figure 6: A determining set of $K_{5:2}$ denoted in black.

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

Figure 7: A characteristic matrix of the determining set in Figure 6.

example, in Figure 6, we see a determining set of $K_{5:2}$, which can be represented easily in the characteristic matrix found in Figure 7. The characteristic matrix communicates information about the members of the determining set and their adjacencies with greater ease than many other methods. One particularly useful aspect of characteristic matrices was demonstrated in [3] by Boutin:

**Lemma 3.1.** *Let* $S = \{V_1, \ldots, V_m\} \subseteq V(K_{n:k})$ *and* $M$ *be a characteristic matrix for* $S$. *Then,* $S$ *is a determining set of* $K_{n:k}$ *if and only if all the columns of* $M$ *are distinct.*

Using facts about adjacencies of vertices in determining sets and the columns of characteristic matrices, we establish several lemmas about the structure of the characteristic matrices of optimal determining sets to find a lower bound on $\mathrm{Det}(K_{n:2})$.

**Lemma 3.2.** *Any determining set of* $K_{n:2}$ *contains no vertex adjacent to every other vertex. That is, in a determining set* $S \subseteq V(K_{n:2})$, *every vertex* $V \in S$ *is non-adjacent to some other vertex* $W \in S$.

*Proof.* Suppose that $S = \{V_1, V_2, \ldots, V_m\}$ is a determining set of $K_{n:2}$ such that, for some $i$, $V_i$ is adjacent to every other vertex in $S$. Let $V_i = \{j,k\}$ for some $1 \leq j,k \leq m$. Since $V_i$ is adjacent to every other vertex in the determining set, $v_i$ will have a 1 in the $j^{th}$ and $k^{th}$ indices, and there will be no other entries in the $j^{th}$ and $k^{th}$ columns of the characteristic matrix of $S$. Hence, columns $j$ and $k$ of the characteristic matrix are indistinct. So $S$ cannot be a determining set by Lemma 3.1. Therefore, if $S$ is a determining set of $K_{n:2}$, it cannot contain a vertex adjacent to every other vertex.

□

An immediate consequence of Lemma 3.2 is that every determining set of size greater than 2 has a pair of non-adjacent vertices. By beginning to gather general information about the structure of these characteristic matrices of determining sets of $K_{n:2}$, we can apply these ideas to find optimally sized determining sets of $K_{n:2}$.

**Definition 6.** Let an $m \times n$ characteristic matrix M be called *nice* if it has the following structure:

$$\mathrm{M} = \left(\begin{array}{ccc|c} & 0 & & \mathrm{A} \\ \hline 1 & 0 & 1 & \multirow{2}{*}{0} \\ 1 & 1 & 0 & \end{array}\right),$$

where A is an $(m-2) \times (n-3)$ matrix with distinct columns.

Any determining set with a nice characteristic matrix also has the property that there exists a fixed pair of non-adjacent vertices of the determining set are adjacent to every other vertex of the determining set. In the nice characteristic matrix, one such pair is $\mathrm{V}_m$ and $\mathrm{V}_m - 1$.

**Lemma 3.3.** *For any determining set* S *of* $\mathrm{K}_{n:2}$ *with* $n \geq 6$*, there is a determining set of equal size* $\mathrm{S}'$ *with a nice characteristic matrix.*

*Proof.* Let S be a determining set of $\mathrm{K}_{n:2}$ with size $m \geq 3$. By Lemma 3.2, there exists at least one pair of non-adjacent vertices in S. Because we can permute the rows of a characteristic matrix without losing any information about the determining set, we may assume those non-adjacent vertices to be $\mathrm{V}_m, \mathrm{V}_{m-1}$. Moreover, define $\{\mathrm{V}_1, \ldots, \mathrm{V}_{m-2}\}$ to be the remaining vertices of S. If one permutes the columns of a characteristic matrix one performs a permutation on $[n]$. This action also induces an automorphism on $\mathrm{K}_{n:2}$, for the vertices of $\mathrm{K}_{n:2}$ are subsets of $[n]$. But since the permutation of the order of the columns in the characteristic matrix does not affect whether those columns are distinct, any permutation of the columns of a characteristic matrix will not change whether the characteristic matrix represents a determining set. Thus, we can permute the columns and rows in such a way to create favorable initial conditions without loss of generality. By Lemma 3.2, there must exist a pair of non-adjacent vertices in S, and that pair of vertices must be expressed by $\{1,3\}$ and $\{1,2\}$ up to some permutation of the columns and rows. To that end, let $\mathrm{V}_m = \{1,3\}$ and $\mathrm{V}_{m-1} = \{1,2\}$. We will show that if some vertex $\mathrm{V}_j \in \mathrm{S}$ is non-adjacent to $\mathrm{V}_m$ or $\mathrm{V}_{m-1}$, $\mathrm{V}_j$ can be exchanged for another vertex $\mathrm{V}_t \in \mathrm{V}(\mathrm{K}_{n:2})$ not contained in S such that $\mathrm{V}_t$ is adjacent to both $\mathrm{V}_m$ and $\mathrm{V}_{m-1}$ and the set with the new vertex, $\mathrm{S}'$, is a determining set.

Suppose that for some $1 \leq j \leq m-2$, $\mathrm{V}_j$ is non-adjacent to $\mathrm{V}_m$ or $\mathrm{V}_{m-1}$. There are four ways $\mathrm{V}_j$ could be non-adjacent to one or both of these two vertices, and we examine them by case.

*Case 1*: $\mathrm{V}_j = \{1, \ell\}, \mathrm{V}_j = \{2, \ell\}$, or $\mathrm{V}_j = \{3, \ell\}$ for $4 \leq \ell \leq n$.

Our general strategy is "moving" the 1 in the $1^{st}, 2^{nd}$, or $3^{rd}$ column to an "open" column (one with a 0) in the same row that will make the new vertex adjacent to $\mathrm{V}_m$ and $\mathrm{V}_{m-1}$. What is actually happening in this process is that we are exchanging the current row vector $v_j$ for another vector $v_t$ such that $v_t$ has a 1 in the index that was formerly "open" and a 0 in the index that formerly had a 1. In doing so, we exchange $\mathrm{V}_t$ for $\mathrm{V}_j$ in S and assign $v_t$ to the $j^{th}$ row of the characteristic matrix, creating a determining set $\mathrm{S}'$ where $\mathrm{V}_t$ is adjacent to $\mathrm{V}_m$ and $\mathrm{V}_{m-1}$.

In this case, there are $n-4$ potential indices in $v_j$ where we can move the 1 in $1^{st}, 2^{nd}$, or $3^{rd}$ column so that $v_j$ represents a vertex adjacent to $\mathrm{V}_m$ or $\mathrm{V}_{m-1}$.

Because each of the columns of M is distinct, there is at most one column, $g$, that is identical to column $\ell$ in every row except row $j$, where $g$ has a 0. Thus, if we were to move the 1 in the $1^{st}, 2^{nd}$, or $3^{rd}$ column to column $g$, the characteristic matrix would no longer represent a determining set because there would be a pair of identical columns. Consequently, we do not choose $g$ as the column to which we can move the 1 in the $1^{st}, 2^{nd}$, or $3^{rd}$ column. No matter if every vertex is adjacent to $\mathrm{V}_m$ or $\mathrm{V}_{m-1}$, or if only $\mathrm{V}_j$ is adjacent to those two vertices, there are at worst $n-5$ potential columns (because we cannot use columns $1,2,3,\ell,g$) to move the 1 in the $1^{st}, 2^{nd}$, or $3^{rd}$ column to make $\mathrm{V}_j$ adjacent to $\mathrm{V}_m$ and $\mathrm{V}_{m-1}$. When $n \geq 6$, there is guaranteed to be one column $h$ that will be distinct from the rest of the columns in the characteristic matrix even if we place a 1 in its $j^{th}$ row. Hence we are always able to exchange $v_j$ for $v_t$ where $v_t$ has a 1 in columns $\ell$ and $h$.

*Case 2*: $\mathrm{V}_j = \{2,3\}$ for $4 \leq \ell \leq n$.

In this case, there are two 1s that are causing $\mathrm{V}_j$ to be non-adjacent to $\mathrm{V}_m$ or $\mathrm{V}_{m-1}$, so we must move both. To address this case, we move a single 1 and then proceed as in case 1. Without loss of generality, we will move the 1 in the $3^{rd}$ index of $v_j$. Similarly to case 1, there is at most one column, $g$, that is identical to column 2 in every row except row $j$, and moving the 1 in the $3^{rd}$ column to column $g$ would make two indistinct columns. Thus there are there are $n-4$ potential columns (because we cannot use columns $1,2,3,g$) to move the 1 in the $3^{rd}$ column. Because $n \geq 6$, there is guaranteed at least one column $i$ where the 1 can be moved while keeping all columns distinct. Without loss of generality, $\mathrm{V}_j$ can be replaced with $\mathrm{V}_t$, where $v_t$ has 1s in the $2^{nd}$ and $i^{th}$ indices. Then proceed similarly to case 1.

Therefore, for any vertex $\mathrm{V}_j \in \mathrm{S}$ adjacent to $\mathrm{V}_m$ or $\mathrm{V}_{m-1}$, there is another vertex of $\mathrm{K}_{n:2}$ that can be exchanged with $\mathrm{V}_j$ while maintaining distinct columns and thus a determining set. After we exchange $\mathrm{V}_t$ for $\mathrm{V}_j$, the above processes can be repeated for any vertex in S that is not adjacent to $\mathrm{V}_m$ or $\mathrm{V}_{m-1}$. Consequently, for any appropriately sized determining set S there is a determining set of equal size $\mathrm{S}'$ such that $\mathrm{S}'$ has a nice characteristic matrix.

□

Lemma 3.3 is the crux of the proof of Theorem 3.1. The lemma shows that given a determining set, we can exchange elements of the determining set so that a nice characteristic matrix obtains for all $n \geq 6$. Knowing this fact, we use induction to establish the lower bound on $\mathrm{Det}(\mathrm{K}_{n:2})$.

**Lemma 3.4.** *When* $n \geq 6$, $\mathrm{Det}(\mathrm{K}_{n:2}) \geq \begin{cases} 2\ell & n = 3\ell \\ 2\ell & n = 3\ell + 1 \\ 2\ell + 1 & n = 3\ell + 2 \end{cases}$

N.B.: This lemma would hold for $\ell \geq 1$ if not for $\mathrm{K}_{4:2}$, the determining number of which can be shown to be 3. In Figure 4, one needs to color one vertex in each pair to identify the effects of any possible automorphism.

*Proof.* We proceed via induction. It has been shown that $\mathrm{Det}(\mathrm{K}_{6:2}) = \mathrm{Det}(\mathrm{K}_{7:2}) = 4$ in [3] and that $\mathrm{Det}(\mathrm{K}_{8:2}) = 5$ in [7]. Suppose for induction on $\ell$ that

$$\mathrm{Det}(\mathrm{K}_{n-3:2}) \geq \begin{cases} 2(\ell - 1) & n = 3(\ell - 1) \\ 2(\ell - 1) & n = 3(\ell - 1) + 1 \\ 2(\ell - 1) + 1 & n = 3(\ell - 1) + 2 \end{cases}$$

Let S be a minimum-sized determining set of $\mathrm{K}_{n:2}$ with size $m$. By Lemma 3.3, there is a determining set of equal size S′ such that S′ has a nice characteristic matrix. Because S′ is a determining set, all columns in the characteristic matrix of S′ must be distinct. So every column in the matrix A—as described in Definition 6—is distinct; hence, A is an $(m-2) \times (n-3)$ matrix with distinct columns and thus is a characteristic matrix for a determining set of $\mathrm{K}_{n-3:2}$ with size $m-2$. By inductive hypothesis, one of the following holds depending on the value of $n$:

$$m - 2 \geq 2(\ell - 1)$$
$$m - 2 \geq 2(\ell - 1)$$
$$m - 2 \geq 2(\ell - 1) + 1$$

Equivalently,

$$m \geq 2\ell$$
$$m \geq 2\ell$$
$$m \geq 2\ell + 1$$

□

### 3.2 A Determining Set of $\mathrm{K}_{n:2}$

**Lemma 3.5.** *If* $n \geq 6$*, then* $\mathrm{K}_{n:2}$ *has a determining set* S *of size* $\begin{cases} 2\ell & n = 3\ell \\ 2\ell & n = 3\ell + 1 \\ 2\ell + 1 & n = 3\ell + 2 \end{cases}$

*Proof.* Let $S \subseteq V(K_{n:2})$. We proceed in 2 cases.

*Case 1*: $n \in \{3\ell, 3\ell+1\}$ for some $\ell \geq 2$.

Let S be identified by the following construction: Select $\{V_0, \ldots, V_m\} \subseteq V(K_{n:2})$ letting $V_{2j} = \{3j+1, 3j+2\}, V_{2j+1} = \{3j+1, 3j+3\}$ for each $0 \leq j \leq \ell - 1$.

In this construction, no 2 columns of the characteristic matrix of S can be equal, for $v_{2j}, v_{2j+1}$ differentiate the columns $3j, 3j+1,$ and $3j+2$ of the characteristic matrix both from one another and from every other column in the characteristic matrix. Hence, when $n \in \{3\ell, 3\ell+1\}$, there exists a determining set S of size $2\ell$.

*Case 2*: $n = 3\ell + 2$ for some $\ell \geq 2$.

Let S be identified by the following construction: Select $\{V_0, \ldots, V_m\} \subseteq V(K_{n:2})$ such that $V_{2j} = \{3j+1, 3j+2\}, V_{2j+1} = \{3j+1, 3j+3\}$ for each $0 \leq j \leq \ell - 1$ and $V_{2\ell} = \{3\ell, 3\ell+1\}$.

Similar to the above argument, $v_{2j}, v_{2j+1}$ differentiate the columns $3j, 3j+1,$ and $3j+2$ of the characteristic matrix. In this case, one additional row is required to differentiate the $3\ell+1^{st}$ and $3\ell+2^{nd}$ columns and this can be done by selecting a row vector $v_{2\ell}$ with 1s in the the $3\ell^{th}$ and $3\ell+1^{st}$ indices. $V_{2\ell}$ in the construction above accomplishes this goal. Therefore, when $n = 3\ell+2$, there exists a determining set of size $2\ell+1$.

□

Thus, Theorem 3.1 is proved as an immediate consequence of Lemmas 3.4 and 3.5. By identifying $\mathrm{Det}(K_{n:2})$, we have provided a new result on a parameter of interest in a family of graphs of interest in graph theory. Moreover, this result lends itself to the further directions discussed in Section 5. By identifying the determining number of this subfamily of 2-distinguishable graphs, we can show just how optimal (or non-optimal) a 2-distinguishing labeling of $K_{n:2}$ is; thus, we turn to an investigation of $\rho(K_{n:2})$.

## 4 Cost of 2-Distinguishing $K_{n:2}$

Several facts and definitions are essential to our proofs in this section.

The *complement of a graph* G, denoted $G^C$, is a graph on V(G) such that two distinct vertices of $G^C$ are adjacent if and only if they are not adjacent in G. One can see that any automorphism of G is also an automorphism of $G^C$, so a labeling that breaks all the automorphisms of G must also break all the automorphisms of $G^C$. The *line graph of* G, denoted L(G), is a graph that represents the adjacencies between the edges of G. The line graph of G is constructed by making a vertex in L(G) for each edge in E(G) and making an edge in L(G) for every two edges of G that share a vertex. The *complete graph on n vertices,* denoted $K_n$, is a graph where each of the $n$ vertices is connected by an edge to all the other vertices in $K_n$. Two vertices of $L(K_n)$ will be subsets of $[n]$ with

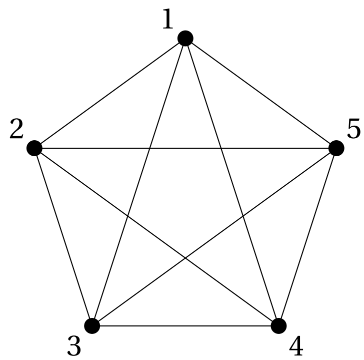


Figure 8: The complete graph $K_5$.

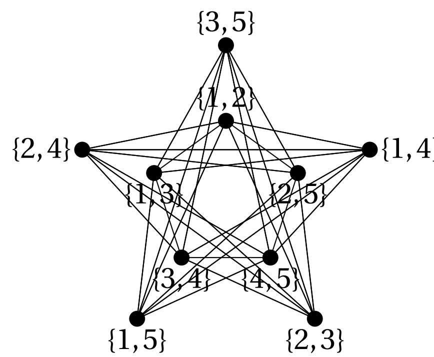


Figure 9: The line graph $L(K_5)$.

size two, adjacent only if those subsets are not disjoint. Simply by definition, therefore, $K_{n:2}$ is isomorphic to $(L(K_n))^C$. One can see an example of this isomorphism visually in Figures 8, 9, and 5.

In previous sections we have referred almost exclusively to distinguishing labelings as colorings of vertices. However, it is equally efficacious to color the edges of a graph to break all the automorphisms of a graph. The definition of an *edge-distinguishing labeling* is completely analogous to that of a distinguishing labeling:

**Definition 7.** For a graph G, a labeling of edges $j : E(G) \to \{1, \ldots, d\}$ is *$d$-edge-distinguishing* if $\varphi \in \mathrm{Aut}(G)$ and $j(\varphi(x)) = j(x)$ for all $x \in E(G)$ implies that $\varphi = id$.

Our strategy for finding $\rho(K_{n:2})$ will begin by edge-distinguishing $K_n$ to distinguish the vertices of $K_{n:2}$. When one colors an edge of $K_n$, the corresponding vertex of $L(K_n)$ is colored identically, and because all edges are fixed in a distinguishing labeling, all the vertices of $L(K_n)$ will be fixed, too. Then, because a distinguishing labeling of a graph is also a distinguishing labeling of the complement of that graph, we have distinguished $(L(K_n))^C$ and consequently $K_{n:2}$.

By finding the fewest number of edges necessary to 2-edge-distinguish $K_n$, we find the fewest number of vertices necessary to 2-distinguish $K_{n:2}$, or $\rho(K_{n:2})$. In this section, we provide two examples of a process for edge-distinguishing $K_n$ and provide an upper and lower bound on $\rho(K_{n:2})$.

An *asymmetric graph* is a graph that by virtue of its structure has a trivial automorphism group. It is always possible, as shown on page 14 of [1], to create an asymmetric subgraph H of $K_n$ that has $n$ edges. By coloring the edges of H black and all other edges gray we would 2-edge-distinguish $K_n$, because only the trivial automorphism can preserve the structure of H and thus the labeling. While labeling the edges of H does 2-edge-distinguish $K_n$, we still seek the minimum complement of a color class over all 2-edge-distinguishing labelings of the edges of $K_n$. One method of reducing the number of edges needed can be seen in the following lemma:

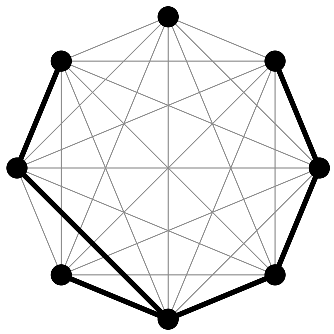

Figure 10: An edge-distinguishing class $K_8$.

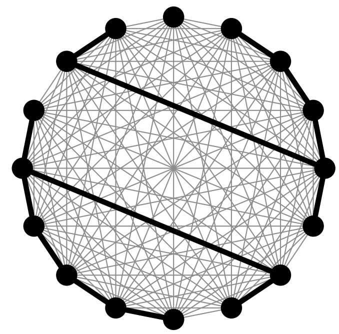

Figure 11: An edge-distinguishing class of $K_{16}$.

**Lemma 4.1.** $\rho(K_{8:2}) = 6$.

*Proof.* Because we have just two color classes to use in this edge-distinguishing labeling, any edge-distinguishing label class of $K_8$ must induce an asymmetric subgraph of $K_8$, for labeling a symmetric subgraph would not fix a label class pointwise and then such a labeling would not be distinguishing by Lemma 2.1. Therefore, we find the smallest possible asymmetric subgraph of $K_8$ that we can edge-distinguish. Given a set of $n$ vertices, the edge-distinguishable asymmetric graph of size $n$ with the fewest number of edges is an asymmetric tree. A *tree* is a graph in which any two vertices are connected by exactly one path. It is simple to see that there is no asymmetric tree on fewer than seven vertices. Therefore, to minimize the number of edges used in a 2-edge-distinguishing labeling of $K_8$, label the 6 edges of an asymmetric tree on 7 vertices black and all other edges of $K_8$ gray, which breaks all automorphisms. In Figure 10, one such labeling is shown. Because there is no edge-distinguishable asymmetric graph on 7 vertices with fewer edges than an asymmetric tree, any labeling with fewer than 6 edges will leave two singleton vertices that can be exchanged by an automorphism. Consequently, $\rho(K_{8:2}) = 6$. □

While the strategy of finding a single asymmetric tree is optimal when $n$ is small, it can be improved as $n$ increases in size. For example, it is possible to edge-distinguish $K_{16}$ with 14 edges using a single asymmetric tree on 15 vertices in a method similar to Lemma 4.1. However, there is a different labeling that uses fewer edges.

**Lemma 4.2.** $\rho(K_{16:2}) = 13$.

*Proof.* The following labeling colors 13 edges: color black the edges of the subgraph of $K_{16:2}$ consisting of three disjoint asymmetric graphs: an isolated vertex, an asymmetric tree on 7 vertices, and an asymmetric tree on 8 vertices. Because there is only one asymmetric tree on seven vertices, this labeling uses the next smallest asymmetric tree to ensure that all automorphisms are broken. Since the tree is the asymmetric graph with the fewest edges and we have used the 2 smallest asymmetric trees, this second labeling minimizes the number of edges needed to distinguish $K_{16}$. By labeling each of the edges in the two disjoint trees black and all other edges of $K_16$ grey, we minimize the number of edges colored in all 2-edge-distinguishing labelings of $K_{16}$. Therefore, $\rho(K_{16:2}) = 13$. In Figure 11, one version of this labeling is shown. □

In Theorem 4.1 we generalize the strategy of using minimally sized disjoint asymmetric trees to establish bounds on $\rho(K_{n:2})$. While there is only one asymmetric tree on seven vertices, when the size of these asymmetric trees grows larger, there might be more than one asymmetric tree up to isomorphism on a given number of vertices. We address this possibility and its implication for the bounds on $\rho(K_{n:2})$ in Theorem 4.2.

As $n$ increases, there are many possible strategies to 2-edge distinguish $K_n$. Each strategy identifies an asymmetric subgraph of $K_n$ and labels its edges black and all other edges gray. One such asymmetric graph is the asymmetric tree on $n-1$ vertices, as done in Lemma 4.1. But this is not always optimal, as seen in Lemma 4.2.

But were one to 2-edge-distinguish $K_{17}$, the only way to maintain an optimal 2-edge-distinguishing labeling is by "adding" a vertex to the largest asymmetric tree in such a way that the tree maintains its asymmetry, which is always possible. By "adding" a vertex, one simply must label an additional edge and append it to a tree that is already asymmetric, and continuously adding vertices in this way will always create an edge-distinguishing labeling. It is only when $n$ becomes sufficiently large (in the case of $n = 17$, $n$ is next large enough when $n = 26$) that we can once again add an additional disjoint asymmetric tree on nine vertices to save an edge. One can see that we "save an edge" because there are more edges in a tree of size 17 than in two trees of size 8 and 9. This process of saving an edge by using an additional disjoint asymmetric tree instead of simply adding an extra vertex can be done periodically as $n$ increases. This process is hard to quantify as $n$ gets very large, so we instead provide a linear bound. Note also that $\rho(K_{6:2}) = \rho(K_{7:2}) = 5$, so this bound begins with $n = 8$.

**Theorem 4.1.** $\frac{6}{7}(n-1) \le \rho(K_{n:2}) \le n-1$ when $n \ge 8$.

*Proof.* Let $j$ be an 2-edge-distinguishing labeling of $K_n$ and let S be the set of vertices labeled by $j$. Establishing an upper bound on the number of vertices labeled by $j$ is made easy by a fact about distinguishing labelings. When $n \ge 8$, it can be seen that $\rho(K_{n:2}) \le n-1$, for one can always isolate one vertex and color an asymmetric graph on the remaining $n-1$ vertices.

To establish the lower bound, we return to asymmetric trees. Since the smallest asymmetric tree is the one on seven vertices, every disjoint tree labeled by $j$ must have at least seven vertices. Thus, because $j$ can always isolate a singleton vertex, we know that $n \ge 7r+1$ where $r$ is the number of disjoint trees labeled by $j$. Similarly, $|E(S)| \ge 6r$. Therefore, $\frac{|E(S)|}{n-1} \ge \frac{6r}{7r}$ and $|E(S)| \ge \frac{6}{7}(n-1)$. Because S is an edge-distinguishing class, this implies $\frac{6}{7}(n-1) \le \rho(K_{n:2})$.

Therefore when $n \geq 8$,

$$\frac{6}{7}(n-1) \leq \rho(\mathrm{K}_{n:2}) \leq n-1.$$

□

It is important to note that this linear lower bound can be improved, but doing so will depend on the size of $n$ and would require a classification of all asymmetric trees, a task outside the scope of this paper.

The establishment of a linear bound, when combined with intuitions about the many potential disjoint trees present in distinguishing classes as $n$ increases, leads to the following theorem.

**Theorem 4.2.** For any $\varepsilon > 0$, there is an $n_0$ such that for $n \geq n_0$ we have $\rho(\mathrm{K}_{n:2}) \geq (1-\varepsilon)n$.

*Proof.* Let $\varepsilon > 0$. Let $t$ be the smallest integer such that $\varepsilon > \frac{1}{t}$ and $t \geq 8$. Because $t$ is finite, there are a finite number of asymmetric trees on fewer than $t$ vertices. Let

$$r = \sum_{i=2}^{t-1} r_i(i-1),$$

where $r_i$ is the number of asymmetric trees on $i$ vertices and $i-1$ denotes the maximum number of edges on those trees. Let $j$ be a 2-edge-distinguishing labeling of $\mathrm{K}_n$. When $n > r$, any disjoint asymmetric tree that will be labeled by $j$ as $n$ increases must have at least $t$ vertices and consequently no fewer than $t-1$ edges. As $n$ increases, since $j$ must also label $r$ edges, $\rho(\mathrm{K}_{n:2}) \geq \frac{t-1}{t}n + r$.

We want to pick an $n_0$ such that the additive constant $r$ can be disregarded to focus on the asymptotic behavior of $\rho(\mathrm{K}_{n:2})$. Essentially, we are going to pick a starting value at which $n$ is sufficiently large to to examine $n$ in isolation. Because as $n$ grows toward infinity, its impact on $\rho(\mathrm{K}_{n:2})$ will tremendously outweigh any number of asymmetric trees that we had to account for when $n$ was small. Because $n > r$, we know that $\frac{t-1}{t}n+r > \frac{t-1}{t}r + r$. So, if we can pick an $n_0$ such that $\frac{t-1}{t}n_0 > \frac{t-1}{t}r + r$, we can ignore the $r$ in our analysis. Equivalently, we will pick $n_0 > (1+\frac{t}{t-1})r$. Because $\frac{t}{t-1}$ is decreasing, it is largest when $t$ is smallest, that is, when $t = 8$. Therefore, pick $n_0$ such that $n_0 > \frac{15}{7}r$.

When $n > n_0$,

$$\rho(\mathrm{K}_{n:2}) \geq \frac{t-1}{t}n = (1-\frac{1}{t})n,$$

and since $\varepsilon > \frac{1}{t}$,

$$\rho(\mathrm{K}_{n:2}) \geq (1-\varepsilon)n.$$

□

## 5 Future Directions

- Investigating the generalized notion of *paint cost of $d$-distinguishing* of $K_{n:2}$, thus seeking to find the smallest number of vertices one needs to label to distinguish $K_{n:2}$ a graph given $d$ colors where $d \geq 2$.
- Identifying what Boutin calls the *frugal distinguishing number* of $K_{n:2}$ [4], which can be understood as the smallest $d$ such that the paint cost of $d$-distinguishing is equal to $\mathrm{Det}(K_{n:2})$.
- Using the paint cost to investigate $\mathrm{Det}(K_{n:k})$.


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