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Recommended Citation

Spector, Reemon (2022) "On the Smallest Non-trivial Action of $\text{SAut}(Fn)$ for Small n ," *Rose-Hulman Undergraduate Mathematics Journal*: Vol. 23: Iss. 2, Article 4.

Available at: <https://scholar.rose-hulman.edu/rhumj/vol23/iss2/4>

On the Smallest Non-trivial Action of $\text{SAut}(F_n)$ for Small n

By *Reemon Spector*

Abstract. In this paper we investigate actions of $\text{SAut}(F_n)$, the unique index 2 subgroup of $\text{Aut}(F_n)$, on small sets, improving upon results by Baumeister–Kielak–Pierro for several small values of n . Using a computational approach for $n \geq 5$, we show that every action of $\text{SAut}(F_n)$ on a set containing fewer than 20 elements is trivial.

1 Introduction

The actions of $\text{Aut}(F_n)$, the automorphism group of the free group of rank n , on sets of small cardinality are not yet very well understood. In particular, the question of the cardinality of the smallest set, X , that $\text{SAut}(F_n)$ acts on nontrivially remains open.

In their paper on non-abelian quotients of $\text{Aut}(F_n)$ [BKP, Theorem 3.16], Baumeister, Kielak, and Pierro give a general lower bound for $|X|$ when $n \geq 7$, as well as specific bounds for $n \in \{3, 4, 5, 6\}$, with the bounds being sharp when $n \in \{3, 4\}$. Further, using a result from Saunders' paper on permutation degrees for Coxeter groups [Sau, Theorem 2.3], we can extract a bound of $2n$ for $n \geq 3$, which is a greater lower bound for the cases $n \in \{7, 8, 9\}$ than the one given in [BKP, Theorem 3.16].

Using a computational proof by exhaustion, we prove [Theorem 3.10](#), which states that any action of $\text{SAut}(F_5)$ on a set containing fewer than 20 elements is trivial. The best previous bound for $n = 5$ was 12. [Corollary 3.12](#) then extends this result of 20 to all $n \geq 5$, which improves on the previous bounds of 14, 14, 16, 18 for $n = 6, 7, 8, 9$ respectively. Currently, the smallest known action coincides with the smallest action of $\text{PSL}_n(\mathbb{Z}/2\mathbb{Z})$, which is on $2^n - 1$ points for non-exceptional values of n . In particular, we have the isomorphisms $\text{PSL}_3(\mathbb{Z}/2\mathbb{Z}) \cong \text{PSL}_2(\mathbb{Z}/7\mathbb{Z})$ and $\text{PSL}_4(\mathbb{Z}/2\mathbb{Z}) \cong A_8$ which dictate the smallest action for $n \in \{3, 4\}$ [BKP].

For a complete picture of the smallest actions of $\text{SAut}(F_n)$, sharp results for the two cases $n \in \{1, 2\}$ are proved in [Proposition 3.1](#) which states that $\text{SAut}(F_1)$ is trivial, and therefore acts trivially on every set; and [Proposition 3.2](#) which states that $\text{SAut}(F_2)$ has a nontrivial action on a set of two elements.

Mathematics Subject Classification. 20E36

Keywords. Automorphism group of free group

Structure of the Paper. In the following section, we provide background information about $\text{SAut}(F_n)$ that will be used in [Section 3](#), where we recapitulate results about the smallest action of $\text{SAut}(F_n)$ for $n \in \{1, 2, 3, 4\}$, then proceed to present new results which build up to [Theorem 3.10](#).

Acknowledgements. The author wishes to thank Dawid Kielak for his invaluable supervision, the anonymous referee for their excellent suggestions, Hertford College and the Crankstart Bursary for their generous funding, the Oxford Mathematical Institute for access to powerful machinery, and Andres Klene for many a meaningful conversation.

2 Background

2.1 Subgroups of $\text{SAut}(F_n)$

We will start by constructing a few automorphisms of F_n , the free group of rank n , and the finite subgroups of $\text{Aut}(F_n)$ generated by these automorphisms. First, pick an ordered basis $A = \{a_1, a_2, \dots, a_n\}$ for F_n , and define $N = \{1, 2, \dots, n\}$.

For each pair of distinct $i, j \in N$, define the *transposition* $\sigma_{ij}: F_n \rightarrow F_n$ as

$$\sigma_{ij}(a_k) = \begin{cases} a_j & \text{if } k = i, \\ a_i & \text{if } k = j, \\ a_k & \text{otherwise.} \end{cases}$$

The transpositions σ_{ij} generate a symmetric group S_n in $\text{Aut}(F_n)$, which acts naturally on the labels of basis elements in A .

In a similar fashion, for each $i \in N$, define the *involution* $\varepsilon_i: F_n \rightarrow F_n$ as

$$\varepsilon_i(a_k) = \begin{cases} a_i^{-1} & \text{if } k = i, \\ a_k & \text{otherwise.} \end{cases}$$

For each pair of distinct $i, j \in N$, the involutions ε_i and ε_j commute, and hence generate a copy of $(\mathbb{Z}/2\mathbb{Z})^n$ in $\text{Aut}(F_n)$. Observe the conjugation action of S_n on $(\mathbb{Z}/2\mathbb{Z})^n$ naturally permutes the indices of the involutions, so preserves $(\mathbb{Z}/2\mathbb{Z})^n$. In fact, since $(\mathbb{Z}/2\mathbb{Z})^n$ intersects S_n trivially, this gives us a wreath product $(\mathbb{Z}/2\mathbb{Z}) \wr S_n$ in $\text{Aut}(F_n)$. We will denote this group by B_n as it is isomorphic to the Coxeter group of type B of rank n .

Next, we turn to the standard representation of S_{n+1} in $\text{Aut}(F_n)$, which we can generate with the transpositions σ_{ij} for $i, j \in N \cup \{n+1\}$ where, for each $i \in N$, the new transposition $\sigma_{i(n+1)}: F_n \rightarrow F_n$ is defined as

$$\sigma_{i(n+1)}(a_k) = \begin{cases} a_i^{-1} & \text{if } k = i, \\ a_k a_i^{-1} & \text{otherwise.} \end{cases}$$

Observe that abelianizing F_n induces a map $\phi: \text{Aut}(F_n) \rightarrow \text{GL}_n(\mathbb{Z})$, which we will refer to as the *canonical map*. By associating the basis A with the standard basis for \mathbb{Z}^n , we see that

- σ_{ij} maps to the identity matrix with rows i and j swapped;
- ε_i maps to the identity matrix with a -1 in position (i, i) ; and
- $\sigma_{i(n+1)}$ maps to the identity matrix where row i has all -1 s.

With this perspective, it is natural to define $\text{SAut}(F_n)$ as the kernel of the determinant map $\det: \text{GL}_n(\mathbb{Z}) \xrightarrow{\det} \{\pm 1\}$ composed with the canonical map ϕ from $\text{Aut}(F_n)$.

We can now see A_n, A_{n+1} and $(\mathbb{Z}/2\mathbb{Z})^{n-1}$ as subgroups of $\text{SAut}(F_n)$, where $A_k = S_k \cap \text{SAut}(F_n)$ and $(\mathbb{Z}/2\mathbb{Z})^{n-1} = (\mathbb{Z}/2\mathbb{Z})^n \cap \text{SAut}(F_n)$. In addition, since $(\mathbb{Z}/2\mathbb{Z})^{n-1}$ is generated by pairs of involutions $\varepsilon_i \varepsilon_j$ and A_n acts naturally on the indices i and j , we also see that $(\mathbb{Z}/2\mathbb{Z})^{n-1} \rtimes A_n \leq \text{SAut}(F_n)$. We will denote this subgroup by D'_n , as it is isomorphic to the derived subgroup of the Coxeter group of type D of rank n . Note that D'_n and B_n act naturally on a set of $2n$ points (or n signed points).

2.2 Gersten's Presentation of $\text{SAut}(F_n)$

A key tool used throughout the paper is the following concise presentation of $\text{SAut}(F_n)$. Following the convention in [BKP], for $n \geq 2$ and distinct $i, j \in \mathbb{N}$, we define the *right transvection* $\rho_{ij}: F_n \rightarrow F_n$, and its analogous *left transvection* $\lambda_{ij}: F_n \rightarrow F_n$ as

$$\rho_{ij}(a_k) = \begin{cases} a_i a_j & \text{if } k = i, \\ a_k & \text{otherwise.} \end{cases}$$

$$\lambda_{ij}(a_k) = \begin{cases} a_j a_i & \text{if } k = i, \\ a_k & \text{otherwise.} \end{cases}$$

In [Ger], Gersten proves that for $n \geq 3$ these transvections generate $\text{SAut}(F_n)$ subject to four families of relations. We formulate Gersten's relations as below noting that each ± 1 can be taken to be either 1 or -1 independently of the choice of the other.

Let $i, j, k \in \mathbb{N}$ be distinct, let $l \in \mathbb{N}$ be distinct from i, k , and define the commutator of a and b as $[a, b] = aba^{-1}b^{-1}$; then

- (r₁) $[\rho_{ij}^{\pm 1}, \rho_{kl}^{\pm 1}] = [\lambda_{ij}^{\pm 1}, \lambda_{kl}^{\pm 1}] = [\rho_{ij}^{\pm 1}, \lambda_{kl}^{\pm 1}] = 1$;
- (r₂) $[\rho_{ij}^{-1}, \rho_{jk}^{-1}] = \rho_{ik}^{-1}$ and $[\lambda_{ij}^{-1}, \lambda_{jk}^{-1}] = \lambda_{ik}^{-1}$;
- (r₃) $[\rho_{ij}^{\pm 1}, \lambda_{il}^{\pm 1}] = 1$; and

$$(r_4) \quad (\lambda_{ij}\lambda_{ji}^{-1}\rho_{ij})^4 = (\rho_{ij}\rho_{ji}^{-1}\lambda_{ij})^4 = 1.$$

Note that (r_4) can be demystified by observing that $\lambda_{ij}\lambda_{ji}^{-1}\rho_{ij} = \varepsilon_i\sigma_{ij}$. We also note that, as a consequence of the commutator relations above, $\text{SAut}(F_n)$ is perfect for all $n \geq 3$.

3 Results

We preface the main results of the paper with a well-known result about $\text{SAut}(F_1)$.

Proposition 3.1. ($n = 1$). $\text{SAut}(F_1)$ is trivial.

Proof. Pick a generating element a_1 for F_1 . The homomorphism $\varphi: F_1 \rightarrow \mathbb{Z}$ given by $\varphi(a_1) = 1$ is bijective, showing that $F_1 \cong \mathbb{Z}$ and hence that $\text{Aut}(\mathbb{Z})$ consists of

- the trivial automorphism ($1 \mapsto 1$); and
- the negation automorphism ($1 \mapsto -1$), which has determinant -1 .

Hence $\text{SAut}(\mathbb{Z}) \cong \text{SAut}(F_1)$ is trivial. □

Proposition 3.1 has the immediate corollary that $\text{SAut}(F_1)$ acts trivially on any set.

Proposition 3.2. ($n = 2$). *The smallest nontrivial action of $\text{SAut}(F_2)$ is on a set of two elements.*

Proof. For $n = 2$, Nielsen [Nie] showed the canonical map $\text{Aut}(F_n) \twoheadrightarrow \text{GL}_n(\mathbb{Z})$ is in fact an isomorphism and hence $\text{SAut}(F_2)$ is isomorphic to $\text{SL}_2(\mathbb{Z})$, a very well-understood group with many interesting properties.

We take an example from Conrad's investigation of $\text{SL}_2(\mathbb{Z})$ [Con, Example 2.5], which draws a connection to the theory of modular forms. Denoting the n^{th} roots of unity by \mathbb{C}_n , we construct an action of $\text{SAut}(F_2)$ on a set of two elements in two steps. First, consider the epimorphism $\chi: \text{SL}_2(\mathbb{Z}) \rightarrow \mathbb{C}_{12}$ given by

$$\chi: \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \exp\left(\frac{2\pi i}{12}((1-c^2)(bd+3(c-1)d+c+3)+c(a+d-3))\right).$$

Second, compose χ with the map $\psi: \mathbb{C}_{12} \rightarrow \mathbb{C}_2$ given by

$$\psi: z \mapsto z^6$$

to get a nontrivial homomorphism $\text{SAut}(F_2) \cong \text{SL}_2(\mathbb{Z}) \twoheadrightarrow \mathbb{C}_2 \cong \mathbb{Z}/2\mathbb{Z}$. □

Proposition 3.3. ($n \in \{3, 4\}$). *The smallest nontrivial actions of $\text{SAut}(F_3)$ and $\text{SAut}(F_4)$ are on sets of 7 and 8 elements respectively.*

Proof. This result is a direct consequence of [BKP, Lemmas 3.1 and 3.3]. □

We now turn to some results that form the foundations upon which the computational aspect of the paper relies.

Lemma 3.4. *Let $n \geq 4$. Given a homomorphism $\alpha: D'_n \rightarrow S_m$ and some $\tau \in S_m$, there is at most one homomorphism $\psi: \text{SAut}(F_n) \rightarrow S_m$ satisfying $\alpha = \psi|_{D'_n}$ and $\tau = \psi(\rho_{12})$.*

Equivalently, any homomorphism $\psi: \text{SAut}(F_n) \rightarrow S_m$ is completely determined by the restriction $\psi|_{D'_n}$ and the image $\psi(\rho_{12})$.

Proof. Crucial observations from [BKP, Lemma 2.3], originally due to Bridson–Vogtmann [BV], tell us that all transvections are conjugate in $\text{SAut}(F_n)$. In particular, conjugating on the right and given distinct $i, j \in \mathbb{N}$ we have

$$\rho_{ij}^{\varepsilon_i \varepsilon_j} = \lambda_{ij}.$$

Since $n \geq 4$, we know A_n acts transitively on the pairs (i, j) , hence given $\sigma \in A_n$ (and excusing a slight abuse of notation) we also have

$$\rho_{ij}^\sigma = \rho_{\sigma(i)\sigma(j)}.$$

As ψ is a homomorphism, these conjugacies also hold in $\text{im}(\psi)$. Hence, since $\varepsilon_i \varepsilon_j, \sigma \in D'_n$, we can generate the images of all transvections given $\psi(\rho_{12})$ and the images of elements in D'_n . This concludes the proof as the transvections generate $\text{SAut}(F_n)$. □

Recall from Gersten’s presentation that $\text{SAut}(F_n)$ is perfect for $n \geq 3$. Hence the image of $\text{SAut}(F_n)$ under ψ must sit inside $A_m \leq S_m$.

Lemma 3.5. *Let $n \geq 5$ and let S be the subgroup of D'_n consisting of all elements of D'_n that fix a_1 and a_2 pointwise, then we have $S = C_{D'_n}(\rho_{12})$.*

Proof. Let $\sigma \in S$ and let $w = l_1 l_2 \dots l_k$ be a fixed word in F_n , where each l_i is a single letter in $\{a_1^{\pm 1}, \dots, a_n^{\pm 1}\}$. Recall both σ and ρ_{12} are automorphisms of F_n so we have

$$\sigma \circ \rho_{12}(w) = \prod_{i=1}^k (\sigma \circ \rho_{12}(l_i)).$$

For each $1 \leq i \leq k$, if $l_i \in \{a_1^{\pm 1}, a_2^{\pm 1}\}$ then $\sigma \circ \rho_{12}(l_i) = \rho_{12}(l_i) = \rho_{12} \circ \sigma(l_i)$. Otherwise, if $l_i \in \{a_3^{\pm 1}, \dots, a_n^{\pm 1}\}$ then $\sigma \circ \rho_{12}(l_i) = \sigma(l_i)$. Note that $\sigma \in D'_n$ so there exist unique $\tau \in A_n$, $\xi \in (\mathbb{Z}/2\mathbb{Z})^{n-1}$ such that $\sigma = \tau \circ \xi$.

Further, $\xi(\{a_3^{\pm 1}, \dots, a_n^{\pm 1}\}) \subseteq \{a_3^{\pm 1}, \dots, a_n^{\pm 1}\}$, so $\sigma(l_i) \in \{a_3^{\pm 1}, \dots, a_n^{\pm 1}\}$ if and only if $\tau(l_i)$ lies in the set $\{a_3^{\pm 1}, \dots, a_n^{\pm 1}\}$. Suppose it does not, then σ^{-1} does not fix one of $a_1^{\pm 1}, a_2^{\pm 1}$, which is a contradiction as S is a subgroup. Hence, $\sigma \circ \rho_{12}(l_i) = \sigma(l_i) = \rho_{12} \circ \sigma(l_i)$ and therefore $\sigma \circ \rho_{12}(w) = \rho_{12} \circ \sigma(w)$. This shows $S \subseteq C_{D'_n}(\rho_{12})$.

Now let $g \in C_{D'_n}(\rho_{12})$ and suppose $g(a_1) \neq a_1$, so we have

$$\rho_{12} \circ g(a_1) = g \circ \rho_{12}(a_1) = g(a_1 a_2) = g(a_1)g(a_2).$$

If $g(a_1) = a_1^{-1}$ then $a_2^{-1} a_1^{-1} = a_1^{-1} g(a_2)$, which contradicts F_n being free as g is a permutation of $\{a_1^{\pm 1}, a_2^{\pm 1}, \dots, a_n^{\pm 1}\}$. Hence $g(a_1) \neq a_1^{\pm 1}$, so we are left with $\rho_{12} \circ g(a_1) = g(a_1) = g(a_1)g(a_2)$, which forces $g(a_2)$ to be trivial, contradicting g being injective.

Alternatively, if $g(a_1) = a_1$ and $g(a_2) \neq a_2$ then $g \circ \rho_{12}(a_1) = g(a_1 a_2) = g(a_1)g(a_2) = a_1 g(a_2)$, while $\rho_{12} \circ g(a_1) = \rho_{12}(a_1) = a_1 a_2$, forcing $g(a_2) = a_2$, which is a clear contradiction. This shows $C_{D'_n}(\rho_{12}) \subseteq S$, which concludes the proof. \square

Corollary 3.6. *If $\psi: \text{SAut}(F_n) \rightarrow A_m$ is a homomorphism, then $\psi(\rho_{12})$ must commute with every element in $\psi(S)$ in the image, that is $\psi(\rho_{12}) \in C_{A_m}(\psi(S))$.*

Definition 3.7. We adopt the terminology from [Lur, Example 1.1.1.1.] and say that two homomorphisms from a group G to a permutation group H are *conjugate* if they are equal up to conjugacy of H . We hence define the *homomorphism classes* between G and H as the equivalence classes under conjugacy.

Remark 3.8. Let ψ be a homomorphism with domain $\text{SAut}(F_n)$. If $\alpha = \psi|_{D'_n}$ and $\beta = \psi|_{A_{n+1}}$, then α and β agree when restricted to A_n , that is $\alpha|_{A_n} = \beta|_{A_n}$.

Consequently, given a homomorphism $\alpha: D'_n \rightarrow A_m$, if there is no homomorphism class representative $\beta: A_{n+1} \rightarrow A_m$ such that $\alpha(A_n)$ and $\beta(A_n)$ are conjugate in A_m , then α cannot be extended to a homomorphism with domain $\text{SAut}(F_n)$. Note that this is a weaker condition than the one in Remark 3.8, but we used this in early versions of the code employed in our proof of Theorem 3.10 as it reduced computation times.

Lemma 3.9. *Let $\psi: \text{SAut}(F_5) \rightarrow A_m$, where $m < 31$ and define $\alpha = \psi|_{D'_5}$ and $\beta = \psi|_{A_6}$. If α or β is not injective, then ψ is trivial.*

Proof. Suppose α is not injective, so that $\ker(\alpha)$ is nontrivial. There exists some nontrivial $\xi \in \ker(\alpha)$ such that $\psi(\xi)$ is central in $\text{im}(\psi)$. By [BKP, Lemma 2.5], we know that ψ factors through $\text{PSL}_5(\mathbb{Z}/2\mathbb{Z})$ or is trivial, in which case we are done.

It is a fact that $\text{PSL}_5(\mathbb{Z}/2\mathbb{Z})$ is simple and has order that is divisible by 31. Hence $m \geq 31$ whenever ψ is nontrivial, so the claim follows when α is not injective.

Suppose β is not injective. Since A_6 is simple, this implies that β is trivial. However, this forces α to be trivial on A_5 by Remark 3.8, hence making α non-injective, which completes the proof. \square

Theorem 3.10. *Any action of $\text{SAut}(F_5)$ on a set containing fewer than 20 elements is trivial.*

Proof. The proof for this Theorem relies entirely on a computer search using [GAP], the code for which is available at [Spe]. Our strategy mainly revolves around using Lemma

3.4, Corollary 3.6, and Lemma 3.9 to attempt to extend a homomorphism from D'_5 to one of $\text{SAut}(F_5)$. The computation is split into three main phases, which we will denote (P_1) , (P_2) and (P_3) .

In (P_1) , we construct the automorphisms defined in Section 2.1 and use them to generate the subgroups A_n, A_{n+1}, D'_n , and S in $\text{SAut}(F_n)$.

In (P_2) , we construct the monomorphism classes from D'_5 to A_{19} as suggested by Lemma 3.9. This list of potential homomorphisms ψ is then tested in (P_3) .

In (P_3) , we rely on Lemma 3.4 and Corollary 3.6 to test whether (r_2) from Gersten's presentation holds in the image of the homomorphism ψ being tested. In particular, we test whether the following relation holds

$$[\psi(\rho_{12})^{-1}, \psi(\rho_{23})^{-1}] = \psi(\rho_{13})^{-1}.$$

The program searches through all potential images $\psi(\rho_{12}) \in C_{A_m}(\psi(S))$ and shows the only image that passes the test above is trivial, thus proving our claim. \square

It is worth noting that when comparing (P_1) with (P_2) and (P_3) , the first phase is by far the shortest, requiring very low levels of CPU and memory usage. However, (P_2) requires a long amount with varying levels of CPU usage, while (P_3) is highly CPU-intensive and takes the longest.

The expected runtime of the program is around twelve hours to run through and terminate. Versions of the code that use Remark 3.8 to test for existence of homomorphisms $\beta: A_{n+1} \rightarrow A_m$ such that $\alpha(A_n)$ and $\beta(A_n)$ are conjugate in A_m require vast (in the ballpark of 256GB for $m = 19$) amounts of memory, which might be a more viable strategy for n and m where the candidate list of homomorphisms $\alpha: D'_n \rightarrow A_m$ is larger. Carrying out these computations required access to powerful computers made available by the Oxford Mathematical Institute.

Proposition 3.11. *Let $n \geq 2$ and let x_n be the cardinality of the smallest set on which $\text{SAut}(F_n)$ acts nontrivially. The sequence $(x_n)_{n \geq 2}$ is increasing.*

Proof. We will show this by proving $x_n \leq x_{n+1}$ for all n . By the sharpness of the results for $n \in \{2, 3\}$, we know this holds for the base case $n = 2$.

Suppose $\psi: \text{SAut}(F_{k+1}) \rightarrow \text{Sym}(X)$ is given where $|X| = x_{k+1}$ and let $\varphi = \psi|_{\text{SAut}(F_k)}$. If φ is trivial then pick an element ξ in $A_n < D'_n$ so that $\varphi(\xi)$ is central in $\text{im}(\psi)$, hence ψ is trivial by [BKP, Lemma 2.5 (3)], which is a contradiction.

Therefore $\varphi: \text{SAut}(F_k) \rightarrow \text{Sym}(X)$ is nontrivial, so $x_k \leq |X| = x_{k+1}$ and the result follows by induction on n . \square

Corollary 3.12. *Any action of $\text{SAut}(F_n)$ on a set containing fewer than 20 elements is trivial.*

Proof. This is immediate by combining the results from Theorem 3.10 and Proposition 3.11. \square

4 Closing Remarks

As pointed out by the author's supervisor Dawid Kielak, one might conjecture that for sufficiently large n , the smallest action of $\text{SAut}(F_n)$ is on a set of m elements, where m is the smallest integer such that there exist monomorphisms $\alpha: D'_n \rightarrow A_m$ and $\beta: A_{n+1} \rightarrow A_m$ satisfying $\alpha|_{A_n} = \beta|_{A_n}$. The inspiration behind this was the role played by B_n and S_{n+1} in rigidity; see [BV].

We also note that the kernel of the determinant map from B_n is a slightly larger subgroup, which we will denote by $K < \text{SAut}(F_n)$. In fact, it is the largest finite subgroup by the Nielsen Realisation Theorem for $\text{Aut}(F_n)$, summarised by [Vog] and originally due to [Cul], [Zim] and [Khr] independently. We conclude the paper with the following question, inspired by Kielak.

Question 4.1. For general n , does there exist an integer m and monomorphisms

$$\beta: A_{n+1} \hookrightarrow A_m \quad \text{and} \quad \gamma: K \hookrightarrow A_m$$

such that $\beta|_{A_n} = \gamma|_{A_n}$, but every action of $\text{SAut}(F_n)$ on a set of m elements is trivial?

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