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Studying Extended Sets from Young Tableaux

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Studying Extended Sets from Young Tableaux

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Abstract. Young tableaux are combinatorial objects related to the partitions of an integer and have various applications in representation theory. They are particularly useful in the study of the fibers arising from the Springer resolution. In recent work of Graham-Precup-Russell, an association has been made between a given row-strict tableau and three disjoint subsets of 1,2,...,n. These subsets are then used in the study of extended Springer fibers, so we call them extended sets. In this project, we use combinatorial techniques to classify which of these extended sets correlate to a valid row-strict or standard tableau and give bounds on the number of extended sets for a fixed size. We are able to identify several global properties of these valid sets, and we further find an algorithm that produces a valid tableau given the extended sets in special cases.

1 Introduction

Young tableaux (singular "tableau") are fairly simple combinatorial objects that have widespread applicability within the field of representation theory. The most well-known application of Young tableaux is in the study of representations of the symmetric group S_n—the group containing all possible permutations of the set $\{1,\ldots,n\}$. See [Ful96] for a thorough treatment of this. The topic using Young tableaux that is most pertinent to this project is the study of Springer fibers.

The Springer correspondence is a way of relating different objects in the area of algebra known as Lie theory. The key to understanding this correspondence was the study of the map called the Springer resolution. The preimages of points from this map are known as Springer fibers. Young tableaux have been helpful in studying these fibers, which in turn has led to broader discoveries in Lie theory. See [CM93] and [Hum95] for overviews of these. In recent work, William Graham, Martha Precup, and Amber Russell have been working on an extended version of the Springer resolution. See [Gra19], [Rus20], and [GPR20] for these results. As part of this, they have developed a connection with Young tableaux that allows them to better understand the extended Springer fibers. In this paper, we will explore the combinatorics related to this connection.

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1.1 Definitions and Background

We first introduce some definitions and examples regarding Young tableaux, starting with the partition of a natural number n. Here our primary references for these terms will be [Ful96], [CM93], and [PT17].

Definition 1.1. Let $n \in \mathbb{N}$. We call $[p_1\ p_2 \dots p_m]$ a **partition of** n if $p_1 + p_2 + \cdots + p_m = n$ and $p_1, p_2, \dots p_m \in \mathbb{N}$. For convenience, we will often write entries that appear with multiplicity greater than one in $[p_1\ p_2 \dots p_m]$ using exponents.

Partitions are considered distinct up to the order of the summands. For example, $1+2$ and $2+1$ are equivalent partitions of 3. Thus, all the partitions of 3 are [3], [2 1], and $[1^3]$. For any positive integer n, we can define a function $p(n)$ that is the number of distinct partitions of n. Determining the values of $p(n)$ can be done by listing these as done here for $n = 3$, and while there is no closed form expression for this function, it is given asymptotically as

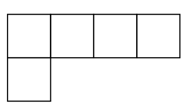
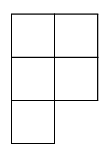
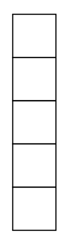
$$p(n) = \frac{1}{4n\sqrt{3}} e^{\left(\pi\sqrt{\frac{2n}{3}}\right)}$$

[Joh12]. See [Gri85] for more of the techniques used to study $p(n)$.

Next we define Young diagrams and Young tableaux, and we provide some examples of each.

Definition 1.2. A **Young diagram** is a visual representation of a partition of an integer n. The diagram is a left-justified set of n boxes organized in rows. Each row corresponds to a summand in the partition of n, and the length of the rows weakly decreases down the diagram, meaning either deacreses or stays the same.

Example 1.3. Some Young diagrams for $n = 5$ are shown here with their corresponding partitions:

[4 1]  $[2^2\ 1]$  $[1^5]$ 

Definition 1.4. A **Young tableau** is a Young diagram with an integer from 1 to n assigned to each box. For our purposes, there will be no repeated integers.

There are several types of Young tableaux, but we will mainly focus on standard and row-strict tableaux.

Definition 1.5. A **standard tableau** is a Young tableau with integers that increase across rows and down columns. A **row-strict tableau** is a Young tableau in which integers increase across rows, but not necessarily down columns. Note that all standard tableaux are also row-strict.

Example 1.6.

$$\begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 4 \\ \cline{1-2} \end{array} \quad \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 \\ \cline{1-1} 5 \\ \cline{1-1} \end{array} \quad \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline 4 \\ \hline 5 \\ \hline \end{array}$$

Some standard tableaux for $n = 5$

$$\begin{array}{|c|c|c|}\hline 3 & 4 & 5 \\ \hline 1 & 2 \\ \cline{1-2} \end{array} \quad \begin{array}{|c|c|c|}\hline 2 & 3 & 5 \\ \hline 1 \\ \cline{1-1} 4 \\ \cline{1-1} \end{array} \quad \begin{array}{|c|}\hline 3 \\ \hline 1 \\ \hline 5 \\ \hline 2 \\ \hline 4 \\ \hline \end{array}$$

Some row-strict tableaux for $n = 5$ that are not standard

We are now able to establish a bit of useful notation. Let T be a given Young tableau of size n. We will impose a labelling on the rows and columns, so that the leftmost column is Column 1 and the numbers increase by 1 from left to right, and the topmost row is Row 1, and the numbers increase by 1 from top to bottom. Then for any $x \in \{1, ..., n\}$, we use $\mathrm{row}(x)$ to denote the number of the row containing the integer x in T, and we use $\mathrm{col}(x)$ to denote the number of the column containing the integer x in T.

1.2 Extended Sets

In the work of Graham–Precup–Russell mentioned earlier, the authors established three sets, I, J and K, associated with a given tableau.[1] We will call these **extended sets** and use the notation above to define the extended sets for a tableau T.

Definition 1.7. Let X_n be the set of integers $\{1, \ldots, n-1\}$. Define the extended sets I, J, and K for a given Young tableau as the following:

$$I = \left\{ i \in X_n \middle| \begin{array}{l} \mathrm{row}(i+1) < \mathrm{row}(i) \text{ and } \mathrm{col}(i+1) = \mathrm{col}(i) + 1; \\ \text{or } \mathrm{col}(i+1) > \mathrm{col}(i) + 1 \end{array} \right\}$$

$$J = \{ j \in X_n | \mathrm{row}(j+1) = \mathrm{row}(j) \text{ and } \mathrm{col}(j+1) = \mathrm{col}(j) + 1 \}$$

$$K = \left\{ k \in X_n \middle| \begin{array}{l} \mathrm{row}(k+1) > \mathrm{row}(k) \text{ and } \mathrm{col}(k+1) = \mathrm{col}(k) + 1; \\ \text{or } \mathrm{col}(k+1) \le \mathrm{col}(k) \end{array} \right\}$$

Essentially, the extended sets partition X_n by placing each element x in I, J, or K based on its position in the corresponding tableau relative to the following element, $x + 1$.

[1] The definitions of these sets is in a portion of this research not publicly available but relayed to the author by Russell.

Example 1.8. Here we see the extended sets for a given row-strict tableau of size $n = 10$.

1	5	6
2	3	10
4	8	
7		
9		

$\mathrm{I} = \{4,7,9\}$ $\quad$ $\mathrm{J} = \{2,5\}$ $\quad$ $\mathrm{K} = \{1,3,6,8\}$

While it is a fairly straightforward task to construct the corresponding extended sets given a tableau, it is more challenging to determine whether a possible collection of extended sets corresponds to a valid tableau. Furthermore, if it does correspond to a tableau, does an algorithm exist that allows us to build said tableau? In this paper, we explore this question, and in doing so obtain some bounds on the number of extended sets for a fixed n, put forth some general results, and outline solutions to our question for some special cases.

2 Number of Possible Extended Sets

Our first question regarding these extended sets is: How many unique extended sets correspond to all possible row-strict tableaux of size n? We start with an obvious upper bound for this value, which is the total number of possible extended sets. This bound is achieved by finding the number of unique divisions of the elements of X_n into 3 subsets. Therefore, our first upper bound is 3^{n-1}.

We now introduce a theorem that will help us set a lower bound on this number of extended sets.

Theorem 2.1. *Let* I, J*, and* K *be disjoint sets which partition* X_n *and have* I *empty. These disjoint sets will be the extended sets for some valid row-strict tableau.*

Proof. Given disjoint sets, let I be empty. We will prove this corresponds to a valid row-strict tableau by construction. For any run $a, a+1, \ldots, b$ in J with $b+1 \in \mathrm{K}$, we construct a row of the tableau of length such that the labels in the row are $a, a+1, \ldots, b, b+1$. Then we order these rows such that the length of the rows weakly decreases down the tableau. Any remaining elements of $\{1, \ldots, n-1\}$ should all be in K. Place each of these in a row of length one below the already-constructed rows. The order of these rows will not matter. Then every element x at the end of a row, including those elements in rows of length one, will be in K, since $\mathrm{col}(x+1) = 1$ by construction, and therefore $\mathrm{col}(x+1) \le \mathrm{col}(x)$. Also, every other element will be in J by construction and the definition of J. Then we have a valid row-strict tableau with no elements in I. $\square$

Since we now know any possible extended sets with an empty I will always produce a valid tableau, the minimum number of extended sets for size n results from placing the elements $\{1, 2, \ldots, n-1\}$ into either J or K. Therefore, a lower bound on our value is 2^{n-1}.

We introduce another theorem to further narrow our bounds.

Theorem 2.2. *In any row-strict or standard tableau, the smallest element not in* J *must be in* K.

Proof. Let $1, 2, \dots, a$ be in J and $x = a+1$, $x \geq 1$, $x \notin$ J. Then by the definition of row-strict and the set J, $\text{row}(1) = \text{row}(2) = \dots = \text{row}(a) = \text{row}(x)$, and $\text{col}(1) = 1, \text{col}(2) = 2, \dots, \text{col}(a) = a, \text{col}(x) = x$. However, since $x \notin$ J, $\text{row}(x+1) \neq \text{row}(x)$, so $\text{col}(x+1) = 1$ by the definition of row-strict. Then $\text{col}(x+1) \leq \text{col}(x)$, and therefore $x \in$ K. □

This theorem allows us to exclude more possible extended sets and improve our upper bound. In order to count the number of possible extended sets to be excluded from our estimate, fix the elements $\{1, 2, \dots, m\}$ in J, and then suppose $m+1 \in$ I. When $m = n-2$, then $n-1 \in$ I, so that accounts for $3^0 = 1$ possible extended set that can be excluded. When $m = n-3$, then $n-2 \in$ I, and so $n-1$ can be in I, J, or K, which accounts for $3^1 = 3$ more possible extended sets to exclude. This continues until $m = 1$, which corresponds to 3^{n-3} excluded possible extended sets. Then lastly, we have the case when J is empty and $1 \in$ I. This accounts for 3^{n-2} possibilities. Then the number of sets excluded due to Theorem 2.2 is

$$3^0 + 3^1 + \cdots + 3^{n-3} + 3^{n-2} = \sum_{i=0}^{n-2} 3^i.$$

All of the above discussion gives us a new upper bound.

Theorem 2.3. *Let n be an integer such that $n \geq 2$. Then there are at most*

$$\frac{3^{n-1}+1}{2}$$

extended sets corresponding to row-strict tableau of size n.

Proof. From counting the number of collections of three disjoint sets partitioning $\{1, \dots, n-1\}$, we started with the bound 3^{n-1}. We can then remove those that we identified as impossible from Theorem 2.2 to get

$$3^{n-1} - \sum_{i=0}^{n-2} 3^i.$$

Since this is a geometric series, we have the compact formula of

$$\sum_{i=0}^{n-2} 3^i = \frac{3^{n-1}-1}{3-1} = \frac{3^{n-1}-1}{2}.$$

So we have

$$3^{n-1} - \frac{3^{n-1}-1}{2} = \frac{3^{n-1}+1}{2}.$$

□

We now present a table containing our lower bound, simple upper bound, improved upper bound, and actual number of unique extended sets for some relatively small values of n.

n	Lower Bound	First UB	Improved UB	Actual # of Unique Extended Sets
2	2	3	2	2
3	4	9	5	5
4	8	27	13	12
5	16	81	41	29

Figure 1: Bounds Versus Actual Number of Unique Extended Sets of Size n

At this time, those are the only actual values for the number of valid extended sets known. We paused our computations here because it would take 90 hand computations of possible extended sets to calculate for $n = 6$ based on our bounds. Alternatively, we could use a program to compute all of the extended sets for all of the row-strict tableaux of size 6. However, there are 1,602 such row-strict tableaux that would need to be generated for such a computation, and this number would grow quickly with n. Because of the connection to the partition function $p(n)$, the question of the asymptotic behaviour of the number of extended sets would be interesting to determine, but we did not explore that here.

3 Results for Row-strict and Standard Tableaux

Now that we have made some investigation into the number of these extended sets, we turn to our overall question of identifying properties that guarantee given extended sets correspond to a valid row-strict or standard tableaux. We were able to produce some general results in Theorem 2.1 and Theorem 2.2. Before we move on and present further results, let us first note some obvious properties of tableaux.

Proposition 3.1. *Let* T *be a row-strict tableau of size n and let* $x \in \mathrm{X}_n$. *If* $\mathrm{row}(x+1) = \mathrm{row}(x)$, *then* $\mathrm{col}(x+1) = \mathrm{col}(x)+1$.

This follows from the fact that in any row-strict tableau, the entries must increase across each row, so consecutive numbers in the same row must be adjacent.

Proposition 3.2. *Let* T *be a row-strict tableau. Then* $\mathrm{col}(1) = 1$.

This again follows from the fact that in any row-strict tableau, entries increase across the rows, and 1 is the least element in our set X_n.

Proposition 3.3. *Let* T *be a standard tableau of size* n *and let* $x \in X_n$. *Then* $\mathrm{col}(x+1) > \mathrm{col}(x)$ *if and only if* $\mathrm{row}(x+1) \le \mathrm{row}(x)$.

This property is a bit less obvious, so we will include a proof.

Proof. In order to see that this property holds, first let $\mathrm{col}(x+1) > \mathrm{col}(x)$ for an element $x \in X_n$ in a standard tableau T. If $\mathrm{row}(x+1) > \mathrm{row}(x)$, then for the elements of the tableau to increase across each row, there must be some element $y > x+1$ such that $\mathrm{row}(y) = \mathrm{row}(x)$, $\mathrm{col}(y) = \mathrm{col}(x+1)$, and $\mathrm{row}(y) < \mathrm{row}(x+1)$. This implies that the elements of the tableau decrease down a column, which is a contradiction since the tableau is standard. Thus, $\mathrm{row}(x+1) \le \mathrm{row}(x)$ as claimed.

Now take $\mathrm{row}(x+1) \le \mathrm{row}(x)$, and again for a contradiction, suppose $\mathrm{col}(x+1) \le \mathrm{col}(x)$. One possibility for this case is $\mathrm{row}(x+1) < \mathrm{row}(x)$ and $\mathrm{col}(x+1) \le \mathrm{col}(x)$, in which case the elements of the tableau decrease down a column, a contradiction. Another possibility is $\mathrm{row}(x+1) = \mathrm{row}(x)$ and $\mathrm{col}(x+1) < \mathrm{col}(x)$, in which case the elements of the tableau decrease across a row, another contradiction. The final possibility is $\mathrm{row}(x+1) = \mathrm{row}(x)$ and $\mathrm{col}(x+1) = \mathrm{col}(x)$. But then x and $x+1$ occupy the same box in the tableau, a contradiction. □

With these properties established, we are now able to state some general results regarding extended sets corresponding to standard and row-strict tableau. Our first result posits a relationship between an element in the first row of a standard tableau and its placement in the corresponding extended sets.

Theorem 3.4. *Let* x *be an element in a standard tableau such that* $\mathrm{row}(x) = 1$. *Then* $x \notin \mathrm{I}$.

Proof. For a contradiction, let $x \in \mathrm{I}$. In either condition for $x \in \mathrm{I}$, $\mathrm{col}(x+1) > \mathrm{col}(x)$. However, according to Property 3.3 for standard tableaux, if $\mathrm{col}(x+1) > \mathrm{col}(x)$, then $\mathrm{row}(x+1) \le \mathrm{row}(x)$. Since there is no row smaller than 1, the only option is $\mathrm{row}(x+1) = \mathrm{row}(x)$, in which case $x \in \mathrm{J}$ by Property 3.1 and the definition of J. Then we have a contradiction, and $x \notin \mathrm{I}$ if x is an element in Row 1 of a standard tableau. □

Our next theorem states the existence of a standard tableau of any shape and any size n such that I is empty in the corresponding extended sets and gives a way to build such a tableau.

Theorem 3.5. *There exists a specific labelling of a tableau of any shape that produces an empty* I *in the extended sets.*

Proof. Let T be a tableau of some fixed shape and let b be the length of the top row of T. Then fill the top row with $1, 2, \ldots, b$ in order. Place $b+1, b+2, \ldots$ in order in the second row, and continue filling that row in the same manner as Row 1. Continue this until the tableau is full. Then $1, 2, \ldots, b-1 \in \mathrm{J}$ by definition, and $b \in \mathrm{K}$ since $\mathrm{col}(b+1) \le \mathrm{col}(b)$. By similar logic, every element before the last spot in each row is in J, and every element at the end of each row is in K. Then I is empty for every such labelling. □

Note that this theorem is similar to Theorem 2.1, but Theorem 2.1 begins with a given collection of possible extended sets with I empty and proves the existence of a corresponding row-strict tableau, while Theorem 3.5 concerns the specific labelling of a given Young diagram in order to produce a standard tableau with a corresponding empty I set.

The next theorem and a corollary that follows are concerned with "runs" of consecutive integers in I and how they relate to the shape and size of the corresponding Young tableau. Before we state and prove these, we must introduce some notation. Define $\langle i, j\rangle$ as a list of consecutive integers from i to j inclusive, where $i \le j$. Also, define $|\langle i, j\rangle|$ as the number of consecutive integers in the run. In other words, $|\langle i, j\rangle| = j - i + 1$. We may now proceed with our results.

Theorem 3.6. *Let $\langle a, a+j\rangle$ be a run of consecutive integers in* I *for a row-strict tableau. Then the tableau must have at least $|\langle a, a+j\rangle|+1$ rows and $|\langle a, a+j\rangle|+1$ columns.*

Proof. To prove our statement, we will directly argue that if $\langle a, a+j\rangle$ is a run in I, then the labels $a, \ldots, a+j+1$ must be in distinct rows and columns of any row-strict tableau. We will use induction on the size of a run in I. We first consider the case of a run of integers in I that is size 1. Suppose this run is $\langle a\rangle$. If $a \in$ I by the first condition, then $\text{row}(a+1) < \text{row}(a)$ and $\text{col}(a+1) = \text{col}(a)+1$, in which case a and $a+1$ are in distinct rows and columns as needed. If $a \in$ I by the second condition, then $\text{col}(a+1) > \text{col}(a)+1$. Let $\text{row}(a+1) = \text{row}(a)$. Then by Property 3.1 of row-strict and standard tableaux, $\text{col}(a+1) = \text{col}(a)+1$, in which case $a \in$ J, a contradiction. So $\text{row}(a+1) \neq \text{row}(a)$, and again a and $a+1$ are in distinct rows and columns.

Now, for our induction hypothesis, suppose if $\langle a, a+j-1\rangle$ is a run in I for some $a \in \text{X}_n$ and $j \ge 1$ then $a, \ldots, a+j$ are in distinct rows and columns of any row-strict tableau. Consider now a run in I of length $j+1$. We will suppose then it is of the form $\langle a, a+j\rangle$ for some $a \in \text{X}_n$. By our induction hypothesis, $a, \ldots, a+j$ are in distinct rows and columns of the tableau, so we need only argue that $a+j+1$ is in a distinct row and column. First, since $a, \ldots, a+j \in$ I, we know $\text{col}(a+j+1) > \text{col}(a+j) > \cdots > \text{col}(a)$. So we know quickly that $a+j+1$ is in a distinct column. Now, suppose $\text{row}(a+j+1) = \text{row}(a+k)$ for some $0 \le k \le j-1$. (We will consider the case of $k = j$ separately.) Then $\text{col}(a+j+1) = \text{col}(a+k)+1$ because the tableau is row-strict and all possible y such that $a+k < y < a+j+1$ are in other rows according to our induction hypothesis. However, we know $\text{col}(a+j+1) > \text{col}(a+j)$ since $a+j \in$ I. Together $\text{col}(a+j+1) \ge \text{col}(a+j)+1$ and $\text{col}(a+j+1) = \text{col}(a+k)+1$ for $k \le j-1$ means $\text{col}(a+k) \ge \text{col}(a+j)$. This contradicts $\text{col}(a+j+1) > \text{col}(a+j) > \cdots > \text{col}(a)$, so we know it cannot be true. This only leaves the possibility that $\text{row}(a+j) = \text{row}(a+j+1)$. If this is true, we must have $\text{col}(a+j+1) > \text{col}(a+j)+1$ since $a+j \in$ I. However, this contradicts Property 3.1. Thus, each element $a, \ldots, a+j+1$ is in a distinct row and column, and the claim follows from induction.

□

While Theorem 3.6 is concerned with the shape of the resulting Young tableau given a certain condition, Corollary 3.7 puts forth a more general result regarding the size n of the tableau given the same condition.

Corollary 3.7. *If there is a run of length k in* I*, then* $n \geq \frac{(k+1)(k+2)}{2}$.

Proof. If there are k consecutive integers in I, there are $k+1$ elements each in a distinct column and distinct row, as shown by Theorem 3.6. Then a new consecutive integer in I will add at least one new column and one new row to the tableau. The first condition for $x \in \mathrm{I}$, which is $\mathrm{row}(i+1) < \mathrm{row}(i)$ and $\mathrm{col}(i+1) = \mathrm{col}(i)+1$, will produce the shape with the smallest n since a new consecutive integer will only add 1 new row and 1 new column. Adding new consecutive integers $a, a+1, a+2, \ldots \in \mathrm{I}$ in this fashion gives a “stair-step" shape that corresponds to the smallest possible n. This leads to a tableau with a top row of length $k+1$, second row of k, and so on for $k+1$ rows. Because this is optimal,

$$n \geq 1+2+3+\ldots+k+1 = \frac{(k+1)(k+2)}{2}.$$

□

Using Corollary 3.7, we are able to immediately dismiss certain possible extended sets as not corresponding to any valid standard or row-strict tableau given their I set. More specifically, if there are k consecutive integers in I for possible extended sets, but

$$n < \frac{(k+1)(k+2)}{2},$$

we know these do not correspond to a valid tableau, no matter the shape or labelling.

4 Hook Shape Tableaux

We now move into discussion on hook shape tableaux and answer our overarching question of which extended sets correspond to valid row-strict and standard tableaux in this specific case. In order to do so, we must first define hook shape.

Definition 4.1. A tableau T is a **hook shape** tableau if the partition of n corresponding to the tableau is of the form $[a\ 1^b]$, where $a \geq 1$, $b \geq 0$. In other words, every element x in a tableau of hook shape is such that either $\mathrm{row}(x) = 1$ and $\mathrm{col}(x) \geq 1$ or $\mathrm{col}(x) = 1$ and $\mathrm{row}(x) \geq 1$.

Before we are able to define the structure of extended sets that give us a valid hook shape tableau, we must introduce a new theorem.

Theorem 4.2. *For a row-strict tableau of hook shape, $x \in \mathrm{K}$ if and only if* $\mathrm{col}(x+1) = 1$.

Proof. Suppose first that an element $x \in X_n$ in in K for a row-strict tableau of hook shape. Then since there is only one column and one row with more than one element in a hook shape tableau, the condition for $x \in K$ must be the second, which is $\text{col}(x+1) \le \text{col}(x)$. If $\text{col}(x+1) = \text{col}(x)$, then both $\text{col}(x) = 1$ and $\text{col}(x+1) = 1$ because Column 1 is the only column with more than one element. If $\text{col}(x+1) < \text{col}(x)$, then $\text{col}(x) \neq 1$ because, if it was, then there would be no column such that $\text{col}(x+1) < \text{col}(x)$ since columns are labeled with positive integers. Since $\text{col}(x) \neq 1$, then $\text{row}(x) = 1$, $\text{col}(x) \ge 2$. Suppose $\text{row}(x+1) = 1$. Then, for the tableau to be row-strict, $\text{col}(x+1) = \text{col}(x)+1$ by Property 3.1, which puts $x \in J$, a contradiction to the fact that $x \in K$. Then $\text{row}(x+1) \neq 1$, and therefore $\text{row}(x) > 1$, which means $\text{col}(x) = 1$ since it's the only column with more than one row.

Now we can prove the converse statement. Suppose $x \in X_n$ and $\text{col}(x+1) = 1$ for a row-strict tableau. Then either $\text{col}(x) = 1$ as well, or $\text{col}(x) > 1$. In either case, $\text{col}(x+1) \le \text{col}(x)$, so $x \in K$ by definition. □

We also introduce one last bit of notation helpful for our full theorem.

Definition 4.3. Let T be a tableau of size n. We will say an element $x \in \{1, 2, \ldots, n\}$ in this tableau T is in **standard position** if $\text{row}(x) = 1$ and $\text{col}(x) = 1$.

We are now finally able to discuss which extended sets produce a valid standard or row-strict tableau of hook shape, and in turn how to build such a tableau from given extended sets. Two cases should be considered. The first is standard tableaux and row-strict tableaux with 1 in standard position. The second is row-strict tableaux with 1 not in standard position. We introduce the following forms of the possible extended sets for these two cases:

Form 1. $K = \{\langle a_0, b_1\rangle, \langle a_1, b_2\rangle, \langle a_2, b_3\rangle, \ldots, \langle a_{i-1}, b_i\rangle\}$ where $a_i > b_i + 1$ and $a_j \le b_{j+1}$
$I = \{b_1 + 1, b_2 + 1, b_3 + 1, \ldots, b_i + 1\}$
$J = X_n \backslash (K \cup I)$

Form 2. $K = \{\langle 1, b_1\rangle, \langle a_1, b_2\rangle, \langle a_2, b_3\rangle, \ldots, \langle a_{i-1}, b_i\rangle\}$ where $a_i > b_i + 1$ and $a_j \le b_{j+1}$
$I = \{b_2 + 1, b_3 + 1, \ldots, b_i + 1\}$
$J = X_n \backslash (K \cup I)$

Theorem 4.4. *Let T be a row-strict tableau of hook shape with 1 in standard position. Then the structure of the extended sets is of Form 1.*

Proof. Let $K = \{\langle a_0, b_1\rangle, \langle a_1, b_2\rangle, \langle a_2, b_3\rangle, \ldots, \langle a_{i-1}, b_i\rangle\}$ in a standard or row-strict tableau of hook shape, with $a_i > b_i + 1$. Any extended sets will have a K set of this form. Then $\text{col}(\{b_1+1, b_2+1, b_3+1, \ldots, b_i+1\}) = 1$ by Theorem 4.2 and $\text{row}(\{b_1+1, b_2+1, b_3+1, \ldots, b_i+1\}) > 1$ since 1 is in standard position. Since $\{b_1+1, b_2+1, b_3+1, \ldots, b_i+1\} \notin K$, $\{b_1+2, b_2+2, b_3+2, \ldots, b_i+2\}$ are not in Column 1 and therefore $\text{row}(\{b_1+2, b_2+2, b_3+2, \ldots, b_i+2\}) = 1$

and $\mathrm{col}(\{b_1+2, b_2+2, b_3+2, \ldots, b_i+2\}) > 1$. It follows that for each b_i, either $\mathrm{row}(b_i+2) < \mathrm{row}(b_i+1)$ and $\mathrm{col}(b_i+2) = \mathrm{col}(b_i+1)+1$ or $\mathrm{col}(b_i+2) > \mathrm{col}(b_i+1)+1$, so by definition, $\{b_1+1, b_2+1, b_3+1, \ldots, b_i+1\} \in \mathrm{I}$.

Now choose some element x such that $x \notin \mathrm{K}$ and x is not in the I set outlined above, and let $\mathrm{row}(x) > 1$, so $\mathrm{col}(x) = 1$. In this case, one option is $\mathrm{col}(x+1) = 1$, in which case $\mathrm{col}(x+1) = \mathrm{col}(x)$, and $x \in \mathrm{K}$ by definition, which is contradiction. The other option for $x+1$ is $\mathrm{col}(x+1) > 1$, which means $\mathrm{row}(x+1) = 1$ by definition of hook shape. Then $x \in \mathrm{I}$ by definition, another contradiction. So $\mathrm{row}(x) = 1$. If $\mathrm{row}(x+1) > 1$ and therefore $\mathrm{col}(x+1) = 1$, then $x \in \mathrm{K}$ by 4.2, another contradiction. Then $\mathrm{row}(x+1) = 1$, and by 3.1, $\mathrm{col}(x+1) = \mathrm{col}(x)+1$, so $x \in \mathrm{J}$ by definition.

□

Theorem 4.5. *The extended sets that correspond to a row-strict tableau of hook shape with 1 not in standard position can take either Form 1 or Form 2.*

Proof. Note that $1 \notin \mathrm{I}$ by Theorem 2.2. If 1 is not in standard position, then $1 \notin \mathrm{J}$ as well. Thus, $1 \in \mathrm{K}$. Then the only other difference between these two forms is the location of b_1+1 in the extended sets. The first form is the case where $b_1+1 \notin \mathrm{J}$, which is of the same form as Theorem 4.4 and follows a similar argument for its proof. The second case occurs when $b_1+1 \in \mathrm{J}$, In this case, b_1+1 is in Column 1 by Theorem 4.2. Also, the condition for $b_1+1 \in \mathrm{J}$ is $\mathrm{row}(b_1+2) = \mathrm{row}(b_1+1)$ and $\mathrm{col}(b_1+2) = \mathrm{col}(b_1+1)+1$, and the only element in Column 1 and also in a row with length greater than 1 is in Row 1. Therefore, for Case 2, $b_2+1, b_3+1, \ldots, b_i+1 \in \mathrm{I}$ follows the same logic as outlined in the proof for Theorem 4.4, except b_1+1 is in standard position instead of 1. □

To build a hook shape tableau of Theorem 4.4 type with valid extended sets of the form above, place 1 in standard position. Then place the elements $x+1$ such that $x \in \mathrm{K}$ in Column 1, in ascending order down the column if the tableau is standard, or in any order if the tableau is row-strict. Then place the remaining elements in ascending order in Row 1.

To build a tableau of the form outlined in Theorem 4.5, first place the element b_1+1 in standard position if the extended sets follow the second case, i.e. if $b_1+1 \in \mathrm{J}$. If $b_1+1 \in \mathrm{I}$, place another element $x+1$ such that $x \in \mathrm{K}$ in standard position. Then build the tableau in the same manner as above, placing the elements in Column 1 in any order since a tableau of hook shape with 1 not in standard position will never be standard.

Example 4.6. The following is a hook shape, row-strict tableau of size $n = 12$ and Theorem 4.5, Case 2 type and its corresponding extended sets.

5	6	8	10	11	12
3					
1					
4					
7					
9					
2					

$\mathrm{I} = \{7, 9\}$ $\mathrm{J} = \{5, 10, 11\}$ $\mathrm{K} = \{1, 2, 3, 4, 6, 8\}$

5 Two-Row Tableaux

We now answer our questions regarding valid extended sets and construction from these extended sets for another special type of tableau: two-row, standard tableaux. We begin by defining these tableaux.

Definition 5.1. A tableau T is a **two-row** tableau if the partition of n corresponding to the tableau is of the form $[a\ b]$, where $a, b \geq 1$.

We now restate an old theorem and put forth a new theorem that relates an element's position in a two-row, standard tableau and its position in the corresponding extended sets. While these theorems are concerned mainly with construction of a tableau, they will be useful later in proving theorems regarding the format of the extended sets for two-row, standard tableaux. First, recall the statement of Theorem 3.4:

Let x be an element in T, a standard tableau. If $\mathrm{row}(x) = 1$, then $x \notin \mathrm{I}$.

The contrapositive of this theorem applied to the two-row case produces a perhaps more useful result for our goal. We'll state this as a corollary to Theorem 3.4.

Corollary 5.2. *If* $x \in \mathrm{I}$ *in a two-row, standard tableau, then* $\mathrm{row}(x) \neq 1$*, so* $\mathrm{row}(x) = 2$.

The next theorem is of a similar form to Theorem 3.4.

Theorem 5.3. *Let x be an element in* T*, a standard tableau with two rows. If* $\mathrm{row}(x) = 2$*, then* $x \notin \mathrm{K}$.

Proof. Let $\mathrm{row}(x) = 2$. The first condition for $x \in \mathrm{K}$ is $\mathrm{col}(x+1) = \mathrm{col}(x)+1$ and $\mathrm{row}(x+1) > \mathrm{row}(x)$, but there is no row greater than 2, so this condition fails. Furthermore, this means $\mathrm{row}(x+1) \leq \mathrm{row}(x)$ The second condition for $x \in \mathrm{K}$ is $\mathrm{col}(x+1) \leq \mathrm{col}(x)$. However, by Property 3.3 of standard tableaux, $\mathrm{row}(x+1) \leq \mathrm{row}(x)$ if and only if $\mathrm{col}(x+1) > \mathrm{col}(x)$, so this condition fails as well. Then $x \notin \mathrm{K}$. □

Again, the contrapositive of this theorem gives another useful result: If $x \in \mathrm{K}$, then $\mathrm{row}(x) \neq 2$, so $\mathrm{row}(x) = 1$.

Using these results, we can now introduce three new theorems that define the format that the extended sets for a valid two-row, standard tableau must take.

Theorem 5.4. *A two-row, standard tableau has no consecutive integers in* I *or* K.

Proof. Let x be an element in I. Then by the contrapositive of Theorem 3.4 above, $\mathrm{row}(x) \neq 1$, so $\mathrm{row}(x) = 2$. If $\mathrm{row}(x+1) = \mathrm{row}(x)$, then $\mathrm{col}(x+1) = \mathrm{col}(x)+1$ by Property 3.1 of standard tableaux, in which case $x \in \mathrm{J}$, which is a contradiction. Then $\mathrm{row}(x+1) \neq \mathrm{row}(x)$, so $\mathrm{row}(x+1) = 1$. Then by Theorem 3.4, $x+1 \notin \mathrm{I}$.

Let x be an element in K. Then by the contrapositive of Theorem 5.3 above, $\mathrm{row}(x) \neq 2$, so $\mathrm{row}(x) = 1$. If $\mathrm{row}(x+1) = \mathrm{row}(x)$, then $\mathrm{col}(x+1) = \mathrm{col}(x)+1$ by Property 3.1 of standard tableaux, in which case $x \in \mathrm{J}$, which is a contradiction. Then $\mathrm{row}(x+1) \neq \mathrm{row}(x)$, so $\mathrm{row}(x+1) = 2$. Then by Theorem 5.3, $x+1 \notin \mathrm{K}$. □

The next two theorems further specify the format of the extended sets in this case.

Theorem 5.5. *Let T be a two-row, standard tableau and x be an element in* I. *If there is a run $\langle x+1, b\rangle \in \mathrm{J}$, where $x+1 \leq b$, such that $\langle x+1, b+1\rangle$ is not also a run in* J, *then $b+1 \in \mathrm{K}$. Otherwise, $x+1 \in \mathrm{K}$.*

Proof. Let $x \in \mathrm{I}$. Then by the contrapositive of Theorem 3.4, $\mathrm{row}(x) \neq 1$, so $\mathrm{row}(x) = 2$. If $\mathrm{row}(\langle x+1, b\rangle) = 2$, then $x \in \mathrm{J}$ because $\mathrm{row}(x+1) = \mathrm{row}(x)$ and $\mathrm{col}(x+1) = \mathrm{col}(x)+1$ since the tableau is standard, which is a contradiction, so $\mathrm{row}(\langle x+1, b\rangle) = 1$. Then $\mathrm{row}(b+1) = 1$ by the definition of J. Thus, by Theorem 3.4 above, $b+1 \notin \mathrm{I}$, and $b+1 \notin \mathrm{J}$, so $b+1 \in \mathrm{K}$.

If no such run $\langle x+1, b\rangle \in \mathrm{J}$ exists, then $x+1 \notin \mathrm{J}$. Additionally, $x+1 \notin \mathrm{I}$ by Theorem 5.4 above. Therefore, $x+1 \in \mathrm{K}$. □

Theorem 5.6 and its proof are of a similar form to Theorem 5.5 and its proof.

Theorem 5.6. *Let T be a two-row, standard tableau and x be an element in* K. *If there is a run $\langle x+1, b\rangle \in \mathrm{J}$, where $x+1 \leq b$, such that $\langle x+1, b+1\rangle$ is not also a run in* J, *then $b+1 \in \mathrm{I}$. Otherwise, $x+1 \in \mathrm{I}$.*

Proof. Let $x \in \mathrm{K}$. Then by the contrapositive of Theorem 5.3, $\mathrm{row}(x) \neq 2$, so $\mathrm{row}(x) = 1$. If $\mathrm{row}(\langle x+1, b\rangle) = 1$, then $x \in \mathrm{J}$ because $\mathrm{row}(x+1) = \mathrm{row}(x)$ and $\mathrm{col}(x+1) = \mathrm{col}(x)+1$ since the tableau is standard, which is a contradiction, so $\mathrm{row}(\langle x+1, b\rangle) = 2$. Then $\mathrm{row}(b+1) = 2$ by the definition of J. Thus, by Theorem 5.3 above, $b+1 \notin \mathrm{K}$, and $b+1 \notin \mathrm{J}$, so $b+1 \in \mathrm{I}$.

If no such run $\langle x+1, b\rangle \in \mathrm{J}$ exists, then $x+1 \notin \mathrm{J}$. Additionally, $x+1 \notin \mathrm{K}$ by Theorem 5.4 above. Therefore, $x+1 \in \mathrm{I}$. □

Our last theorem gives a result regarding the construction of a two-row, standard tableau. By construction, we mean creating a Young tableau by placing the numbers $1, 2, \ldots, n$ into a Young diagram in increasing order.

Theorem 5.7. *The bottom row of a two-row standard tableau will never exceed the length of the top row during construction of the tableau.*

Proof. Take a standard, two-row tableau partway through its construction in which the bottom row is longer than the top row. Then there exists an element x in the top row such that $x+1$ is in the bottom row, and $\text{col}(x+1) > \text{col}(x)$. Since a valid, completed tableau must have a top row with a length greater than or equal to that of the bottom row, at some point during construction, an element y such that $y > x+1$ must be placed in the top row such that $\text{col}(y) = \text{col}(x+1)$. Then the tableau decreases down columns, which means it's standard, and we have a contradiction, so the bottom row of a two-row standard tableau can never exceed the length of the top row during construction. □

If the conditions given in Theorems 5.4-5.6 are met, we may begin to try to construct a two-row, standard tableau. Start by placing 1 in Column 1, Row 1 since the tableau is standard. Continue across each row with the elements $2, 3, \ldots, n$ by placing the element in Row 2 if the element is in I in the extended sets and in Row 1 if the element is in K in the extended sets (by Theorems 3.4 and 5.3). For each run $\langle x, b\rangle \in \text{J}$ such that $x \le b$ and $\langle x, b+1\rangle$ is not a run in J, look at the element $b+1$. If $b+1$ is in K, place $\langle x, b\rangle$ in Row 1. If $b+1$ is in I, place $\langle x, b\rangle$ in Row 2. Continue with this pattern. If ever the length of the bottom row exceeds the length of the top row, then by Theorem 5.7, the given extended sets are invalid and no standard tableau can be constructed.

Example 5.8. The following is a two-row, standard tableau of size $n = 15$, along with its corresponding extended sets. Note that the format of the extended sets follows the conditions set by Theorems 5.4-5.6 and that the positions of the elements in the tableau follow from Theorems 3.4 and 5.3. Also note that Theorem 5.7 holds as the tableau is being constructed.

1	2	5	6	9	10	12	14	15
3	4	7	8	11	13			

$\text{I} = \{4, 8, 11, 13\}$ $\quad$ $\text{J} = \{1, 3, 5, 7, 9, 14\}$ $\quad$ $\text{K} = \{2, 6, 10, 12\}$

6 Conclusion

In this research regarding Young tableaux and their extended sets, we've been able to produce some general results for standard and row-strict tableaux and made counting arguments for the number of possible extended sets for a tableau size n. We've also been able to answer our overarching question in some special cases by specifying the format of extended sets for valid hook shape and two-row tableaux, and in turn discussed how to build such tableaux.

The main question of being able to tell if extended sets correspond to any valid row-strict tableaux is still unanswered, a lot of which has to do with the large number of possible unique sets, along with the number of possible configurations of a tableau of a given size. Some progress has been made in solving the two-row, row-strict case, and an algorithm has been proposed for building a standard tableau given extended sets we know are valid. However, our overarching problem remains unsolved and requires further research.

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