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Using Differential Equations to Model a Cockatoo on a Spinning Wheel as part of the SCUDEM V Modeling Challenge

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Cover Page Footnote

We would like to thank our coach Anthony Stefan for giving us guidance on mathematical writing. We would also like to thank Dr. Brian Winkel for inviting us to present at the 2021 SIMIODE EXPO. Finally, we thank SCUDEM for the valuable experience outside of the classroom for both years that we participated.

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Using Differential Equations to Model a Cockatoo on a Spinning Wheel as part of the SCUDEM V Modeling Challenge

By *Miles Pophal*, *Henry Bae*, and *Chenming Zhen*

Abstract. For the SCUDEM V 2020 virtual challenge, we received outstanding distinction for modeling a bird perched on a bicycle wheel utilizing the appropriate physical equations of rotational motion. Our model includes both theoretical calculations and numerical results from applying the Heaviside function for the swing motion of the bird. We provide a discussion on: our model and its numerical results, the overall limitations and future work of the model we constructed, and the experience we had participating in SCUDEM V 2020.

1 Introduction

As part of the SCUDEM V challenge, students are given three weeks to model real-life scenarios in either the life sciences, social sciences, or the physical sciences using differential equations. We selected the physics problem due to the team's interest in working on the unique modeling problem and our strengths in the physical sciences (for the problem statement we selected, see Appendix 7, and for all problems, see [1]) . The provided video [4] (also see Figure 1) shows a cockatoo with its claws latched onto the outer edge of a bicycle wheel frame, it begins to fall down, then spin around and back up by tucking in or extending its body. Our goal was to develop a mathematical equation describing the motion of the bird on this wheel. Below in Figure 1 we see the cockatoo at two positions on the bicycle wheel.

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Keywords. differential equations, modeling, competition

(a) Cockatoo Extended (b) Cockatoo Tucked

Figure 1: Cockatoo in extended (left) and tucked (right) positions along the bicycle wheel.

The video [4] shows the Cockatoo starting at the apex (i.e., the top) of the wheel and initializing a rotation with some movement. It maintains this form until it's located at the bottom of the wheel and then it tucks in until the apex again, where it repeats this process/motion. In Section 2, we seek to explain the problem concisely and apply natural assumptions for an effective model. In Subsection 2.1 we model the scenario with equations of motion and then reduce our model to an idealized case. Then in Subsection 2.2, we obtain an equation for a numerical estimate for the maximal speed of the bird. Next in Subsection 2.3, we demonstrate the model via simulations in MATLAB's Simulink software with Simscape libraries using different initial conditions and time intervals. We discuss in Section 3 the limitations to our model as well as what future directions we could take to improve our model. Finally in Section 4, we provide a discussion on the overall virtual SCUDEM experience by giving personal testimonies from each team member. The video of our presentation that received *outstanding* distinction for the SCUDEM V challenge is available online at [2].

2 Bird Perched on a Bicycle Wheel

When the bird is on the wheel it maneuvers itself in order to continue its motion. We can liken this to how a person swings on a swing, where we tuck and extend our legs depending on our location. For our purposes, we consider an apparatus consisting of a small mass and a piston which can extend and contract to change its center of mass, thereby changing the moment of inertia of the system. It is important that our piston changes state based on its position, as that mimics the swing we are familiar with, but as we find out later this exact position is difficult to find based on time. In order to model the system effectively we take the approach of using torques to derive our model's differential equations.

2.1 Assumptions and Our Model

The following are our assumptions for representing both the physical and the mathematical aspects of our model. First, we assume the wheel and apparatus are a rigid body system (i.e., it does not deform during the physical motion of our system) and we are only modeling the natural motion of the apparatus in the clockwise direction. Second, we assume the time between piston states has a negligible impact on the motion (i.e., the system exerts instantaneous transition). Next, we assume an external torque is generated when the apparatus extends (e.g., the cockatoo in Figure 1), so there exists forces acting against our system and some forces acting along with it at times, as shown in the free body diagram of Figure 2 below.

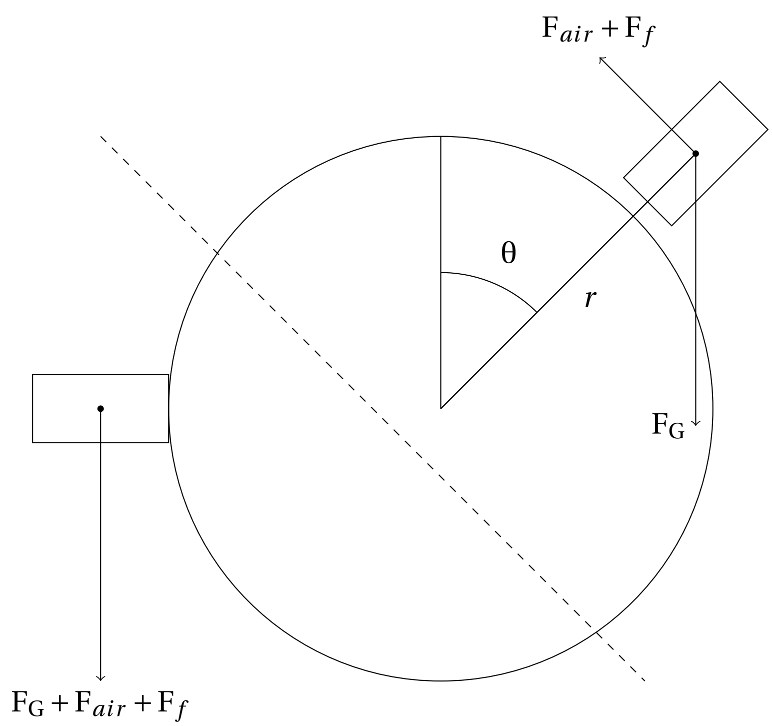


Figure 2: Free body diagram of mass on a wheel.

Our next assumption is that the points of action of the forces acting on the mass are close enough together to be counted as the same. We can see Figure 2 has forces for gravity F_G, air resistance F_{air} and friction F_f, which are all of the forces we consider since the cockatoo generates motion from the force of gravity pulling the wheel down. The right hand side of the dashed line shows a moment in time where the apparatus is extended, with the force of gravity in a downward direction. On the left hand side of the dashed line, at some other time, the apparatus tucks in as gravity goes against the upward motion.

Before we proceed to our model's governing equation, we use the following equations of motion from [3].

- Linear velocity v and acceleration a:

$$v = r(\theta)\frac{d\theta}{dt}, \quad a = r(\theta)\frac{d^2\theta}{dt^2} = r(\theta)\alpha.$$

- Angular position $\theta(t)$, measured from the vertical position as shown in Figure 2.
- Center of mass coordinate of the apparatus:

$$x(\theta) = r(\theta)\sin(\theta), \quad y(\theta) = r(\theta)\cos(\theta).$$

  Here, $r(\theta)$ is the distance from the axis of rotation (the axle) to the center of mass of the whole system.

- General torque:

$$\tau = \vec{r} \times \vec{\mathrm{F}} \Rightarrow |\tau| = |r||\mathrm{F}|\sin(\theta).$$

We know from [3, p. 307] that the equation of motion we are interested in is

$$\sum \tau_i = \mathrm{I}\alpha \iff \sum \tau_i - \mathrm{I}\alpha = \mathbf{0}, \tag{1}$$

which allows us to recover a differential equation in $\theta(t)$ angle as a function of time. Since the torques are from a cross product and the zero vector is the only vector uniquely determined by its magnitude we can rewrite (1) as follows

$$\sum \mathrm{sgn}_i |\tau_i| - \mathrm{I}|\alpha| = 0, \tag{2}$$

where the sgn term represents the torque acting with or against our motion. Since most of these torques are perpendicular to the the apparatus, we can deduce $|\tau| = |r||\mathrm{F}|$. Now, we characterize torques starting with the ones that always resist motion, such as the coefficients for air resistance k and friction μ. Air resistance is usually proportional to the square of linear velocity, or

$$|\tau_{air}| = r(\theta)(kv^2) = kr^3\left(\frac{d\theta}{dt}\right)^2.$$

Similarly, friction is taken to be proportional to the normal force F_n. Thus,

$$|\tau_f| = \mu r \mathrm{F}_n = \mu m v^2 = m\mu r^2\left(\frac{d\theta}{dt}\right)^2.$$

We consider a torque by gravity, but this changes depending on the the location of the apparatus (see Figure 2). Therefore, we use the usual equation

$$|\tau_g| = mgr\sin(\theta).$$

Finally, we calculate the net torque $I\alpha$ by using the moment of inertia formula

$$I\alpha = I\frac{d^2\theta}{dt^2} = mr^2\frac{d^2\theta}{dt^2},$$

where m represents the mass of the system. Now all that remains is to plug these equations back into (2), hence,

$$-mr^2(\theta)\frac{d^2\theta}{dt^2} - [kr^3(\theta) + m\mu r^2(\theta)]\left(\frac{d\theta}{dt}\right)^2 + mgr(\theta)\sin(\theta) = 0.$$

Notice we still have not defined $r(\theta)$ yet, but we assume it is nonzero and reduce our equation by one factor to get our second order ordinary differential equation (ODE) that models our system:

$$-mr(\theta)\frac{d^2\theta}{dt^2} - [kr^2(\theta) + m\mu r(\theta)]\left(\frac{d\theta}{dt}\right)^2 + mg\sin(\theta) = 0. \tag{3}$$

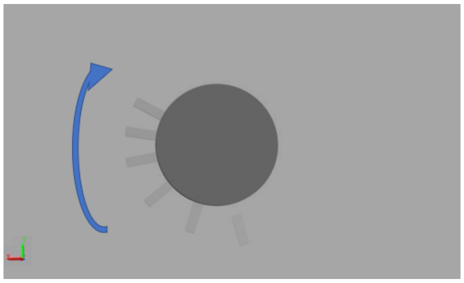
By our assumption that the air resistance and friction in the system are negligible, since our problem is just a bird on a bicycle wheel, we consider a simplified model in the following ideal case when these terms (k and μ) vanish

$$r(\theta)\frac{d^2\theta}{dt^2} = g\sin(\theta). \tag{4}$$

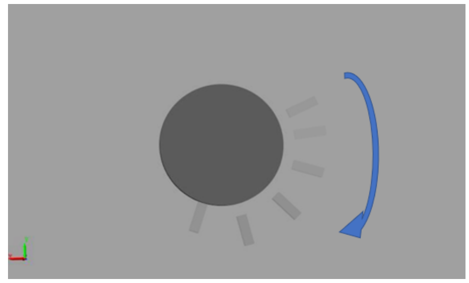
We discuss in Subsection 2.2 the maximal velocity in both equations (3) and (4), which later helps determine our maximal speed as part of the SCUDEM problem. We expect there to be no closed form solution as our ODE (3) is autonomous and nonlinear, although homogeneous. With no general theory to solve this ODE type, we will later turn to numerical results in Subsection 2.3. Finding the radius function is beyond the scope of this paper (due to the complexity of variational calculus), but we can represent it by the following Heaviside function

$$r(\theta) = r_1 + r_2 H(\sin(\theta)) = \begin{cases} r_1 & -1 \le \sin(\theta) \le 0, \\ r_1 + r_2 & 0 < \sin(\theta) \le 1, \end{cases}$$

where r_1 and r_2 are chosen such that r_1 is the distance from the axle to the center of mass for the contracted apparatus and $r_1 + r_2$ is the distance from the axle to the center of mass for the extended apparatus. This Heaviside function represents the change in the objects radius affecting energy on the descent by increasing the radius and minimizes the energy lost on the ascent by decreasing the radius (the object 'tucking in') and is a well defined function since sin has range $[-1, 1]$.



(a) Contraction



(b) Extension

Figure 3: Apparatus extension illustration.

To visualize the motion of the apparatus contraction and extension we used MATLAB Simulink software with the help of Simscape libraries. To model our system in Simulink we created a gravity field, a revolving wheel with internal friction damping, and an extendable rectangular block. The block follows the motion described in the Heaviside function with a force proportional to angular velocity squared simulating the air resistance. With the built-in ODE solver, the motion of the rectangular block simulating the bird does match our expectations. In Figure 3, with one frame per second superimposed, we illustrate the apparatus contraction and extension. In (a) one can see the rectangular block's shadow decelerates on ascent by contracting and in (b) accelerates on descent by extending. As the system gains energy with every new revolution, we are confident in our assumptions and the overall model, which we justify with numerical simulations in Subsection 2.3.

2.2 Maximal Speed

In this section we investigate whether a maximum speed is reached on the time interval $[0, t]$. We know a maximal angular speed is reached on this interval with $\theta(t)$ being continuous on a compact set. The maximum must occur on either the boundary or the interior, where the derivative (angular acceleration) is 0. In our problem statement there was no final time requested to approximate the maximal speed and we have to define the velocity at $t = 0$ as an initial condition for our ODE. Since we can set the angular acceleration to 0 in the interior, we solve for angular velocity as

$$-[kr^2(\theta) + m\mu r(\theta)]\left(\frac{d\theta}{dt}\right)^2 + mg\sin(\theta) = 0,$$

and with some rearranging, we have

$$\frac{d\theta}{dt} = \pm\sqrt{\frac{mg\sin(\theta)}{kr^2(\theta) + m\mu r(\theta)}}.$$

Although this quantity can be a negative value we are only interested in the magnitude, and hence we only consider its positive value. Furthermore, we include the radius factor for linear velocity as

$$v = r(\theta)\frac{d\theta}{dt} = r(\theta)\sqrt{\frac{mg\sin(\theta)}{kr^2(\theta) + m\mu r(\theta)}} = \sqrt{\frac{mgr^2(\theta)\sin(\theta)}{kr^2(\theta) + m\mu r(\theta)}},$$

which simplifies down to

$$v = \sqrt{\frac{mg\sin(\theta)}{k + \frac{m\mu}{r(\theta)}}}.$$

In order to maximize the speed, we take $\sin(\theta) = 1$ and maximize $r(\theta)$ to get

$$v_{max} = \sqrt{\frac{mg}{k + \frac{m\mu}{r_1 + r_2}}}. \tag{5}$$

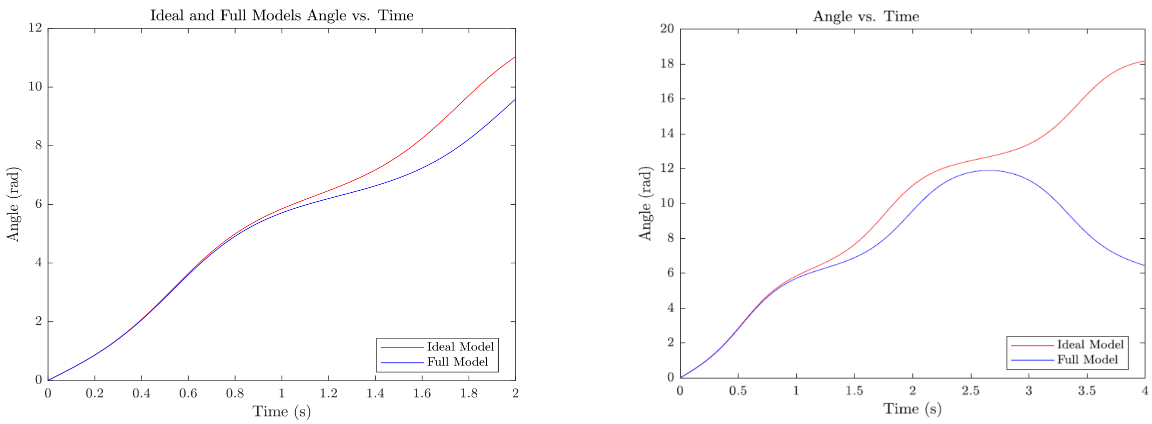
We would like to mention that v_{max} in equation (5) applied to the ideal model of equation (4) causes the denominator to be 0, which we would expect from the ideal physical model (i.e., that the speed is unbounded). We obtain a bound on the speed in the next section from the conditions we set and the formulas derived in this section (i.e., $v_{max} = 16.8575\,\mathrm{m/s}$).

2.3 Numerical Estimates

In this section we test our model with numerical simulations. First, we define the following conditions:

- $r_1 = 0.7\,\mathrm{m}$
- $r_2 = 0.1\,\mathrm{m}$
- $m = 5\,\mathrm{kg}$
- $g = 9.8\,\mathrm{m\,s^{-2}}$
- $k = 0.001\,\mathrm{kg\,m^{-1}}$
- $\mu = .01$
- $\theta(0) = 0\,\mathrm{rad}$
- $\frac{d\theta}{dt}(0) = 4\,\mathrm{rad\,s^{-1}}$

and initially let $0 \le t \le 2$ seconds. We also consider a longer time interval of $0 \le t \le 4$. We choose the initial velocity at $4\,\mathrm{rad\,s^{-1}}$ to illustrate the divergence in the full model and ideal the model. To ensure close numerical estimates coming from our conditions we used a fourth order Runge-Kutta method to simulate our system. We plot $\theta(t)$ as the time ranges across the two intervals, as shown in Figure 4 below.



(a) Shorter time interval

(b) Longer time interval

Figure 4: Comparison of time intervals for the full model (blue) and the ideal model (red).

The results of our simulation are what we would expect, where as time increases there are more resisting forces on the full model compared to the ideal model. At longer time intervals our numerical scheme becomes less reliable, which we discuss in more detail in Section 3. This unreliability is further supported by a plot of $\cos(\theta(t))$ and time, which gives an indication of the $y-$coordinate of motion, as seen in Figure 5.

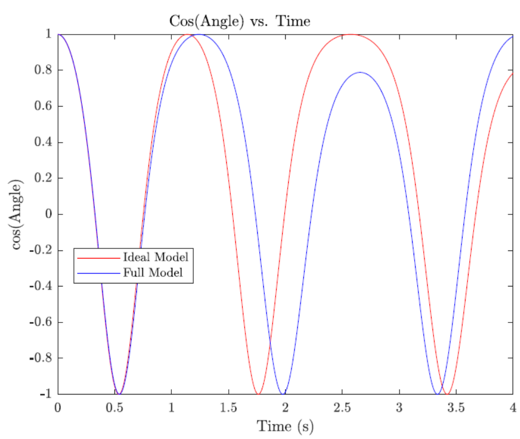
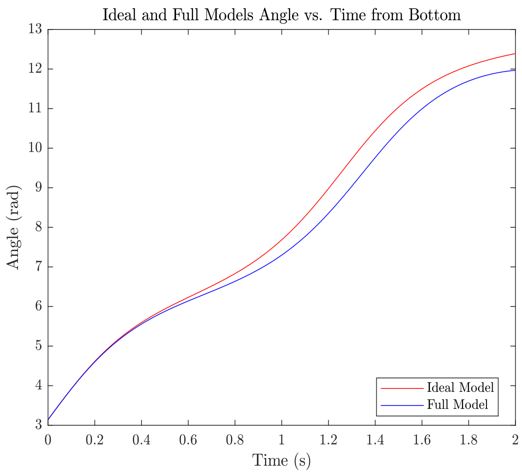
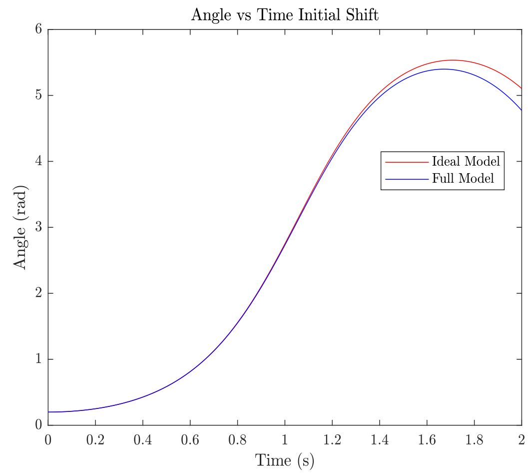


Figure 5: Numerical estimates of $\cos(\theta)$ for the full model (blue) and the ideal model (red).

At $t = 4$ we begin to see a decline in the peak of the ideal model y-coordinate, where the peak corresponds to the extension of the apparatus, this is unexpected behavior and validates that longer time estimates become unreliable. One explanation may be the imperfect correspondence of the angle $\theta(t)$ keeping track of revolutions while $\cos(\theta(t))$ loses this information. We also tested the possibility of regaining motion from different starting positions and initial velocities, as outlined in Figure 6 (a) and (b).



(a) $\theta(0) = \pi$ rad, $\frac{d\theta}{dt}(0) = 8\,\mathrm{rad\,s^{-1}}$.



(b) $\theta(0) = 0.2\,\mathrm{rad}$, $\frac{d\theta}{dt}(0) = 0$.

Figure 6: Different initial conditions comparison.

In Figure 6 (a) we observe the expected increase after increasing the initial velocity from $4\,\mathrm{rad\,s^{-1}}$ to $8\,\mathrm{rad\,s^{-1}}$. In Figure 6 (b) we observe slower motion by removing initial velocity to 0, hence, the simulation failed to perform a full revolution. We note that the apex point of the wheel $[\theta(t) = 0]$ is unstable as a slight deviation in either direction starts motion. In order to ensure our apparatus only starts motion with nonzero initial conditions we tested the scheme with no initial displacement or velocity. Here we expect no motion at all where both models should be a single point at the top of the wheel, and as we expect, Figure 7 illustrates the axis of rotation (axle), and the position remaining unchanged.

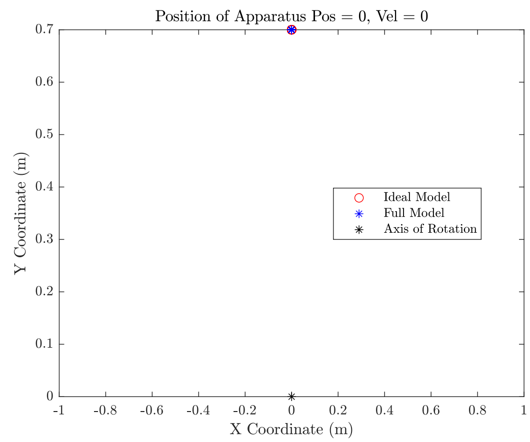


Figure 7: Stationary test case at $\theta(t) = 0$.

3 Limitations and Future Directions

There are various limitations to our model. First, the actual friction coefficient and air resistance coefficient can only be obtained through repeated experimentation, not by our simulation. Second, longer intervals cause our numerical estimates to become unreliable because we used a fourth-order scheme with local error $O(h^4)$. For example, when we doubled our interval length we potentially introduced up to 16 times the error. Third, the center of gravity and the moment of inertia of the system was assumed to be piecewise constant (from our Heaviside function) but in reality the bird would change its radial weight distribution. Lastly, the bird tucks and extends based on position and not its velocity. This is a limitation as when the bird can not make a full rotation, our model does not act to gain the energy in order to compensate for the failed rotation. For future directions, data collection from experiments and including them into our simulations would produce a more accurate simulation to test our model. We are also interested in how different numerical schemes, which offer more robust approximations over longer intervals, would compare to the 4th order Runge-Kutta scheme.

4 Personal Testimonies

"This wasn't my first time participating in the SCUDEM challenge, and one thing I love is the absolute freedom you have in how you approach the problem. Last year, I tackled a problem with a discrete difference equation instead of a differential equation. This year there were many options, for instance, I initially thought about approaching our problem with a work-energy approach but that would have resulted in integrals all over the place, but when my teammates suggested working with torques, our model popped into my head and I realized they suggested the perfect approach to the problem. After that, everything clicked and the natural progression felt great. The SCUDEM challenge during the pandemic was an interesting experience because we didn't have to travel to present our model, but we had to work through presenting in an online environment. It's harder to give body language in the form of support or an indication for progression between my team and the audience while presenting online. We still made it work and I'm glad we took part in the SCUDEM challenge. It's such a great feeling to have your model on your mind constantly. I found myself thinking about what it might miss or might capture incorrectly and I enjoyed thinking about it, knowing it wasn't some homework for class. You can gain great appreciation for real work, look back, and be proud of what you did."

-Miles Pophal, Applied Mathematics Major

"I have participated in last year's competition and both times I worked on the physics related problems. Both problems really interested me, but comparing to the last year's asteroid landing problem, the birding perching on a bicycle wheel is more down to earth. We could have even just found a bicycle wheel and experimented our model if the time permitted. What amazes me the most about the SCUDEM challenge is that a wide variety of applications are brought to students' attention and the participants get a feel of doing applications. Since I am pursuing an Aerospace Engineering degree as my second major, the asteroid landing problem piqued my interest, therefore, I was excited to participate again in the SCUDEM challenge. What I enjoy about SCUDEM the most is how I use what I've learned in my classes to solve problems that are practical in real life and interesting to me, which are more rich than textbook problems."

-Chenming Zhen, Mathematical Science/Aerospace Engineering Major

"I loved the teamwork aspect of the SCUDEM Challenge. Modeling a problem together is a transformative experience compared to independent problem solving as done in many traditional classrooms. Having the chance to work with familiar teammates we already knew each others strengths and weaknesses. For instance, Miles was familiar with LaTeX for formatting the presentation while Chenming was familiar with Simulink for modeling the motion of the bird. It was also an advantage for us to be in the same area, and Florida Tech was generous enough to allow us to work together in a quiet area for extended periods of time. My favorite aspect of working on this problem was having discussions about approaching and modeling the problem. Every step of the way, we would have each one of us present their ideas to the group. We would then challenge each others ideas, then conclude which one of them we thought would work best. This was an effective process to check for any mistakes in our thinking and correct any major errors. I loved how I learned to be a better problem solver by having my previous methods challenged by my peers who have intuition on this problem. "

-Henry Bae, Mathematical Science Major

5 Acknowledgement

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7 Appendix

SCUDEM V 2020 Problem B - Spinning a Wheel

A popular video [4] has been shared in various social media forums of a bird that likes to perch on the edge of a bicycle wheel and move its body so that the wheel spins through a full revolution. The movement of the bird is similar to the way a person can change their position on a swing in a way that can increase the amplitude of their oscillations. The bird can do this from a still start but is starting while at the top of the wheel.

Can you replicate and model this phenomenon? Assume that you have a bicycle wheel mounted vertically, and a small device is mounted on the wheel. The device can move a small mass attached to the wheel and hence the wheel. Is it possible to find a way to move the mass so that the wheel and the mass can rotate? Is it possible to do this if the mass only moves in the tangential direction or can you also do so with a radial component as well?

Your results should include the following:

- A complete description of your apparatus and assumptions.
- A complete description of how it moves.

- A complete description of how it moves.
- Describe the equations of motion.
- Describe the necessary initial conditions to increase the amplitude of any oscillations. For example, can this occur if the apparatus is initially completely still with the weight at the bottom, or do you have to impart some initial rotational velocity or a specific position?
- Does the motion of the object have to change as the wheels angular velocity changes?
- Describe the maximum speed that the wheel can spin.

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