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Cover Page Footnote

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A New Method to Compute the Hadamard Product of Two Rational Functions

By *Ishan Kar*

Abstract. The Hadamard product (denoted by $*$) of two functions $\mathrm{A}(x)$ and $\mathrm{B}(x)$ with their power series expansions $\mathrm{A}(x) = a_0 + a_1x + a_2x^2 + \cdots$ and $\mathrm{B}(x) = b_0 + b_1x + b_2x^2 + \cdots$ is the power series $\mathrm{A}(x) * \mathrm{B}(x) = a_0b_0 + a_1b_1x + a_2b_2x^2 + \cdots$. Although it is well known that the Hadamard product of two rational functions corresponds to a rational function, a closed form expression of the Hadamard product of rational functions has not been found. Since any rational power series can be expanded by partial fractions as a polynomial plus a sum of power series of the form $\frac{1}{(1-ax)^{m+1}}$, to find the Hadamard product of any two rational power series it is sufficient to find the Hadamard product

$$\frac{1}{(1-ax)^{m+1}} * \frac{1}{(1-bx)^{n+1}} = \frac{(1+ax)^m * (1+bx)^n}{(1-abx)^{m+n+1}}.$$

The Hadamard product of negative powers of quadratic polynomials have also been derived.

1 Introduction

The Hadamard product [1, 5, 15] of two functions $\mathrm{A}(x)$ and $\mathrm{B}(x)$ with their power series expansions $\mathrm{A}(x) = \sum_{n=0}^{\infty} a_nx^n = a_0 + a_1x + a_2x^2 + \cdots$ and $\mathrm{B}(x) = \sum_{n=0}^{\infty} b_nx^n = b_0 + b_1x + b_2x^2 + \cdots$ is the power series defined by

$$\mathrm{A}(x) * \mathrm{B}(x) = \sum_{n=0}^{\infty} a_nb_nx^n = a_0b_0 + a_1b_1x + a_2b_2x^2 + \cdots.$$

For example if $\mathrm{A}(x) = \frac{1}{1-2x} = 1 + 2x + 2^2x^2 + \cdots$ and $\mathrm{B}(x) = \frac{1}{1-3x} = 1 + 3x + 3^2x^2 + \cdots$ then

$$\mathrm{A}(x) * \mathrm{B}(x) = 1 + 2\cdot 3x + 2^2\cdot 3^2x^2 \cdots = \sum_{n=0}^{\infty} 2^n3^nx^n = \frac{1}{1-6x}.$$

The Hadamard product [5] is different from the ordinary product of $\mathrm{A}(x)$ and $\mathrm{B}(x)$, where the coefficient of x^n is the convolution of the sequences of a_n and b_n.

$$\mathrm{A}(x) * \mathrm{B}(x) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} a_kb_{n-k}x^k = a_0b_0 + (a_1b_0 + a_0b_1)x + (a_2b_0 + a_1b_1 + a_2b_0)x^2 + \cdots.$$

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The Hadamard product is commutative and distributive over addition. That is, for rational functions $\mathrm{A}(x), \mathrm{B}(x)$, and $\mathrm{C}(x)$

$$\text{Commutative: } \mathrm{A}(x) * \mathrm{B}(x) = \mathrm{B}(x) * \mathrm{A}(x)$$
$$\text{Distributive: } \mathrm{A}(x) * \{\mathrm{B}(x) + \mathrm{C}(x)\} = \mathrm{A}(x) * \mathrm{B}(x) + \mathrm{A}(x) * \mathrm{C}(x).$$

Some interesting examples of Hadamard product are

- $\mathrm{A}(x) = \frac{1}{1-x} = 1 + x + x^2 + \cdots$ acts as an identity in the Hadamard product, where $\mathrm{A}(x) * \mathrm{B}(x) = \mathrm{B}(x)$ for any $\mathrm{B}(x)$.
- If $\mathrm{A}(x) = \frac{1}{1-\lambda x}$, then $\mathrm{A}(x) * \mathrm{B}(x) = \mathrm{B}(\lambda x)$.
- For $\mathrm{A}(x) = \frac{x}{x^2+1} = \sum_k (-1)^k x^{2k+1}$ and $\mathrm{B}(x) = \frac{x}{x^2+1} = \sum_k -x^{2k+1}$, gives Hadamard product $\mathrm{C}(x) = \mathrm{A}(x) * \mathrm{B}(x) = -\sum_k (-1)^k x^{2k+1}$, so $\mathrm{C}(x) = -\mathrm{A}(x)$.

2 Prior Methods and Applications

Although useful, the analytic solution is complex involving Cauchy's residue theorem [6, Eqn. 11 on p. 230][10, 12]

$$\mathrm{A}(x) * \mathrm{B}(x) = \frac{1}{2\pi i} \oint_{\mathrm{C}} \mathrm{A}(s)\mathrm{B}(\frac{x}{s})\frac{ds}{s} = \sum_{i=1}^{k} \mathrm{res}_{s=s_i}\{\frac{\mathrm{A}(s)\mathrm{B}(x/s)}{s}\}.$$

If $\mathrm{A}(x)$ converges for $|x| < a$ and $\mathrm{B}(x)$ converges for $|x| < b$ then the integral formula is valid for $|x| < ab$, where C is the circle $|s| = (\mathrm{A} + |x|/\mathrm{B})/2$. The integral can be evaluated in terms of residues where k is the number of different poles s_i of $\mathrm{A}(s)\mathrm{B}(x/s)/s$ within C. Residue computation is complex for pole s_i of multiplicity $m > 1$.

$$\mathrm{res}_{s=s_i}\{\frac{\mathrm{A}(s)\mathrm{B}(x/s)}{s}\} = \frac{1}{(m-1)!} \lim_{s\to s_i} \frac{d^{m-1}}{ds^{m-1}} \left[(s-s_i)^m \frac{\mathrm{A}(s)\mathrm{B}(x/s)}{s} \right].$$

Salvy and Zimmerman [18] describe a Maple implementation of a method for computing the differential equations satisfied by Hadamard products of holonomic power series (series that satisfy a linear differential equation with polynomial coefficients). Shapiro [14] gave a combinatorial proof using tilings of the Hadamard product of generating functions for Chebyshev polynomials given by

$$\frac{1}{1-ax-x^2} * \frac{1}{1-cx-x^2} = \frac{1-x^2}{1-acx-(2+a^2+c^2)x^2-acx^3+x^4}. \tag{1}$$

Kim [8, 7] extended this tilings based combinatorial method to find the Hadamard product of

$$\frac{1}{1-ax-x^2} * \frac{x^m}{1-cx-x^n}, \quad m, n \in \mathbb{Z}_{\geq 2}.$$

The Hadamard product finds its way in numerous applications. Gessel [4] uses Hadamard product to evaluate sums of products of Fibonacci numbers. Bragg [2] cites numerous applications of Hadamard products in theoretical and applied mathematics. Prodinger and Selkirk [13] used it to find an explicit formula for the sum of Tetranacci numbers. Potekhina and Tolovikov [11], Zhilinskii [19] applied the Hadamard product of rational functions to probabilistic problems and molecular physics respectively. In signal processing, we can apply our formula to the z-transform of the product of two signals and also in Parseval's theorem [17, 12].

Although it has been proven by partial fraction expansion [15, Proposition 4.2.5 on p. 542], [16] that the Hadamard product of two rational generating function $A(x) * B(x)$ is also rational, an explicit formula for the product of two rational functions has not been derived. In this paper we present a formula for the Hadamard product of two rational functions.

3 The New Method

A generating function is a way of encoding an infinite sequence of numbers by treating them as the coefficients of a power series. A rational generating function $A(x)$, where the degree of the numerator is less than the degree of the denominator, has a unique partial fraction expansion to represent $A(x)$ as a linear combination of [9]

$$A(x) = \sum_{i=1}^{M} \left[\frac{A_{i,0}}{(1-a_i x)} + \frac{A_{i,1}}{(1-a_i x)^2} + \cdots \frac{A_{i,m_i}}{(1-a_i x)^{m_i+1}} \right],$$

where $1/a_i$ are the poles with multiplicity $m_i + 1$ of $A(x)$ and a_i can be real or complex. The Hadamard product after the partial fraction expansion of two rational functions $A(x), B(x)$ is given as

$$A(x) * B(x) = \sum_{i=1}^{M} \sum_{m=0}^{m_i} \frac{A_{i,m}}{(1-a_i x)^{m+1}} * \sum_{j=1}^{N} \sum_{n=0}^{n_j} \frac{B_{j,n}}{(1-b_j x)^{n+1}} \tag{2}$$

$$= \sum_{i=1}^{M} \sum_{m=0}^{m_i} \sum_{j=1}^{N} \sum_{n=0}^{n_j} \frac{A_{i,m}}{(1-a_i x)^{m+1}} * \frac{B_{j,n}}{(1-b_j x)^{n+1}}. \tag{3}$$

Since the Hadamard product of two rational functions $A(x)$ and $B(x)$ is the sum of the Hadamard products of the terms in their partial fraction expansion, it is sufficient to find a formula for $\frac{1}{(1-ax)^{m+1}} * \frac{1}{(1-bx)^{n+1}}$ to find the explicit formula for the Hadamard product $A(x) * B(x)$. Although this formula will suffice for all rational functions, to improve speed of computation, in Section 5 we will derive a formula for rational functions with a power of a quadratic.

Theorem 3.1. *If* $\mathrm{A}(x)$ *and* $\mathrm{B}(x)$ *are rational functions of* x*, then*

$$[y^m z^n]\frac{1}{\mathrm{A}(x)-y} * \frac{1}{\mathrm{B}(x)-z} = \frac{1}{\mathrm{A}(x)^{m+1}} * \frac{1}{\mathrm{B}(x)^{n+1}} \tag{4}$$

where the symbol $[y^m z^n] f(y,z)$ *will mean the coefficient of* $y^m z^n$ *in the series* $f(y,z)$ *and Hadamard products are with respect to* x*.*

Proof.

$$\begin{aligned}\frac{1}{\mathrm{A}(x)-y} * \frac{1}{\mathrm{B}(x)-z} &= \frac{1}{\mathrm{A}(x)\left(1-\frac{y}{\mathrm{A}(x)}\right)} * \frac{1}{\mathrm{B}(x)\left(1-\frac{z}{\mathrm{B}(x)}\right)} \\ &= \sum_{m=0}^{\infty}\frac{y^m}{\mathrm{A}(x)^{m+1}} * \sum_{n=0}^{\infty}\frac{z^n}{\mathrm{B}(x)^{n+1}}\end{aligned}$$

This is by power series expansion. By equating the coefficients of $y^m z^n$ on both sides, we get

$$[y^m z^n]\frac{1}{\mathrm{A}(x)-y} * \frac{1}{\mathrm{B}(x)-z} = \frac{1}{\mathrm{A}(x)^{m+1}} * \frac{1}{\mathrm{B}(x)^{n+1}}. \tag{5}$$

□

4 Hadamard Product of negative powers of linear factors

By the binomial theorem $\frac{1}{(1-ax)^{m+1}} = \sum_{k=0}^{\infty}\binom{m+k}{m}a^k x^k$ and $\frac{1}{(1-bx)^{n+1}} = \sum_{j=0}^{\infty}\binom{n+j}{n}b^j x^j$ where $n, m \in \mathbb{Z}_{\geq 0}$. For simple Hadamard products ($m = n = 0$) it is easy to observe

$$\frac{1}{1-ax} * \frac{1}{1-bx} = \sum_{n=0}^{\infty}(abx)^n = \frac{1}{1-abx}. \tag{6}$$

To prove this for $m, n > 0$, we will use Vandermonde's identity $\binom{x+y}{z} = \sum_{m=0}^{z}\binom{x}{m}\binom{y}{z-m}$, the cancellation identity $\binom{x}{y}\binom{y}{z} = \binom{x}{z}\binom{x-z}{y-z}$ where $y, z \in \mathbb{Z}_{\geq 0}$, and the binomial theorem.

Theorem 4.1.

$$\frac{1}{(1-ax)^{m+1}} * \frac{1}{(1-bx)^{n+1}} = \frac{\sum_{j=0}^{m}\binom{m}{j}\binom{n}{j}(abx)^j}{(1-abx)^{n+m+1}} = \frac{(1+ax)^m * (1+bx)^n}{(1-abx)^{m+n+1}} \tag{7}$$

Proof.

$$\frac{1}{1-ax-y} * \frac{1}{1-bx-z} = \frac{1}{(1-y)(1-z)\{1-\frac{abx}{(1-y)(1-z)}\}} \quad \text{from eq. (6)}$$

$$= \frac{1}{(1-abx)\left(1-\frac{y+z-yz}{1-abx}\right)} = \sum_{k=0}^{\infty}\frac{(y+z-yz)^k}{(1-abx)^{k+1}} \quad \text{by power series expansion.}$$

By **theorem 3.1** we have to find the coefficient of $y^m z^n$ of this equation.

$$[y^m z^n]\frac{1}{1-ax-y} * \frac{1}{1-bx-z} = \sum_{k=0}^{\infty} \frac{\sum_{l=0}^{k} [y^m z^n]\binom{k}{l}(y+z)^{k-l}(-yz)^l}{(1-abx)^{k+1}}$$

$$= \sum_{k=0}^{\infty} \frac{\sum_{l=0}^{k} \binom{k}{l}(-1)^l [y^{m-l} z^{n-l}](y+z)^{k-l}}{(1-abx)^{k+1}}.$$

In order for the coefficient of $y^{m-l}z^{n-l}$ in $(y+z)^{k-l}$ to be nonzero, we must have $m+n-2l = k-l$, so $k = m+n-l$. Also, since the value of k will vary from the maximum of m and n denoted by $m \vee n$ to $m+n$, the value of l will vary from 0 to the minimum of m and n, denoted by $m \wedge n$. Substituting this value for k in the last formula

$$\frac{1}{(1-ax)^{m+1}} * \frac{1}{(1-bx)^{n+1}} = \sum_{l=0}^{m\wedge n} \frac{(-1)^l \binom{m+n-l}{l}\binom{m+n-2l}{n-l}}{(1-abx)^{m+n+1-l}}$$

$$= \sum_{l=0}^{m\wedge n} \frac{1}{(1-abx)^{m+n+1}} \binom{m+n-l}{m}\binom{m}{l}(-1)^l(1-abx)^l \text{ by Cancellation identity}$$

$$= \sum_{l=0}^{m\wedge n} \frac{1}{(1-abx)^{m+n+1}} \sum_{j=0}^{m} \binom{n}{j}\binom{m-l}{m-j}\binom{m}{l}(-1)^l(1-abx)^l \text{ by Vandermonde's identity}$$

$$= \sum_{j=0}^{m\wedge n} \binom{n}{j}\binom{m}{m-j} \frac{1}{(1-abx)^{m+n+1}} \sum_{l=0}^{j} \binom{j}{l}(-1)^l(1-abx)^l \text{ by Cancellation identity}$$

$$= \frac{\sum_{j=0}^{m\wedge n}\binom{n}{j}\binom{m}{j}(abx)^j}{(1-abx)^{m+n+1}} = \frac{(1+ax)^m * (1+bx)^n}{(1-abx)^{m+n+1}}. \qquad \square$$

The **theorem 4.1** can also be derived from the binomial coefficient identity (see Gessel and Stanton Eqn. 1 on p. 87 [3])

$$\binom{n+k}{n}\binom{m+k}{m} = \sum_{j=0}^{m\wedge n} \binom{n}{j}\binom{m}{j}\binom{n+m+k-j}{n+m}.$$

In order to prove the theorem, at first we do a power series expansion of the negative powers of the linear equations. Applying the identity above in eq. (9) and reversing the power series expansion we can prove eq. (7).

$$\frac{1}{(1-ax)^{m+1}} * \frac{1}{(1-bx)^{n+1}} \tag{8}$$

$$= \sum_{k=0}^{\infty} \binom{m+k}{m}\binom{n+k}{n} a^k b^k x^k = \sum_{k=0}^{\infty} \sum_{j=0}^{m \wedge n} \binom{m}{j}\binom{n}{j}\binom{n+m+k-j}{n+m} a^k b^k x^k \tag{9}$$

$$= \sum_{j=0}^{m \wedge n} \binom{n}{j}\binom{m}{j} (abx)^j \sum_r \binom{n+m+r}{n+m} (abx)^r = \frac{\sum_{j=0}^{m} \binom{m}{j}\binom{n}{j}(abx)^j}{(1-abx)^{n+m+1}} \tag{10}$$

$$= \frac{(1+ax)^m * (1+bx)^n}{(1-abx)^{m+n+1}}. \tag{11}$$

Putting this formula back into eq. (3) we will get the formula for the Hadamard product of two rational functions.

$$\mathrm{A}(x) * \mathrm{B}(x) = \sum_{i=1}^{\mathrm{N}} \sum_{n=0}^{n_i} \sum_{j=1}^{\mathrm{M}} \sum_{m=0}^{m_j} \frac{\mathrm{A}_{i,m+1}\mathrm{B}_{j,n+1}(1+a_i x)^m * (1+b_j x)^n}{(1-a_i b_j x)^{n+m+1}}$$

For simple fractions with a zero at origin, the Hadamard product of the partial fraction expansion can be cumbersome. For example, the partial fraction expansion of $\frac{x^3}{(1-ax)^5}$ $= -\frac{1}{a^3(1-ax)^2} + \frac{3}{a^3(1-ax)^3} - \frac{3}{a^3(1-ax)^4} + \frac{1}{a^3(1-ax)^5}$ has four partial fractions. So, the Hadamard product of that with another similar fraction will create too many terms. Some identities to solve the Hadamard product of partial fractions with zero at origin proven similar to eq. (11) and $p, q, u, v \in \mathbb{Z}_{\geq 0}, v \leq q$ are given below.

$$\frac{x^v}{(1-ax)^{q+1}} * \frac{1}{(1-bx)^{p+1}} = \frac{x^v(1+ax)^{q-v} * (1+bx)^{p+v}}{(1-abx)^{p+q+1}}$$

$$\frac{x^{u+v}}{(1-ax)^{q+1}} * \frac{x^u}{(1-bx)^{p+1}} = \frac{x^{u+v}(1+ax)^{q-v} * x^u(1+bx)^{p+v}}{(1-abx)^{p+q+1}}$$

In my new method, given two rational functions $\mathrm{A}(x)$ and $\mathrm{B}(x)$, we can find $\mathrm{A}(x) * \mathrm{B}(x)$ by following three simple algebraic steps (example shown in table 1). The two rational functions $\mathrm{A}(x)$ and $\mathrm{B}(x)$ undergo a partial fraction expansion. The problem gets reduced to the sum of the Hadamard product of the partial fractions of the two functions. Hadamard multiplication of each partial fraction is obtained by using an explicit formula derived in **theorem 4.1**. Composing the generating function $\mathrm{A}(x) * \mathrm{B}(x)$ by the sum of each Hadamard product of the partial fractions.

5 Hadamard Product of negative powers of quadratic

By the fundamental theorem of algebra, if a function is rational, we can write the denominator as $(1-a_1x)^{j_1} \cdots (1-a_m x)^{j_n}(1-b_1x+c_1x^2)^{k_1} \cdots (1-b_m x+c_m x^2)^{k_m}$, where $a_1, \ldots, a_n$;

Partial Fraction Expansion	Hadamard Product of Partials	Sum of the results
$\frac{9x^2-8x+3}{-9x^3+15x^2-7x+1} = \frac{1}{1-x}+\frac{2}{(1-3x)^2}$ $\frac{4x-1}{-8x^3+12x^2-6x+1} = \frac{-2}{(1-2x)^2}+\frac{1}{(1-2x)^3}$	$\frac{1}{1-x}*\frac{-2}{(1-2x)^2}=\frac{-2}{(1-2x)^2}$ $\frac{2}{(1-3x)^2}*\frac{-2}{(1-2x)^2}=\frac{-4(1+6x)}{(1-6x)^3}$ $\frac{1}{1-x}*\frac{1}{(1-2x)^3}=\frac{1}{(1-2x)^3}$ $\frac{1}{(1-2x)^3}*\frac{2}{(1-3x)^2}=\frac{2(1+12x)}{(1-6x)^4}$	$\frac{-2}{(1-2x)^2}+\frac{-4(1+6x)}{(1-6x)^3}+\frac{1}{(1-2x)^3}+\frac{2(1+12x)}{(1-6x)^4} = \frac{4032x^5-3216x^4+1168x^3-336x^2+64x-3}{(1-6x)^4(1-2x)^3}$

Table 1: Example of computing the Hadamard Product of two rational functions

$b_1,\cdots,b_m; c_1,\cdots,c_m$ are real numbers and $j_1,\cdots,j_n; k_1,\cdots,k_m$ are positive integers but the terms $1-b_ix+c_ix^2$ are the irreducible quadratic factors which correspond to pairs of complex conjugate or irrational roots. Then, the partial fraction decomposition of proper rational functions $\mathrm{A}(x)$ and $\mathrm{B}(x)$, where the degree of the denominator is greater than that of the numerator is

$$\mathrm{A}(x)=\sum\frac{\mathrm{A}_{im}}{(1-a_ix)^{m+1}}+\sum\frac{\mathrm{P}_{jm}+\mathrm{Q}_{jm}x}{(1-a_jx+b_jx^2)^{m+1}},$$

$$\mathrm{B}(x)=\sum\frac{\mathrm{B}_{kn}}{(1-b_kx)^{n+1}}+\sum\frac{\mathrm{R}_{kn}+\mathrm{S}_{kn}x}{(1-c_kx+d_kx^2)^{n+1}}.$$

$$\begin{aligned}\mathrm{A}(x)*\mathrm{B}(x)=&\sum\frac{\mathrm{A}_{im}}{(1-a_ix)^{m+1}}*\frac{\mathrm{B}_{kn}}{(1-b_kx)^{n+1}}\\&+\sum\frac{\mathrm{A}_{im}}{(1-a_ix)^{m+1}}*\frac{\mathrm{R}_{kn}+\mathrm{S}_{kn}x}{(1-c_kx+d_kx^2)^{n+1}}\\&+\sum\frac{\mathrm{P}_{jm}+\mathrm{Q}_{jm}x}{(1-a_jx+b_jx^2)^{m+1}}*\frac{\mathrm{R}_{kn}+\mathrm{S}_{kn}x}{(1-c_kx+d_kx^2)^{n+1}}\\&+\sum\frac{\mathrm{P}_{jm}+\mathrm{Q}_{jm}x}{(1-a_jx+b_jx^2)^{m+1}}*\frac{\mathrm{B}_{kn}}{(1-b_kx)^{n+1}}\end{aligned}$$

Again, we can find $\mathrm{A}(x)*\mathrm{B}(x)$ by finding the formula for $\frac{1}{(1-ax)^{m+1}}*\frac{px+q}{(1-bx+cx^2)^{n+1}}$ and $\frac{px+q}{(1-ax+bx^2)^{m+1}}*\frac{rx+s}{(1-cx+dx^2)^{n+1}}$ in addition to eq. (7). $\frac{1}{(1-ax+bx^2)^{n+1}}*\frac{1}{(1-cx+dx^2)^{m+1}}$ gives us the Hadamard product of generating functions for two different convolved generalized Fibonacci sequence.

Theorem 5.1.

$$\frac{1}{(1-ax)^{m+1}}*\frac{px+q}{(1-bx+cx^2)^{n+1}}$$
$$=\sum_{k=m\wedge n}^{m+n}\frac{\sum\limits_{s=n}^{k}\sum\limits_{r=0}^{k-s}\left\{(-1)^{n+m+s}\binom{k}{s}\binom{k-s}{r}\binom{s}{n}(abx)^r\right\}\left\{apx\binom{2s}{m-r}+q\binom{2s+1}{m-r}\right\}}{(1-abx+a^2cx^2)^{k+1}}$$

Proof. We can easily prove that $\frac{1}{1-\lambda x} * f(x) = f(\lambda x)$ by power series expansion. Using this formula we can get

$$\begin{aligned}
&\frac{1}{1-ax-y} * \frac{px+q}{1-bx+cx^2-z} = \frac{1}{(1-y)(1-\frac{a}{1-y}x)} * \frac{px+q}{(1-z)(1-\frac{b}{1-z}x+\frac{c}{1-z}x^2)} \\
&= \frac{1}{(1-y)(1-z)} \cdot \frac{1}{1-\frac{ax}{1-y}} * \frac{px+q}{1-\frac{bx}{1-z}+\frac{cx^2}{1-z}} = \frac{1}{(1-y)(1-z)} \cdot \frac{\frac{ap}{1-y}x+q}{1-\frac{abx}{(1-y)(1-z)}+\frac{a^2x^2}{(1-y)^(1-z)}} \\
&= \frac{apx+q(1-y)}{(1-abx+a^2cx^2)(1-\frac{1-abxy-(1-y)^2(1-z)}{1-abx+a^2cx^2})} \\
&= \sum_k \frac{\{apx+q(1-y)\}\{1-abxy-(1-y)^2(1-z)\}^k}{(1-abx+a^2cx^2)^{k+1}}.
\end{aligned}$$

From **theorem 4.1**, we know

$$\begin{aligned}
&\frac{1}{(1-ax)^{m+1}} * \frac{px+q}{(1-bx+cx^2)^{n+1}} \\
&= [y^m z^n] \sum_k \frac{\{apx+q(1-y)\}\{1-abxy-(1-y)^2(1-z)\}^k}{(1-abx+a^2cx^2)^{k+1}} \\
&= \sum_k \frac{[y^m z^n] apx \sum_s \sum_r \frac{k!}{s!r!(k-r-s)!}(-1)^r(abxy)^r(-1)^s(1-y)^{2s}(1-z)^s}{(1-abx+a^2cx^2)^{k+1}} \\
&\qquad + \sum_k \frac{[y^m z^n] q \sum_s \sum_r \frac{k!}{s!r!(k-r-s)!}(-1)^r(abxy)^r(-1)^s(1-y)^{2s+1}(1-z)^s}{(1-abx+a^2cx^2)^{k+1}} \\
&= \sum_{k=m\vee n}^{m+n} \frac{apx \sum_{s=n}^{k} \sum_{r=0}^{k-s} (-1)^{n+m+s}\binom{k}{s}\binom{k-s}{r}\binom{s}{n}\binom{2s}{m-r}(abx)^r}{(1-abx+a^2cx^2)^{k+1}} \\
&\qquad + \sum_{k=m\wedge n}^{m+n} \frac{q \sum_{s=n}^{k} \sum_{r=0}^{k-s} (-1)^{n+m+s}\binom{k}{s}\binom{k-s}{r}\binom{s}{n}\binom{2s+1}{m-r}(abx)^r}{(1-abx+a^2cx^2)^{k+1}} \\
&= \sum_{k=m\vee n}^{m+n} \frac{\sum\limits_{s=n}^{k} \sum\limits_{r=0}^{k-s} \{(-1)^{n+m+s}\binom{k}{s}\binom{k-s}{r}\binom{s}{n}(abx)^r\}\{apx\binom{2s}{m-r}+q\binom{2s+1}{m-r}\}}{(1-abx+a^2cx^2)^{k+1}}
\end{aligned}$$

□

Theorem 5.2.

$$\frac{px+q}{(1-ax+bx^2)^{m+1}} * \frac{rx+s}{(1-cx+dx^2)^{n+1}}$$
$$= \sum_{k=m\wedge n}^{m+n} \sum_{l,t,u,v,w\geq 0}^{l+t+u+v+w=k} \frac{\frac{k!}{l!t!u!v!w!} d_1^l (-a^2dx^2)^t (-bc^2x^2)^u d_2^v (-1)^w}{\{1-acx+(a^2d+bc^2-2bd)x^2-abcdx^3+b^2d^2x^4\}^{k+1}}$$
$$\times \left\{ n_1 \binom{u+v+2w+1}{m}\binom{t+v+2w+1}{n} + axqr\binom{u+v+2w}{m}\binom{t+v+2w+1}{n} \right.$$
$$\left. +cxps\binom{u+v+2w+1}{m}\binom{t+v+2w}{n} + n_2\binom{u+v+2w}{m}\binom{t+v+2w}{n} \right\}$$

where $n_1 = qs+xpr$, $n_2 = -x^2(adps+bcqr+bdqs)-bdprx^3$, $d_1 = 1-acx+(a^2d+bc^2-2bd)x^2$ *and* $d_2 = acx+2bdx^2$.

Proof. As before, we need to find $[y^m z^n]\frac{px+q}{1-ax+bx^2-y} * \frac{rx+s}{1-cx+dx^2-z}$. We can do a partial fraction expansion of $\frac{px+q}{1-ax+bx^2}$ by using $a_1+a_2=a, a_1a_2=b$. We can then use $\frac{1}{1-\lambda x} * f(x) = f(\lambda x)$ and combine the results.

$$\frac{px+q}{1-ax+bx^2} * \frac{rs+x}{1-cx+dx^2} = \frac{px+q}{(1-a_1x)(1-a_2x)} * \frac{rx+s}{1-cx+dx^2}$$
$$= \frac{p+qa_1}{a_1-a_2}\cdot\frac{ra_1x+s}{1-a_1cx+a_1^2dx^2} - \frac{p+qa_2}{a_1-a_2}\cdot\frac{ra_2x+s}{1-a_2cx+a_2^2dx^2}$$
$$= \frac{qs+x(pr+aqr+cps)-x^2(adps+bcqr+bdqs)-bdprx^3}{1-acx+(a^2d+bc^2-2bd)x^2-abcdx^3+b^2d^2x^4}.$$

By using this formula, we can write

$$\frac{px+q}{(1-ax+bx^2)^{m+1}} * \frac{rx+s}{(1-cx+dx^2)^{n+1}} = [y^m z^n]\frac{px+q}{1-ax+bx^2-y} * \frac{rx+s}{1-cx+dx^2-z}$$
$$= [y^m z^n]\frac{px+q}{(1-y)(1-\frac{a}{1-y}x+\frac{b}{1-y}x^2)} * \frac{rx+s}{(1-z)(1-\frac{c}{1-z}x+\frac{d}{1-z}x^2)}$$
$$= [y^m z^n]\frac{1}{(1-y)(1-z)}$$
$$\times \frac{qs+xpr+\frac{a}{1-y}xqr+\frac{c}{1-z}xps-\frac{x^2(adps+bcqr+bdqs)+bdprx^3}{(1-y)(1-z)}}{1-\frac{acx}{(1-y)(1-z)}+\frac{a^2dx^2}{(1-y)^2(1-z)}+\frac{bc^2x^2}{(1-y)(1-z)^2}-\frac{2bdx^2}{(1-y)(1-z)}-\frac{abcdx^3}{(1-y)^2(1-z)^2}+\frac{b^2d^2x^4}{(1-y)^2(1-z)^2}}$$
$$= [y^m z^n]\frac{1}{1-acx+(a^2d+bc^2-2bd)x^2-abcdx^3+b^2d^2x^4}\times$$
$$\left\{\frac{(qs+xpr)(1-y)(1-z)+axqr(1-z)+cxps(1-y)-x^2(adps+bcqr+bdqs)-bdprx^3}{1-\frac{1-acx-2bdx^2-(1-y)^2(1-z)^2+(acx+2bdx^2)(1-y)(1-z)+(a^2dz+bc^2y)x^2}{1-acx+(a^2d+bc^2-2bd)x^2-abcdx^3+b^2d^2x^4}}\right\}$$

For simplicity $n_1 = qs + xpr$, $n_2 = -x^2(adps + bcqr + bdqs) - bdprx^3$, $\operatorname{den}(x) = 1 - acx + (a^2d + bc^2 - 2bd)x^2 - abcdx^3 + b^2d^2x^4$, $d_1 = 1 - acx + (a^2d + bc^2 - 2bd)x^2$ and $d_2 = acx + 2bdx^2$.

$$= [y^m z^n] \frac{n_1(1-y)(1-z) + axqr(1-z) + cxps(1-y) + n_2}{\operatorname{den}(x)[1 - \frac{d_1 - a^2dx^2(1-z) - bc^2x^2(1-y) + d_2(1-y)(1-z) - (1-y)^2(1-z)^2}{\operatorname{den}(x)}]}$$

$$= [y^m z^n]\{n_1(1-y)(1-z) + axqr(1-z) + cxps(1-y) + n_2\} \times \sum_{k=m\wedge n}^{m+n} \frac{\{d_1 - a^2dx^2(1-z) - bc^2x^2(1-y) + d_2(1-y)(1-z) - (1-y)^2(1-z)^2\}^k}{\operatorname{den}(x)^{k+1}}$$

Expanding by binomial theorem

$$\{d_1 - a^2dx^2(1-z) - bc^2x^2(1-y) + d_2(1-y)(1-z) - (1-y)^2(1-z)^2\}^k$$

$$= \sum_{l,t,u,v,w\geq 0}^{\substack{l+t+u\\+v+w=k}} \frac{k!}{l!t!u!v!w!} d_1^l(-a^2dx^2)^t\{-bc^2x^2(1-y)\}^u\{d_2(1-y)(1-z)\}^v\{-(1-y)^2(1-z)^2\}^w$$

and simplifying

$$= [y^m z^n]\{n_1(1-y)(1-z) + axqr(1-z) + cxps(1-y) + n_2\} \times \sum_{k=m\wedge n}^{m+n} \sum_{l,t,u,v,w\geq 0}^{l+t+u+v+w=k} \frac{\frac{k!}{l!t!u!v!w!} d_1^l(-a^2dx^2)^t(-bc^2x^2)^u d_2^v(-1)^w(1-y)^{u+v+2w}(1-z)^{t+v+2w}}{\operatorname{den}(x)^{k+1}}$$

$$= \sum_{k=m\wedge n}^{m+n} \sum_{l,t,u,v,w\geq 0}^{l+t+u+v+w=k} \frac{\frac{k!}{l!t!u!v!w!} d_1^l(-a^2dx^2)^t(-bc^2x^2)^u d_2^v(-1)^w}{\operatorname{den}(x)^{k+1}} \times [y^m z^n]\{n_1(1-y)^{u+v+2w+1}(1-z)^{t+v+2w+1} + axqr(1-y)^{u+v+2w}(1-z)^{t+v+2w+1} + cxps(1-y)^{u+v+2w+1}(1-z)^{t+v+2w} + n_2(1-y)^{u+v+2w}(1-z)^{t+v+2w}\}$$

$$= \sum_{k=m\wedge n}^{m+n} \sum_{l,t,u,v,w\geq 0}^{l+t+u+v+w=k} \frac{\frac{k!}{l!t!u!v!w!} d_1^l(-a^2dx^2)^t(-bc^2x^2)^u d_2^v(-1)^w}{\{1 - acx + (a^2d + bc^2 - 2bd)x^2 - abcdx^3 + b^2d^2x^4\}^{k+1}} \times \left\{ n_1\binom{u+v+2w+1}{m}\binom{t+v+2w+1}{n} + axqr\binom{u+v+2w}{m}\binom{t+v+2w+1}{n} + cxps\binom{u+v+2w+1}{m}\binom{t+v+2w}{n} + n_2\binom{u+v+2w}{m}\binom{t+v+2w}{n} \right\}$$

□

Factoring irreducible polynomials (over the rationals) of degree 3 or 4 is complicated and of degree greater than 4 impossible in terms of explicit formulas. So, the method of **theorem 3.1** is not going to be easy to apply when denominators have irreducible factors of degree greater than 2.

6 Conclusion

This novel method provides a three-step simple method to replace the complex computation of the Hadamard product of two rational generating function. If the denominator of the rational function has many roots, then the number of Hadamard multiplication of the partials and the sum of products can grow exponentially. To improve computation speed we have derived identities for the Hadamard product of negative powers of quadratic. But still number of computation is equal to the product of the number of factors in the denominator of the two rational functions.

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