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Winning Strategy for Multiplayer and Multialliance Geometric Game

By *Jingkai Ye*

Abstract. The Geometric Sequence with common ratio 2 is one of the most well-known geometric sequences. Every term is a nonnegative power of 2. Using this popular sequence, we can create a Geometric Game which contains combining moves (combining two copies of the same terms into the one copy of next term) and splitting moves (splitting three copies of the same term into two copies of previous terms and one copy of the next term). For this Geometric Game, we are able to prove that the game is finite and the final game state is unique. Furthermore, we are able to calculate the upper bound and lower bound of the length of Geometric Game. We are also able to prove some interesting results in terms of the winning strategy of 2-player games, and some special cases of multiplayer games and multialliance games.

1 Introduction

1.1 Overall Guideline

This paper focuses on the winning strategy for multiplayer and multialliance Geometric Game. In Section 1, we discuss where the motivation of Geometric Game comes from. More specifically, as the Geometric Game stems from the Fibonacci Zeckendorf Game, this section covers the rules of Zeckendorf Game, the rules of Geometric Game and the connections between these two games. In Section 2 and 3, we show some basic properties of the Geometric Game. In particular, for Section 2, we show that the Geometric Decomposition for any starting number is unique and the Geometric Game is finite. For Section 3, we discuss the properties of combining moves and splitting moves of the Geometric Game, and we find an upper bound and a lower bound in terms of the length of Geometric Game. In Section 4, 5 and 6, we discuss the winning strategies of Geometric Game. In particular, for section 4, we focuses on the winning strategy of 2-player games. For Section 5, we focuses on the winning strategy of multiplayer games, and in Section 6 we focuses on the winning strategy of multialliance games.

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1.2 The Zeckendorf Game

Flint and Miller [3, 4] have created Fibonacci Zeckendorf Game based on the Fibonacci sequence and Zeckendorf Theorem [14]. The motivation of designing Geometric Game comes from the rules and some results of Fibonacci Zeckendorf Game. There are lots of papers related to the Fibonacci Sequence and Zeckendorf Game, such as [1, 2, 5, 6, 7, 8, 9, 10, 11, 12, 13]. The concept of Zeckendorf Decomposition comes from a theorem proved by Zeckendorf [14], which states that every positive integer can be uniquely written as the sum of non-adjacent and distinct Fibonacci numbers. We call this Zeckendorf Decomposition of n. In order to clearly describe the rules of Zeckendorf Game, we cite from [4] as the description of the Zeckendorf game. In Zeckendorf game, one of the most important concepts is the Zeckendorf Decomposition.

First of all, we introduce the concept of the Fibonacci Sequence. For any positive integer k, we use F_k to denote the k-th term of the Fibonacci Sequence. In this paper, the Fibonacci Sequence is defined as: $\mathrm{F}_1 = 1, \mathrm{F}_2 = 2$, and $\mathrm{F}_{i+1} = \mathrm{F}_i + \mathrm{F}_{i-1}$ for and $i \geq 2$.

Then we introduce some notations. By $\{1^n\}$ or $\{{\mathrm{F}_1}^n\}$ we mean n copies of 1, or F_1. If we have 5 copies of F_1, 3 copies of F_2, and 6 copies of F_4, we write either $\{{\mathrm{F}_1}^5 \wedge {\mathrm{F}_2}^3 \wedge {\mathrm{F}_4}^6\}$ or $\{1^5 \wedge 2^3 \wedge 5^6\}$.

After introducing these notations, we introduce the concept of Zeckendorf Decomposition. Edourad Zeckendorf [14] proved that every positive integer n can be written uniquely as the sum of distinct non-consecutive Fibonacci numbers. This result is known as the Zeckendorf's Theorem [14], and this sum is the Zeckendorf Decomposition of n. For example, the Zeckendorf Decomposition of 11 is $\{{\mathrm{F}_3}^1 \wedge {\mathrm{F}_5}^1\}$, or $\{3^1 \wedge 8^1\}$.

The Zeckendorf Game is defined as follows.

Definition 1.1 (The Zeckendorf Game)**.** At the beginning of the game, there are n copies of 1's. Let $\mathrm{F}_1 = 1, \mathrm{F}_2 = 2$, and $\mathrm{F}_{i+1} = \mathrm{F}_i + \mathrm{F}_{i-1}$; therefore the initial list is $\{{\mathrm{F}_1}^n\}$. On each turn, a player can do one of the following moves.

1. Combining moves:
   (a) if the list contains two consecutive Fibonacci numbers, $\mathrm{F}_{i-1}, \mathrm{F}_i$, these can be combined to create F_{i+1}. We denote this move by $\mathrm{F}_{i-1} \wedge \mathrm{F}_i \to \mathrm{F}_{i+1}$.
   (b) if the list contains two copies of F_1, then F_1 and F_1 can be combined to create one copy of F_2, denoted by $1 \wedge 1 \to 2$.
2. Splitting moves: if the list has two of the same Fibonacci number, $\mathrm{F}_i, \mathrm{F}_i$, where $i \geq 2$, then
   (a) if $i = 2$, a player can change $\mathrm{F}_2, \mathrm{F}_2$ to $\mathrm{F}_1, \mathrm{F}_3$, denoted by $2 \wedge 2 \to 1 \wedge 3$, and
   (b) if $i \geq 3$, a player can change $\mathrm{F}_i, \mathrm{F}_i$ to $\mathrm{F}_{i-2}, \mathrm{F}_{i+1}$, denoted by $\mathrm{F}_i \wedge \mathrm{F}_i \to \mathrm{F}_{i-2} \wedge \mathrm{F}_{i+1}$.

The players alternate moving. The game ends when one player moves to create the Zeckendorf decomposition.

1.3 Geometric Game with Common Ratio 2

Stemming from the results of the Zeckendorf Game, we create the Geometric Game based on the Geometric sequence of ratio 2. The Geometric sequence of common ratio 2 is one of the most common sequences. We define this Geometric sequence as $a_1 = 1$, $a_2 = 2$, $a_3 = 4$, and $a_{n+1} = 2a_n$. In this paper, everything related to Geometric Game refers to the Geometric Game based on the Geometric Sequence of powers of 2.

We first introduce some notation. The notation will mirror that of the Fibonacci-Zeckendorf Game. By $\{1^n\}$ or $\{{a_1}^n\}$ we mean n copies of 1, or a_1.By $\{2^i\}$ or $\{{a_2}^i\}$ we mean i copies of 2, or a_2.

Definition 1.2 (The Geometric Game with Common Ratio 2)**.** The game starts with a positive integer n, where n is a finite positive integer. At the beginning of the game, there are n copies of 1's. Let $a_1 = 1, a_2 = 2$, and $a_{i+1} = 2a_i$; therefore the game always starts with n copies of a_1. On each turn, a player can do one of the following moves.

1. Combining moves: if a game state contains at least two copies of Geometric number a_i, then we can combine two of them to create a_{i+1}. We denote this move by $a_i \wedge a_i \to a_{i+1}$, or $2a_i \to a_{i+1}$.

2. Splitting moves: if a game state has three of the same Geometric number greater than 1, a_i, a_i, a_i, where $i \geq 2$, then we can split them into one a_{i+1} term and two a_{i-1} terms. We denote this move by $a_i \wedge a_i \wedge a_i \to a_{i+1} \wedge a_{i-1} \wedge a_{i-1}$, or $3a_i \to a_{i+1} \wedge 2a_{i-1}$.

The players alternate moving. The game ends when one player moves to create the Geometric decomposition, which will be introduced in the following section.

In the following results, p represents the number of players in the Geometric game. Furthermore, for combining moves, when $i = 1$, we can denote it as $1 \wedge 1 \to 2$; when $i = 2$, we can denote it as $2 \wedge 2 \to 4$. For splitting moves, when $i = 2$, we can denote it as $2 \wedge 2 \wedge 2 \to 1 \wedge 1 \wedge 4$.

1.4 Game State and Geometric Decomposition with Common Ratio 2

In a Geometric Game, the game states can be classified as follows: beginning game state, mid-game states and final game state. For an arbitrary starting number n, the beginning game state is the state that only contains n copies of 1.

The final state of the game is defined as Geometric Decomposition with Ratio 2. More specifically, Geometric Decomposition is the final game state where there no term of the Geometric Sequence (with ratio 2) has two or more copies. In other words, in an arbitrary game state, for all the geometric numbers a_i, if the game state has 0 or 1 copy of a_i, then the game state is a geometric decomposition and the game ends. If the game state has 2 or more copies of a_i for some positive integer i, then the game state is not a geometric decomposition and the game will continue. For example, a game state consisting of 1 copy of a_1, 1 copy of a_4 and 1 copy of a_6 is a Geometric Decomposition, and a game consisting of 3 copies of a_1, 2 copies of a_2 and 4 copies of a_6 is not a Geometric Decomposition. In the rest of the paper, everything about Geometric Decomposition refers to the Geometric Decomposition based on the Geometric sequence of powers of 2.

The mid-game states include all the game states that players can possibly reach by taking combining moves or splitting moves (or both), except the beginning game state and Geometric Decomposition. For example, if the initial state is $\{1^5\}$, players can combine 2 1s and reach the state of $\{1^3+2^1\}$. Note that $\{1^3+2^1\}$ is not the Geometric Decomposition of 5, so $\{1^3+2^1\}$ is a mid-game state for $n=5$.

1.5 Connections between the Zeckendorf Game and the Geometric Game

In many aspects, the design of Geometric Game is similar to the Fibonacci Zeckendorf Game. For instance, both games have combining and splitting moves. Furthermore, both games are based on well-known sequences: Zeckendorf Game is based on Fibonacci sequence, and Geometric Game is based on geometric sequence of ratio 2. As a result, we can find a correspondence between binary representations of an integer and the Geometric Game as compared to the Zeckendorf Decomposition and the Zeckendorf Game. Besides, from the Zeckendorf's Theorem proved by Edouard Zeckendorf [14], every positive integer n can be uniquely written as the sum of distinct non-consecutive Fibonacci numbers. This sum is known as the Zeckendorf Decomposition of n. Similarly, in Geometric Game with common ratio 2, as it appears later in the paper as Theorem 2.1, for every starting number n, the Geometric Decomposition can be written uniquely as the sum of distinct terms in the Geometric sequence with ratio 2.

1.6 A Game consisting of p players

In general, a game consisting of p players ($p \geq 2$) is defined as follows: p players, which are Players $1,2,3,4,\ldots,p$, take turns to play the game. In other words, following the rules of the game, each player takes one step each time. More specifically, the game is started by Player 1, followed by Player 2, Player 3 and so on. When Player p takes one step, Player 1 takes the next step and the game keeps going on. The player who takes the last step of the game is the winner of the game.

Usually, if the game consists of 3 or more players, and we will call it "multiplayer game". If the the p players are divided into 2 or more alliances, we will call it "multialliance game". In the multialliance game, each player takes turns playing the game as mentioned before, and if Player m ($1 \le m \le p$) is the player who takes the last step, then player m's alliance wins the game.

1.7 Winning strategy

In a finite 2-player game, if Player 1 has a winning strategy of the game, then no matter what Player 2 does in each of Player 2's turns, Player 1 will always be able to make a move in response and ultimately win the game. Similarly, if Player 2 has a winning strategy of the game, then no matter what Player 1 does in each of Player 1's turns, Player 2 can always make a move in response and ultimately win the game.

In a finite multiplayer game consisting of p players ($p \ge 3$), if Player m has a winning strategy of the game, then no matter what all the other players do, Player m can always take some moves according to the game rules to win the game, and any paths that Player m can win the game is called a winning path for Player m. In other words, for every possible situation of the other $p-1$ players' moves, Player m can always take a combination of moves as a response of the other $p-1$ players' moves and win the game, and there exists at least one winning path for Player m if Player m has a winning strategy.

1.8 Key Techniques

Later in this paper, we have some new theorems in terms of the upper bound and lower bound of the Geometric Game, winning strategy for 2-player game, multiplayer games and multialliance games. Some techniques are very useful and helpful to prove these theorems. In Section 2, a very important technique is the use of monovariant. In this paper, a monovariant refers to a variable that is always non-increasing or non-decreasing from the beginning of the game to the end of the game. Later in Section 2, we will show that both the sum of indices and the number of terms are monovariants in the Geometric Game.

In Section 5, an important technique is to use the stealing strategy. In short, the stealing strategy means that suppose Player m has a winning strategy in a game, if we could find a way such that some player other that Player m could always steal the Player m's winning strategy by taking certain moves, then we get a contradiction, so we can conclude that Player m does not have a winning strategy.

2 Basic Properties of Geometric Game

This section includes some interesting and important properties of the Geometric Game and Geometric Decomposition. More specifically, the following 2 theorems show that for any arbitrary positive integer n, the Geometric Decomposition of n is always unique and Geometric Game on n is always finite.

2.1 The Geometric Decomposition is Unique

Theorem 2.1. *For every starting number n(initial state of the game), there exists a unique Geometric Decomposition.*

Proof. We prove Theorem 2.1 by proving Lemma 2.2 and Lemma 2.3. Note that in the proof of the following two lemmas, 2^i and 2^j are actual numbers, where 2^i means "two to the power of i", and 2^j means "two to the power of j". □

Lemma 2.2. *For every starting number n, the Geometric Decomposition always exists.*

Proof. We prove Lemma 2.2 by using strong induction. When $n = 1, 2, 3$ and 4, the Decomposition is easy to find $(1 = 1, 2 = 2, 3 = 1+2, 4 = 4)$, so Lemma 2.2 is true for $n = 1, 2, 3, 4$.

Suppose the Geometric Decomposition exists for all $n \leq k-1$, where $k \geq 5$. For $n = k$, if k is in the form of 2^i, where i is a positive integer, then $k = 2^i$ itself is a Geometric decomposition for k. If k is not in the form of 2^i, then there exists a positive integer j such that $2^j < k < 2^{j+1}$. As a result $1 \leq k - 2^j < k$. From our previous assumption for induction, the Geometric Decomposition of $k-2^j$ exists. Note that in the 2^j is not a term in the Geometric Decomposition of $k-2^j$ because $2^j < k$. Therefore, the Geometric Decomposition of $k-2^j$ and the term 2^j together form the Geometric Decomposition of k, so the Geometric Decomposition of k exists. In other words, Lemma 2.2 is true for $n = k$. Therefore, by using strong induction, Lemma 2.2 is proved. □

Lemma 2.3. *For every starting number n, the Geometric Decomposition is unique.*

Proof. We prove Lemma 2.3 by using strong induction. When $n = 1, 2, 3$ and 4, the Decomposition is easy to check that the Geometric Decomposition is unique. In other words, Lemma 2.3 is true for $n = 1, 2, 3, 4$.

Suppose the Geometric Decomposition exists for all $n \leq k-1$, where $k \geq 5$. For $n = k$, we consider the following 2 cases:

Case 1. If k is in the form of 2^i, where $i > 2$, then we can prove that 2^i is the unique Geometric Decomposition by contradiction: suppose there is a different Geometric Decomposition other than 2^i itself, then the Geometric Decomposition of $k = 2^k$ must contain at least two terms and largest term in the Geometric Decomposition will be at most 2^{i-1}. Therefore, the largest possible sum of the Geometric Decomposition is $1 + 2 + 2^2 + 2^3 + \ldots + 2^{i-1} = 2^i - 1$. As $2^i - 1 < 2^i = k$, this contradicts with the fact that the largest possible sum of Geometric Decomposition is at least k. Therefore, 2^i is the unique Geometric Decomposition.

Case 2. If k is not in the form of 2^i, then there exists a positive integer j such that $2^j < k < 2^{j+1}$. As a result $1 \leq k - 2^j < k$, and so we need to prove that in any Geometric Decompositions for k, 2^j must be a term in the Geometric Decomposition.

We can prove this by contradiction: suppose 2^j is not in the Geometric Decompositions of k. Note that $k < 2^{j+1}$, so any term greater than or equal to 2^{j+1} is not in the Geometric Decomposition of k. As 2^j is not in the Geometric Decomposition of k, then the largest possible term in the Geometric Decomposition is 2^{j-1}. Also note that in the Geometric Decomposition, the coefficient of every term in the Geometric sequence is at most 1. Therefore, the largest possible sum of the Geometric Decomposition is $1+2+2^2+2^3+\ldots+2^{j-1} = 2^j - 1$. As $2^j - 1 < 2^j < k$, the largest possible sum of the Geometric Decomposition is less than k, which contradicts with the fact that the largest possible sum of Geometric Decomposition is at least k. Therefore, 2^j must be a term in any Geometric Decompositions of k.

Note that from the definition of Geometric Decomposition, the coefficient of any terms in the Geometric Decomposition can be at most 1. As we proved that 2^j must be a term in any Geometric Decompositions of k, the coefficient of any 2^j in the Geometric Decomposition will be exactly 1.

Therefore, if we suppose that there are different Geometric Decompositions for k, then there are different Geometric Decompositions for $k-2^j$. However, this contradicts with the assumption in the strong induction that Geometric Decomposition for $k-2^j$ is unique since $1 \le k-2^j < k$. As a result, k has a unique Geometric Decomposition in this case.

According to the 2 cases, Lemma 2.3 is true for $n = k$. Therefore, by using strong induction, Lemma 2.3 is proved.

□

After proving the uniqueness of Geometric Decomposition on n, the following theorem shows that the Geometric Game on any arbitrary n is always finite.

2.2 The Geometric Game is Finite

In this section, we prove that every game terminates in a finite number of moves at the Geometric Decomposition. We prove a number of lemmas along the way.

Lemma 2.4. *After each combining move, the number of terms always decreases by* 1.

Proof. Since all the combining moves are in the form of $2a_i = a_{i+1}$ (i is any positive integers), each combining move combines two terms into one term. Therefore, after each combining move, the number of terms always decreases by 1. □

Lemma 2.5. *After each splitting move, the number of terms does not change.*

Proof. Since all the splitting moves are in the form of $3a_i = a_{i+1} + 2a_{i-1}$ ($i \ge 2$), each splitting move splits three terms into three other terms. Therefore, after each splitting move, the number of terms do not change. □

From Lemma 2.4 and Lemma 2.5, now we are able to prove Lemma 2.6 and Lemma 2.7, which are essential to determine what our splitting and combining moves do to the sum of indices.

Lemma 2.6. *After each splitting moves, the sum of indices on all of the terms decreases by* 1.

Proof. Since all the splitting moves are in the form of $3a_i = a_{i+1} + 2a_{i-1}$ ($i \geq 2$), the sum of indices of all terms on the left hand side is $3i$, and the sum of indices of all terms on the right hand side is $(i+1)+2(i-1) = i+1+2i-2 = 3i-1$. Therefore, after each splitting move, the sum of indices of all the terms decreases by 1. □

Lemma 2.7. *After each combining move, the sum of indices on all of the terms never increases.*

Proof. Since all the combining moves are in the form of $2a_i = a_{i+1}$ (i is any positive integers), the sum of indices of all terms on the left hand side is $2i$, and the sum of indices of all terms on the right hand side is $(i+1)$. Note that when $i \geq 1$, $2i \geq i+1$, so for all the positive integer i, the sum of indices of all the terms never increases after each combining move. □

Now we need to prove the following lemma, which states that in the Geometric Game, the number of terms will always decrease by 1 within finitely many steps.

Lemma 2.8. *The number of terms will be decreased from $k+1$ to k within finitely many steps, where* ($p \leq k \leq n-1$).

Proof. First of all, there are n copies of a_1 at the beginning of the game, so at any state of the Geometric Game, the sum of indices of all terms never exceeds n. From Lemma 2.6 and Lemma 2.7, we know that throughout the Geometric Game, the sum of indices of all the terms never increases. Therefore, after reaching the state of $k+1$ terms, the sum of indices of all terms is at most n, which is finite. According to Lemma 2.6, each splitting move will decrease the sum of indices by one. Since the sum of indices of all terms can never be negative, there cannot be infinitely many consecutive splitting moves before the first combining move. According to Lemma 2.4, each combining move decreases the number of terms by 1. Therefore, The number of terms will be decreased from $k+1$ to k within finitely many steps, where ($p \leq k \leq n-1$). □

Theorem 2.9. *Every game terminates in a finite number of moves at the Geometric Decomposition.*

Proof. From Lemma 2.8, we know that the number of terms will always decrease by 1 within finitely many steps. As the number of terms in the Geometric Game can never be negative, every Geometric Game terminates within a finite number of moves. Therefore, Theorem 2.9 is proved. □

A very important concept is monovariant. In this paper, a monovariant refers to a variable that is always non-increasing or always non-decreasing from the beginning to the end of the Geometric Game. It is worth noting that from Lemma 2.4 and Lemma 2.5, we know that the number of terms in the Geometric Game is a non-increasing monovariant; from Lemma 2.6 and Lemma 2.7, we also know that the sum of indices is a non-increasing monovariant. Furthermore, after proving 2.9, we can prove the following Lemma.

Lemma 2.10. *Suppose the Geometric Decomposition of n has p terms. If the Geometric Game of n reaches the state consisting of exactly p terms, then this state must be the Geometric Decomposition, which is the final state of the game.*

Proof. We prove this lemma by contradiction. Suppose there is a game state consisting of p terms and this state is not the Geometric Decomposition of n. As a result, in this game state, there exists a term a_i in the geometric sequence such that this game state has at least two copies of a_i. As a result, through combining move, we can combine these two copies of a_i into one a_{i+1}. According to Lemma 2.4, after this combining move, there will be only $p-1$ terms in total. From then on, according to Lemma 2.4 and Lemma 2.5, no matter what moves we do towards the Geometric Decomposition, the number of terms will never increase, so the Geometric Decomposition will of n be at most p terms, which contradicts with the assumption that the Geometric Decomposition of n has p terms. Therefore, by contradiction, Lemma 2.10 is proved. □

3 Upper and Lower Bound of the Length of Geometric Game

This section shows the upper bound and lower bound for the length of Geometric Game. More specifically, when the Geometric Decomposition of n consists of m terms, the upper bound of Geometric Game never exceeds $2n-6$, and the lower bound of Geometric Game on n is always $n-m$, and we will prove them later in this section. First of all, we will prove that in any Geometric Game, the number of combining moves is always fixed.

3.1 Number of Combining Moves in Geometric Game is Fixed

Theorem 3.1. *For any arbitrary n, the total number of combining moves in the Geometric Game of Ratio 2 is fixed. More specifically, if the final decomposition of n has m terms, then the number of combining moves is always $n-m$.*

Proof. According to Theorem 2.1, every positive integer n has a unique final decomposition, so the Geometric Decomposition of an arbitrary n has m terms. In other words, the final state of Geometric Game has m terms. Also note that the initial state has n terms, which are n 1's. According to Lemma 2.4 and 2.5, every combining move decreases the number of terms by 1 and every splitting move do not change the number of terms.

Therefore, there will be exactly $n-m$ moves to decrease the number of terms from n to m. In other words, the total number of combining moves for n is always $n-m$, which is a fixed number for the Geometric Game on n. □

3.2 Geometric Game is Playable with only Combining Moves

Theorem 3.2. *For any arbitrary n, the Geometric Game on n can be played by only doing combining moves for all players.*

Proof. We prove this theorem by using strong induction. First of all, the statement is trivial for $n = 1,2,3,4$, so the theorem is true for $n \le 4$.

Suppose the statement is true for all $n \le k-1 (k \ge 5)$. In other words, for all $n \le k-1$, the Geometric Game on n can be played by using only combining moves.

For $n = k$, we have two cases. First, if k is in the form of 2^i, where i is a positive integer, then the final decomposition of k is k it self. Note that we can combine 2 copies of 2^{i-1} to get 2^i, which is the final decomposition of k. As $2^{i-1} \le 2^i = k$, we have $2^{i-1} \le k-1$. Therefore, according to the strong induction, we can get to 1 copy of 2^{i-1} starting from 2^{i-1} copies of 1 by only using combining moves. So for the game of $n = k = 2^i$, since the initial game state has 2^i copies of 1, we can at first get to 1 copy of 2^{i-1} starting from 2^{i-1} copies of 1. Then, we can use the other 2^{i-1} copies of 1 to get to the second copy of 2^{i-1} using only combining moves. Finally, we can combine the two copies of 2^{i-1} to get the 2^i, which is the final decomposition of k. Therefore, when $k = 2^i$, then the game can be played using only combining moves.

For the second case, if k is not in the form of 2^i, then the final decomposition of k has at least 2 terms, and every term in the final decomposition is less than k. Therefore, in the final decomposition of k, every term is at most $k-1$. According to the strong induction, starting with only 1s, we can reach each of these terms by using only combining moves. Therefore, for the Geometric Game on k we can first get to largest term in the final decomposition starting with 1s by using only combining moves. Then we can get the second largest term from the remaining 1s by using only combining moves, and we continue this process until we get all the terms in the final decomposition using only combining moves. After this, we are done. Therefore, when k is not in the form of 2^i, then the Geometric Decomposition of k can also be reached by using only combining moves.

As a result, for both cases, the Geometric Game can be played using only combining moves, so the statement is true for $n = k$. Therefore, by using strong induction, Theorem 3.2 is proved. □

As a result of Theorem 3.1 and Theorem 3.2, we can now determine the lower bound of Geometric Game on n, which is stated in the following theorem.

3.3 Lower Bound of the Length of Geometric Game

Theorem 3.3. *For any arbitrary n, the total number of steps of Geometric Game reaches the lower bound by only doing combining moves. More specifically, when the final decomposition of n has m terms, the lower bound of Geometric Game on n is $n-m$.*

Proof. According to 3.1, the total number of combining moves in every Geometric Game on n is fixed, which is always $n-m$, where m is the number of terms in the final decomposition of n. As a result, the lower bound of Geometric Game on n is at least $n-m$.

Furthermore, according to 3.2, Geometric Game on n can be played by only combining moves. In this case, the game only contains combining moves. As there are no splitting moves in the game, the total number of steps is exactly $n-m$. Therefore, the lower bound of Geometric Game on n is $n-m$, so Theorem 3.3 is proved. □

3.4 Upper Bound of the Length of Geometric Game

Theorem 3.4. *For any $n \geq 6$, the total number of steps in the Geometric Game on n never exceeds $2n-6$. In other words, the upper bound of game length never exceeds $2n-6$ for any $n \geq 6$. Furthermore, we can never achieve this upper bound of $2n-6$ when n is not in the form of 2^p, where p is a non-negative integer.*

Note that in Theorem 3.4 and the following proof of this theorem, 2^p is an actual number meaning "2 to the power of p".

Proof. When $n \geq 6$, at the beginning of the game, the sum of indices is n. Note that the smallest sum of indices needed for a splitting move is 6, which corresponds to splitting 3 a_2's. According to Lemma 2.6 and Lemma 2.7, the sum of indices throughout the whole game never increases, and each splitting move decreases the sum of indicies by 1. Therefore, the upper bound splitting moves in Geometric game never exceeds $n-6+1=n-5$.

Furthermore, at the beginning of the game, the total number of terms in the initial game state is n. In the Geometric Decomposition of any positive integer n, there will be at least 1 term in the final game state. According to Lemma 2.4 and Lemma 2.4, the number of terms throughout the geometric game never increases, and each combining move decreases the number of terms by 1. Therefore, the maximum possible number of combining moves in Geometric game is $n-1$.

Since the maximum possible total number of steps never exceeds the sum of maximum possible number of combining moves and maximum splitting moves, the upper bound of the game length never exceeds $(n-5)+(n-1)=2n-6$.

Furthermore, if n is not in the form of 2^p (p is a non-negative integer), then the final decomposition of n will contain at least 2 terms since any terms in the Geometric Sequence is a power of 2. Therefore, as there are n terms at the beginning and each combining move decreases the number of terms by 1, the maximum possible number of combining moves will be $n-2$. As we proved earlier, the maximum number of splitting move is $n-1$. Therefore, the total number of steps will not exceed $(n-2)+(n-5) = 2n-7 < 2n-6$. In other words, the upper bound of the game length cannot reach $2n-6$ if n is not in the form of 2^p. As a consequence, the maximum total number of steps for n can possibly be $2n-6$ only if n is in the form of 2^p. □

4 Winning Strategy of 2-Player Game

In this section, we will determine which player has a winning strategy for a two-player Geometric Game on n.

4.1 Winning Strategy of 2-player Game

Theorem 4.1. *For any positive integer* $2 \le n \le 49$*, suppose the final decomposition of* n *consists of* m *terms. In the* 2*-player Geometric Game on* n*, if* $n-m$ *is odd, then Player* 1 *always has a winning strategy; if* $n-m$ *is even, then Player* 2 *always has a winning strategy.*

Proof. Proving this theorem is equivalent of proving the following statement:

Player k $(1 \le k \le 2)$ always has a winning strategy for any $2 \le n \le 49$: when $n-m$ is odd, $k=1$; when $n-m$ is even, $k=2$.

Therefore, in all the following proof, the value of k is determined as follows: when $n-m$ is odd, $k=1$; when $n-m$ is even, $k=2$.

First of all, we need to prove the following lemma:

Lemma 4.2. *For a geometric Game on* n*, at any state before reaching the final decomposition, there is always at least one available combining move.*

Proof. For any state before reaching the final decomposition, as the state is not the final decomposition, there exists at least one term of Geometric Sequence that has at least 2 copies, and we can call this term a_i. Therefore, from this state, the player can combine two copies of a_i into one copy of a_{i+1}, so there is always at least one available combining move. Therefore, 4.2 is proved. □

According to Lemma 4.2, every time it is Player k's turn to make a move, there is always at least one combining move available for Player k. As a result, we can design Player k's strategy of playing the Geometric Game as follows.

Definition 4.3 (Player k's Strategy)**.** Player k only takes combining moves each time. In other words, Player k never takes splitting moves. From any state of the game except the final decomposition, Player k chooses combining moves as follows:

1. For all terms a_i in the Geometric Sequence where $i \geq 2$, Player k chooses the term a_j that has most copies and combines 2 copies of a_j in to one copy of a_{j+1}.

2. For any terms a_i, where $i \geq 2$, if there are two terms or more than two terms have same number of copies and all these terms have most copies among all terms a_i ($i \geq 2$). Then, among these terms that have most copies, Player k chooses the term a_j that has the largest index j, and combines 2 copies of a_j in to one copy of a_{j+1}.

3. If every term a_i ($i \geq 2$) has only 1 copy, then Player k combines two copies of a_1 into one copy of a_2.

In order to prove that Player k's strategy described in Definition 4.3 is the winning strategy for Player k, we need to prove the following lemma.

Lemma 4.4. *In a Geometric Game on any $2 \leq n \leq 49$, when Player k plays the game according to the strategy described in the Definition 4.3, after each of Player k's move, there are no splitting moves available for the opponent player, which is Player $k+1 \pmod 2$.*

Proof. Note that for any $n \geq 46$, if $k = 1$ and Player 1 plays the game according to the definition 4.3, computing by hand, we can find that Player 2 will be able to make a splitting move as early as Player 2's 21st step. We can also find that the smallest integer such that $n \geq 46$ and $k = 1$ is $n = 50$. Therefore, in the following proof, we focus on the integers $2 \leq n \leq 49$.

Computing by hand, we can verify that for any $2 \geq n \geq 49$, if $k = 1$ and Player 1 plays the game according to the Definition 4.3, then Player 2 never has a chance to perform a splitting move; if $k = 2$ and Player 2 plays the game according to the Definition 4.3, then Player 1 never has a chance to perform a splitting move.

Therefore, we have proved that the statement of Lemma 4.4 is true.

□

According to Lemma 4.4, for any $2 \leq n \leq 49$, if Player k follows the strategy, then the game will only contains combining moves. According to Theorem 3.1, the total number of combining moves is fixed, which is always $n - m$, so Player k will always have a winning strategy. Therefore, Theorem 4.1 is proved.

□

5 Multiplayer Game

In this section, we move from the 2-player game to multiplayer game, which consists of 3 or more players. The general strategy of proving all the properties and lemmas in Theorem 5.1 is essentially very similar, which is proving by contradiction. More specifically, we assume that 1 player has a winning strategy. Then we construct a sequence of moves for the other $p - 1$ players, and we also find a shorter sequence of moves so that one of the other $p-1$ players can use the shorter sequence of moves. Then we use this stealing strategy to get the contradiction.

5.1 Winning Strategy of 5 or More than 5 Player Game

Theorem 5.1. *When $n \geq 28$, for any $p \geq 5$, no player has a winning strategy.*

Proof. In all the following proofs, all player numbers are calculated $\bmod p$.

To prove Theorem 5.1, we introduce the following property:

Property 5.2. Suppose Player m has a winning strategy $(1 \leq m \leq p)$. For any $p \geq 2$, if Player m is not the player who takes step 1 listed below, then any winning path of Player m does not contain the following 2 consecutive steps:

Step $1: 2 \wedge 2 \wedge 2 \rightarrow 1 \wedge 1 \wedge 4$.

Step $2: 1 \wedge 1 \rightarrow 2$.

Proof. Suppose Player m has a winning strategy and there is a winning path that contains these 5 consecutive steps. Then there exists a Player a where $1 \leq a \leq p,\ a \neq m$, such that Player a can take step 1 and Player $a+1 \pmod p$ can take step 2. Note that instead of doing $2 \wedge 2 \wedge 2 \rightarrow 1 \wedge 1 \wedge 4$, Player a can do $2 \wedge 2 \rightarrow 4$. Then the game state is the same for Player $m-1$ as it was for Player m when Player m had a winning strategy, so Player $m-1$ now has a winning strategy and this is a contradiction. Therefore, by using stealing strategy, Property 5.2 holds. □

We now prove Theorem 5.1 by splitting it into the following 2 lemmas, Lemma 5.3 and Lemma 5.4.

Lemma 5.3. *When $n \geq 20$, for any $p \geq 6$, no player has a winning strategy.*

Proof. Suppose Player m has a winning strategy $(1 \leq m \leq p)$.

Consider the following two cases.

Case 1. If $m \geq 6$, then Player $1,2,3,4,5$ can do the following:

Player $1: 1 \wedge 1 \rightarrow 2$.

Player $2: 1 \wedge 1 \rightarrow 2$.

Player $3: 1 \wedge 1 \rightarrow 2$.

Player $4: 2 \wedge 2 \wedge 2 \rightarrow 1 \wedge 1 \wedge 4$.

Player $5: 1 \wedge 1 \rightarrow 2$.

The steps for Player 4 and 5 contradict Property 5.2, so Player m does not have winning strategy for any $m \geq 6$.

Case 2. If $m \leq 5$, then after Player m's first move, Player $m+1, m+2, m+3, m+4, m+5$($\pmod p$) can do the following:

Player $m+1: 1 \wedge 1 \rightarrow 2$.

Player $m+2: 1 \wedge 1 \rightarrow 2$.

Player $m+3 : 1 \wedge 1 \rightarrow 2$.
Player $m+4 : 2 \wedge 2 \wedge 2 \rightarrow 1 \wedge 1 \wedge 4$.
Player $m+5 : 1 \wedge 1 \rightarrow 2$.

The steps for Player $m+4$ and $m+5$ contradict Property 5.2, so Player m does not have winning strategy for any $m \le 5$.

By Case 1 and Case 2, Lemma 5.3 is proved. □

Lemma 5.4. *When $n \ge 28$, for $p = 5$ no player has a winning strategy.*

Proof. Suppose Player m has a winning strategy ($1 \le m \le 5$). After Player m's first move, Players $m+1$, $m+2$, $m+3$, $m+4$ ($\pmod{p}$) can do the following in the next 2 rounds (if $m = 5$, we can start the following process from the first step of the game):

Player $m+1 : 1 \wedge 1 \rightarrow 2$ (Step 1).
Player $m+2 : 1 \wedge 1 \rightarrow 2$ (Step 2).
Player $m+3 : 1 \wedge 1 \rightarrow 2$ (Step 3).
Player $m+4 : 1 \wedge 1 \rightarrow 2$ (Step 4).
Player m: Player m can do anything (Step 5).
Player $m+1 : 1 \wedge 1 \rightarrow 2$ (Step 6).
Player $m+2 : 1 \wedge 1 \rightarrow 2$ (Step 7).
Player $m+3 : 2 \wedge 2 \wedge 2 \rightarrow 1 \wedge 1 \wedge 4$ (Step 8).
Player $m+4 : 1 \wedge 1 \rightarrow 2$ (Step 9).

Note that Step 5 removes at most three 2's, but Step 1, Step 2 and Step 3 generate four 2's in total, so there will be at least one 2 remaining after step 5. Therefore, as Step 6 and Step 7 has generated two 2's, there will be at least three 2's left after Step 7, that's why Player $m+3$ is always able to do $2 \wedge 2 \wedge 2 \rightarrow 1 \wedge 1 \wedge 4$ in Step 8. As Step 8 and Step 9 contradict Property 5.2, we have a contradiction. Therefore, by using a stealing strategy, Lemma 5.4 is proved. □

By Lemmas 5.3 and 5.4, Theorem 5.1 is proved. □

5.2 Winning Strategy of 3-player Game

Theorem 5.5. *When $n \ge 8$ and $p = 3$, Player 2 never has a winning strategy.*

Proof. Suppose Player 2 has a winning strategy.

When $n \ge 8$, At the beginning of the game, Player 1 and Player 2 both must do $1 \wedge 1 \rightarrow 2$ as Player 1's first step and Player 2's first step. After these two steps, we can let Player 3 also do $1 \wedge 1 \rightarrow 2$ as Player 3's first step. Then, we can let Player 1 do $2 \wedge 2 \wedge 2 \rightarrow 1 \wedge 1 \wedge 4$ as Player 1's second step. After the 4 steps mentioned above, the game state is $\{1^{n-4} + 4^1\}$. As a result, Player 2 can only do $1 \wedge 1 \rightarrow 2$ as Player 2's second step. Therefore, if Player 2 has a winning strategy, then Player 2 must have a winning strategy for any paths starting with the following 5 steps:

Player $1 : 1 \wedge 1 \rightarrow 2$.
Player $2 : 1 \wedge 1 \rightarrow 2$.
Player $3 : 1 \wedge 1 \rightarrow 2$.
Player $1 : 2 \wedge 2 \wedge 2 \rightarrow 1 \wedge 1 \wedge 4$.
Player $2 : 1 \wedge 1 \rightarrow 2$.

Note that among these 5 steps, Player 1 takes the 4^{th} step, so Player 2 is the not the player who takes step 4. As a result, the last 2 steps contradict Property 5.2. Therefore, by contradiction, we have proved that Theorem 5.5 is true for any $n \geq 8$. Thus, Theorem 5.5 is proved. □

5.3 Winning Strategy of 4-player Game

Theorem 5.6. *When $n \geq 10$ and $p = 4$, Player 1 and Player 2 never have a winning strategy.*

Proof. Suppose Player 1 has a winning strategy.
When $n \geq 10$, At the beginning of the game, Player 1 and Player 2 both must do $1 \wedge 1 \rightarrow 2$ as Player 1's first step and Player 2's first step. After these two steps, we can let Player 3 also do $1 \wedge 1 \rightarrow 2$ as Player 3's first step and we can let Player 4 do $2 \wedge 2 \wedge 2 \rightarrow 1 \wedge 1 \wedge 4$ as Player 4's first step. After the 4 steps mentioned above, the game state is $\{1^{n-4} + 4^1\}$. As a result, Player 1 can only do $1 \wedge 1 \rightarrow 2$ as Player 1's second step. Therefore, if Player 1 has a winning strategy, then Player 1 must have a winning strategy for any paths starting with the following 5 steps:
Player $1 : 1 \wedge 1 \rightarrow 2$.
Player $2 : 1 \wedge 1 \rightarrow 2$.
Player $3 : 1 \wedge 1 \rightarrow 2$.
Player $4 : 2 \wedge 2 \wedge 2 \rightarrow 1 \wedge 1 \wedge 4$.
Player $1 : 1 \wedge 1 \rightarrow 2$.

Note that among these 5 steps, Player 4 takes the 4^{th} step. Therefore, Player 1 is the not the player who takes step 4. As a result, the last two steps violate Property 5.2. So by contradiction, we have proved that in a 4-player game, Player 1 never has a winning strategy for any $n \geq 10$.

Now, suppose Player 2 has a winning strategy. When $n \geq 10$, At the beginning of the game, Player 1 and Player 2 both must do $1 \wedge 1 \rightarrow 2$ as Player 1's first step and Player 2's first step. After these two steps, we can let Player 3 also do $1 \wedge 1 \rightarrow 2$ as Player 3's first step and we can let Player 4 do $2 \wedge 2 \wedge 2 \rightarrow 1 \wedge 1 \wedge 4$ as Player 4's first step. After the 4 steps mentioned above, the game state is $\{1^{n-4} + 4^1\}$. As a result, Player 1 can only do $1 \wedge 1 \rightarrow 2$ as Player 1's second step. Therefore, if Player 2 has a winning strategy, then Player 2 must have a winning strategy for any paths starting with the following 5 steps:
Player $1 : 1 \wedge 1 \rightarrow 2$.

Player $2:1\wedge 1\to 2$.
Player $3:1\wedge 1\to 2$.
Player $4:2\wedge 2\wedge 2\to 1\wedge 1\wedge 4$.
Player $1:1\wedge 1\to 2$.

Note that among these 5 steps, Player 4 takes the 4^{th} step. Therefore, Player 2 is the not the player who takes step 4. As a result, the last 2 steps violate Property 1. So by contradiction, we have proved that in a 4-player game, Player 2 never have a winning strategy for any $n\geq 10$.

Therefore, in a 4-player game, neither Player 1 nor Player 2 has a winning strategy for any $n\geq 10$. Thus, Theorem 5.6 is proved. □

6 Multialliance Game

In this section, we move from the multiplayer game to multialliance game. In the Geometric Game, if there are p players playing the game and these p players are divided into 2 or more alliances, then we call it multialliance game. In the multialliance game, each palyer plays one step every turn, and every player take turns playing the game. If Player m is the player who takes the last step, then Player m's alliance wins the game. The alliances all take their turns consecutively. In this section, we focus on some special cases of the multialliance game. Note that for all theorems in this section, n must satisfy some lower bound because we need enough number of 2 to make use of Property 5.2 and Property 6.2 listed below.

6.1 Winning Strategy of Multialliance Game

Theorem 6.1. *When there are t alliances (t teams) and each alliance has $k=t-1$ consecutive players, if $t\geq 5$ and $n\geq max\{2k^2+10k,160\}$, then no alliance has a winning strategy.*

Proof. For the following proofs, team 0 is same as team t (under $\mod t$).

Note that the player after Player tk next is player is Player 1, so we treat Player tk and Player 1 as two consecutive players. In the proof of Theorem 6.1, without loss of generality, we assume that team 1 has Player $1,2,3,\dots,k$; team 2 has Player $k+1,k+2,\dots,2k$; team 3 has Player $2k+1,2k+2,\dots,3k$ etc.

Now we need to prove the following property.

Property 6.2. Suppose team m has a winning strategy ($1\leq m\leq t$). When there are $t\geq 2$ teams and each team has exactly $k=t-1$ consecutive players, if none of the k players in Part 1 listed below belong to team m, then all the winning paths for team m do not contain the following $2k$ consecutive steps:

Consecutive k players all do: $2\wedge 2\wedge 2\to 1\wedge 1\wedge 4$ (Part 1).

Next k consecutive players all do: $1 \wedge 1 \to 2$ (Part 2).

Proof. Suppose team m has a winning strategy and there is a winning path for team m that contains such $2k$ consecutive steps. Then there exists an integer q ($1 \le q \le p$) such that Player q belongs to team m and takes the last step of the game.

For the k players in Part 1 listed in 6.2, instead of doing $2 \wedge 2 \wedge 2 \to 1 \wedge 1 \wedge 4$, they can all do $2 \wedge 2 \to 4$. By doing so, Player $q-k \pmod p$ now becomes the player who takes the last step. Note that team m has k players, so Player $q-k \pmod p$ belongs to team $m-1 \pmod t$. Therefore, team $m-1(\bmod t)$ now has a winning strategy, which contradicts with our previous assumption. As a result, by using stealing strategy, Property 6.2 is proved. □

Now we prove Theorem 6.1 by splitting it into the following 2 cases.

Case 1. When $n \ge 2k^2 + 10k$, $t \ge 6$ and $k = t-1$, no team has winning strategy.

Proof. Assume that team m has a winning strategy ($1 \le m \le t$). Note that the last player in team m is Player mk, so the first player after team m is Player $mk+1 \pmod p$. Also, since there are $t-1 = k$ other teams, and each team has k players, where $k \ge 6-1 = 5$. Therefore, there are $k^2 \ge 5k$ consecutive players from other teams. After all the members of team m's first move, the consecutive $5k$ other teams can do the following:

(If $m = t$, we start the following steps from the first step of Player 1.) In all the following, given players' numbers are mod p.

From Player $mk+1$ to $(m+1)k$ all do: $1 \wedge 1 \to 2$.

From Player $(m+1)k+1$ to Player $(m+2)k$ all do: $1 \wedge 1 \to 2$.

From Player $(m+2)k+1$ to Player $(m+3)k$ all do: $1 \wedge 1 \to 2$.

From Player $(m+3)k+1$ to Player $(m+4)k$ all do: $2 \wedge 2 \wedge 2 \to 1 \wedge 1 \wedge 4$.

From Player $(m+4)k+1$ to Player $(m+5)k$ all do: $1 \wedge 1 \to 2$.

Since all these $5k$ players are not from team m, the steps from $(m+3)k+1$ to Player $(m+5)k$ contradict with property 6.2, so team m does not have winning strategy. Therefore, Case 1 is proved. □

Case 2. When $n \ge 160$, for any $t = 5$ and $k = 4$, no team has winning strategy.

Proof. Suppose team m has a winning strategy ($1 \le m \le 5$). Note that the game has 5 alliances and 20 players in total, so all the players' numbers listed below are under mod 20, and all the teams' numbers listed below are under mod 5.

Note that team m has Players $4m-3, 4m-2, 4m-1, 4m$, while the other players are not in team m. Therefore, we can design a path, where players not in team m take some certain steps, so that no matter what steps players in team m take to win the game, players not in team m can use a stealing strategy and lead to a contradiction. The path is designed as follows. After Player $4m$'s (last player from team m) first move, we can do the following first:

(if $m = 5$, the same following process can start from the first step of Player 1.)
From $4m+1$ to Player $4m+16$ all do: $1 \wedge 1 \to 2$ (Step 1 to 16).
Player $4m-3$: anything (Step 17).
Player $4m-2$: anything (Step 18).
Player $4m-1$: anything (Step 19).
Player $4m$: anything (Step 20).
From $4m+1$ to Player $4m+16$ all do: $1 \wedge 1 \to 2$ (Step 21 to 36).
Player $4m-3$: anything (Step 37).
Player $4m-2$: anything (Step 38).
Player $4m-1$: anything (Step 39).
Player $4m$: anything (Step 40).
From $4m+1$ to Player $4m+16$ all do: $1 \wedge 1 \to 2$ (Step 41 to 56).
Player $4m-3$: anything (Step 57).
Player $4m-2$: anything (Step 58).
Player $4m-1$: anything (Step 59).
Player $4m$: anything (Step 60).
From $4m+1$ to Player $4m+4$ all do: $2 \wedge 2 \wedge 2 \to 1 \wedge 1 \wedge 4$ (Step 61 to 64).
From $4m+5$ to Player $4m+8$ all do: $1 \wedge 1 \to 2$ (Step 65 to 68).
(Note: step $17, 18, 19, 20, 37, 38, 39, 40, 57, 58, 59, 60$ can be anything because they are controlled by team m.)

Note that from Step 1 to step 16, each step generates one copy of 2, so these 16 steps in total generate 16 copies of 2. From Step 17 to Step 20, each step can take away at most 3 copies of 2, so these 4 steps can take away at most 12 copies of 2. Therefore, from Step 1 to Step 20, there are at least 4 copies of 2 remaining.

Similarly, from step 21 to step 36, 16 copies of 2 are generated in total. Step $37, 38, 39, 40$ can take away at most 12 copies of 2. As there are at least 4 copies of 2 remaining after step 20, there are at least 8 copies of 2 remaining after Step 40.

Furthermore, from step 41 to step 56, 16 copies of 2 are generated in total. Step $57, 58, 59, 60$ can take away at most 12 copies of 2. As there are at least 8 copies of 2 remaining after step 40, there are at least 12 copies of 2 remaining after Step 60.

Since there are at least 12 copies of 2 remaining after Step 60, we have enough numbers of 2 to perform $2 \wedge 2 \wedge 2 \to 1 \wedge 1 \wedge 4$ from Step 61 to Step 64. Since the steps from Step 61 to Step 68 violate property 6.2, we have a contradiction. Therefore, Case 2 is proved.

□

Therefore, according to Case 1 and Case 2, Theorem 6.1 is proved. □

6.2 Winning Strategy of 2-alliance Game

Theorem 6.3. *When $n \geq 12k^2 + 13k$ and there are 2 alliances in the Geometric Game, if one alliance consists of $3k+1$ consecutive players (and we call it big alliance), and the other alliance has k consecutive players, then the big alliance always has a winning strategy.*

Proof. In the following proof of Theorem 6.3, we define the start of the first round as the first player in the big alliance's first move, and we define one round as all the $4k+1$ players in the game take turns playing the game and each player makes exactly one move in each round.

In the first $3k$ round, we let all players from the big alliance all do $1 \wedge 1 \to 2$. In round $2k+1$, we let the first k players from the big alliance all do $2 \wedge 2 \wedge 2 \to 1 \wedge 1 \wedge 4$, and the second k players from the big alliance all do $1 \wedge 1 \to 2$.

Suppose the small alliance has a winning strategy. In each of the first $3k$ rounds, since the big alliance has $3k+1$ consecutive players and each step of $1 \wedge 1 \to 2$ generates one copy of 2, so each round generates $3k+1$ copies of 2. Furthermore, in each of the first $3k$ rounds, since the small alliance has k consecutive players and each step can take away at most three copies of 2, the small alliance in each round can take away at most $3k$ copies of 2 in total. Therefore, for every round, there will be at least 1 copy of newly generated 2 remaining. (Note that here, in a round, the newly generated 2 means the 2 generated in this round.) As a result, after $3k$ rounds, there will be at least $3k$ copies of 2 remaining, so we have enough copies of 2 for the first k players of the big alliance all perform $2 \wedge 2 \wedge 2 \to 1 \wedge 1 \wedge 4$ in round $3k+1$.

Since the steps of the first $2k$ players in the big alliance in round $3k+1$ violate Property 6.2, now we have a contradiction. Therefore, by using stealing strategy, the big alliance always has a winning strategy. □

7 Conclusion and Future Direction

Stemming from the rules and some interesting results of Fibonacci Zeckendorf Game, we have created a Geometric Game based on the Geometric Sequence of ratio 2 along with combining moves and splitting moves. In this game, we proved that the game is finite, and we also proved the existence and uniqueness of the final Geometric Decomposition on n. Then, we found out the upper bound and lower bound on n. Furthermore, we proved that the winning strategy of 2-player game depends on the parity of $n-m$, where m is the number of terms on n. Then we also proved some results about the winning strategy of some special cases of multiplayer games and multialliance games by using the stealing strategy.

Meanwhile, in the future, there are some interesting directions that we can work on in terms of this Geometric game. One of the future directions is for a 2-player Geometric

Game on n, for any $n \geq 50$, we can try to find a general pattern about which player has a winning strategy. Furthermore, for the 2-player Geometric Game on an arbitrary n, how we can find a general method to tell which player has a winning strategy without calculating the number of terms m. In other words, how we can find a general method to tell which player has a winning strategy as a function of n. We might also further tighten the upper bound of the Geometric Game on an arbitrary n, find a tighter upper bound and when we achieve that upper bound. Besdies, in the 3-player Geometric Game on an arbitrary n, we can explore whether player 1 has a winning strategy and whether Player 3 has a winning strategy; in the 4-player Geometric Game on an arbitrary n, we can explore whether Player 3 has a winning strategy and whether Player 4 has a winning strategy. Furthermore, we can also try to improve the lower bound on the size of big alliance in Theorem 6.3 to guarantee that the big alliance always has a winning strategy.

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