Rose-Hulman Undergraduate Mathematics Journal

Volume 23
Issue 1

Article 1

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ℝose-Hulman Undergraduate Mathematics Journal
Volume 23, Issue 1, 2022

Additional Fay Identities of the Extended Toda Hierarchy

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Abstract. The focus of this paper is the extended Toda Lattice hierarchy, an infinite system of partial differential equations arising from the Toda lattice equation. We begin by giving the definition of the extended Toda hierarchy and its explicit bilinear equation, following Takasaki's construction. We then derive a series of new Fay identities. Finally, we discover a general formula for one type of Fay identity.

1 Introduction and Previous Research Results

Given a differential equation that describes a physical process, an integrable hierarchy is formed by adding infinitely many differential equations that can be solved simultaneously. This results in an infinite system of differential equations for which one solution exists. Some famous examples of integrable hierarchies are the KP hierarchy and the Toda Lattice hierarchy [10]. Such systems can be encoded by one single equation, known as the *bilinear equation*, and the solution to such system is called the τ-function.

In this paper, we will mainly discuss the extended Toda hierarchy (ETH). Based on the explicit bilinear equation for the ETH from [6], we get a series of *Fay identities*, which are equations satisfied by known solutions to the system.

This project is motivated by the equivalence between integrable hierarchies and their Fay identities. For example, in [7] the authors proved the equivalence between the Toda lattice hierarchy and a certain set of Fay identities. Our hope is that this project is a step towards discovering a similar result for the extended Toda hierarchy. We try to provide inspirations for the proof of such equivalence by giving a generalized form of the Fay identities for the ETH. Through future work, we might find out such equivalence and possibly do the same for the extended bigraded Toda hierarchy (EBTH), which is a generalization of ETH.

In Section 2, we will introduce the background knowledge about the ETH. In Section 3, we will state what we already know about the Fay identities and how they were derived. In Section 4, we will discuss how we derive further to get multiple new Fay identities. In Section 5, we will provide the conclusion and possible future work.

[*]Thanks to everyone who has helped me.
Mathematics Subject Classification. 11A41
Keywords. Toda hierarchies, Integrable systems

2 Extended Toda Hierarchy

An *integrable hierarchy* is defined as a system of infinitely many differential equations that commute with an original equation. This means the infinite system of equations has a common solution of the form $\mathrm{L} = \Lambda + v_n(\boldsymbol{t}) + e^{u_n(\boldsymbol{t})}\Lambda^{-1}$ where $\boldsymbol{t} = (t_1, t_2, t_3, \ldots)$.

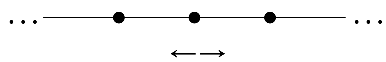

Figure 1: Charged particles moving on an infinite lattice

The original Toda lattice equation was introduced to model the movement of charged particles on a one-dimensional lattice (see Figure 1). In [9], the equation was written as

$$\frac{d^2 q_n}{dt^2} = e^{q_{n-1}-q_n} - e^{q_n - q_{n+1}} \tag{2.1}$$

where $q_n = q_n(t)$ is the displacement of the n^{th} particle of a crystal from its equilibrium position at time t.

Define the operator Λ by $\Lambda f_n(t) = f_{n+1}(t)$, and a Lax operator

$$\mathrm{L} = \Lambda + v_n + e^{u_n}\Lambda^{-1} \tag{2.2}$$

where $v_n = v_n(t)$, $u_n = u_n(t)$. Then the Toda lattice equation (2.1) can be written in Lax form as

$$\frac{d\mathrm{L}}{dt} = [\mathrm{L}_+, \mathrm{L}] \tag{2.3}$$

Where $[\mathrm{X}, \mathrm{Y}] = \mathrm{XY} - \mathrm{YX}$ is called the *commutator bracket*. If we make the substitution

$$v_n = -\frac{dq_n}{dt}, \qquad u_n = q_{n-1} - q_n,$$

then (2.3) is equivalent to (2.1).

The *discrete* Toda lattice hierarchy is defined to be:

$$\frac{\partial \mathrm{L}}{\partial t_m} = [(\mathrm{L}^m)_+, \mathrm{L}], \ m = 1, 2, \ldots \tag{2.4}$$

with the Lax operator L defined in equation (2.2).

To generalize the Toda lattice hierarchy, the discrete variable n in the Lax operator in (2.4) is replaced by a continuous variable s to get the *interpolated* Toda hierarchy, where in [5], the Lax operator is defined to be

$$\mathrm{L} = \Lambda + v(s) + e^{u(s)}\Lambda^{-1} \tag{2.5}$$

with the remaining equations of the hierarchy defined as in (2.4), except with the discrete variable n replaced by the continuous variable s. Here, $\Lambda f(s) = e^{\partial_s} f(s) = f(s+1)$. From [2], after such replacing, we are able to add more equations to the hierarchy, with respect to a new set of variables $\boldsymbol{x} = (x_0, x_1, x_2, \dots)$ with $x_0 = s$. These are called *the Logarithm variables*. The resulting collection of equations is known as the *extended Toda hierarchy* (ETH) and is defined by Carlet, Dubrovin, and Zhang in [2] with the equations (2.4) and the new equations:

$$\partial_{x_m} \mathrm{L} = [(2\mathrm{L}^m \log \mathrm{L})_+, \mathrm{L}] \quad m = 0, 1, 2, \dots. \tag{2.6}$$

where the definition of $\log \mathrm{L}$ and a rigorous explanation for (2.6) can be found in [2].

A property of integrable hierarchies is that the equation system can be expressed in various forms so that we can use various methods to solve the system. For example, the interpolated Toda hierarchy has the following *wave operators* [10]:

$$\mathrm{W} = 1 + \sum_{n=1}^{\infty} w_n(s)\Lambda^{-n}, \qquad \bar{\mathrm{W}} = \sum_{n=0}^{\infty} \bar{w}_n(s)\Lambda^{n}, \quad \text{with } \bar{w}_n(s) \neq 0 \tag{2.7}$$

such that $\mathrm{L} = \mathrm{W}\Lambda\mathrm{W}^{-1} = \bar{\mathrm{W}}\Lambda^{-1}\bar{\mathrm{W}}^{-1}$. There also exist wave functions [10]

$$\psi(s,z) = \mathrm{W} z^s e^{\xi(\boldsymbol{t},z)/2}, \qquad \bar{\psi}(s,z) = \bar{\mathrm{W}} z^s e^{-\xi(\bar{\boldsymbol{t}},z^{-1})/2} \tag{2.8}$$

where

$$\xi(t,z) = \sum_{n=1}^{\infty} \boldsymbol{t}_n z^n. \tag{2.9}$$

Using this definition, one can verify that the wave functions ψ, ψ^* satisfy the eigenvalue equations $\mathrm{L}\psi = z\psi$ and $\mathrm{L}\bar{\psi} = z^{-1}\bar{\psi}$. Furthermore, any function that satisfies this eigenvalue equation and the auxiliary linear equations is a solution to the ETH [5].

Finally, (2.4) can be expressed in terms of a single function $\tau(\boldsymbol{t})$, called the *tau function* [4] satisfying

$$\psi(s,z) = \frac{\tau(s, \boldsymbol{t} - [z^{-1}])}{\tau(s,\boldsymbol{t})} z^s e^{\xi(\boldsymbol{t},z)/2} \tag{2.10}$$

where $[z] = \left(z, \frac{z^2}{2}, \frac{z^3}{3}, \dots\right)$. One can write a single *bilinear equation* in terms of the tau function which encodes the entire hierarchy [10].

These results can also be extended to the ETH, in particular, using the same wave operator as (2.7) and the following wave functions ψ:

$$\begin{aligned} \psi(s,\boldsymbol{t},\boldsymbol{x},z) &= \mathrm{W} z^{s+\xi(\boldsymbol{x},z)} e^{\xi(\boldsymbol{t},z)/2} \\ &= \mathrm{W}\chi \end{aligned} \tag{2.11}$$

Finally, there exists a tau function such that

$$\psi(s,\boldsymbol{t},\boldsymbol{x},z) = \frac{\tau(s,\boldsymbol{t}-[z^{-1}],\boldsymbol{x})}{\tau(s,\boldsymbol{t},\boldsymbol{x})}\chi \tag{2.12}$$

From [5], a function τ is a tau function that solves the ETH if and only if it satisfies the following bilinear equation:

$$\begin{aligned}&\oint \frac{dz}{2\pi i} z^{n+s'-s} e^{\xi(\boldsymbol{t'}-\boldsymbol{t},z)/2} \tau(s'-\xi(\boldsymbol{a},z), \boldsymbol{t'}-[z^{-1}], \boldsymbol{x}+\boldsymbol{a}) \\ &\qquad \times \tau(s-\xi(\boldsymbol{b},z), \boldsymbol{t}+[z^{-1}], \boldsymbol{x}+\boldsymbol{b}) \\ =& \oint \frac{dz}{2\pi i} z^{-n+s'-s} e^{\xi(\boldsymbol{t}-\boldsymbol{t'},z^{-1})/2} \tau(s'+1-\xi(\boldsymbol{a},z^{-1}), \boldsymbol{t'}-[z], \boldsymbol{x}+\boldsymbol{a}) \\ &\qquad \times \tau(s-1-\xi(\boldsymbol{b},z^{-1}), \boldsymbol{t}-[z], \boldsymbol{x}+\boldsymbol{b})\end{aligned} \tag{2.13}$$

for any set of variables $\boldsymbol{a} = (a_1, a_2, \ldots)$, $\boldsymbol{b} = (b_1, b_2, \ldots)$, and all $n \in \mathbb{Z}_{\geq 0}$, $s' - s \in \mathbb{Z}$, $t_i \in \mathbb{C}$.

Remark 2.1. Note that we can replace $z^{\pm k}$ by $f(z^{\pm 1}) = \sum_{k=0}^{\infty} f_k z^{\pm k}$ for any formal power series $f(z^{\pm 1})$ by taking linear combinations of the integrals in the bilinear equation (see [5], Section 2.4).

By finding a τ that satisfies this equation, we can work backward to find a solution to the system, so we see that the infinite system can be encoded by the single bilinear equation. We will be focusing on this bilinear equation in the following sections.

3 Fay Identities for the ETH

3.1 Introduction

From [8], every τ-function that is a solution of the Toda lattice hierarchy satisfies a specific equation called the differential Fay identity. We obtain these Fay identities by making certain substitutions in the bilinear equation (which the tau function is known to satisfy), and then simplifying. In [7], Takasaki and Takebe showed that, conversely, if a tau function satisfies a collection of Fay identities, it must satisfy the bilinear equation. Therefore, it is natural to think that the ETH would possibly preserve such property, but we save this for future work.

3.2 Background and Cauchy's Residue Formulas

For the remainder of this paper (similar to equation (2.5)), we will use the notation $e^{t\partial_x}$ to denote shifting the variable x by some $t \in \mathbb{C}^*$ [3]. To shift a function $f(x)$ by some t, we apply the operator

$$e^{t\partial_x} f(x) = \left(1 + (t\partial_x) + \frac{(t\partial_x)^2}{2!} + \cdots\right) f(x) = f(x+t). \tag{3.1}$$

Next, we define the residue of $f(z) = \sum f_i z^i$ to be f_{-1} the coefficient of z^{-1}. We denote this

$$\mathrm{Res}_z \sum f_i z^i = \oint \frac{dz}{2\pi i} f(z)$$

Now consider the residue

$$\oint \frac{dz}{2\pi i} \frac{z^k}{1 - z\lambda^{-1}} f(z) \quad \text{where} \quad f(z) = \sum_{i=0}^{\infty} f_i z^{-i}$$

Substituting $f(z)$, We have

$$\begin{aligned}
&\oint \frac{dz}{2\pi i} \frac{z^k}{1 - z\lambda^{-1}} f(z) \\
=&\oint \frac{dz}{2\pi i} \frac{z^k}{1 - z\lambda^{-1}} \sum_{i=0}^{\infty} f_i z^{-i} \\
=&\oint \frac{dz}{2\pi i} \left(\sum_{n=0}^{\infty} \lambda^{-n} z^n\right)\left(\sum_{i=0}^{\infty} f_i z^{k-i}\right)
\end{aligned}$$

To compute the reside, we look for the coefficient of z^{-1}, so we need $n + k - i = -1$. Hence the residue becomes

$$\begin{aligned}
&\sum_{n=0}^{\infty} f_{k+n+1} \lambda^{-n} \\
=&\lambda^{k+1} \sum_{n=0}^{\infty} f_{k+n+1} \lambda^{-(n+k+1)} \\
=&\lambda^{k+1} \left(f(\lambda) - \sum_{i=0}^{k} f_i \lambda^{-i}\right)
\end{aligned}$$

Using this, we can compute

$$\oint \frac{dz}{2\pi i} \frac{z^k}{1 - z\lambda^{-1}} f(z) = \lambda^{k+1} \left(f(\lambda) - \sum_{i=0}^{k} f_i \lambda^{-i}\right) \quad \text{if} \quad f(z) = \sum_{i=0}^{\infty} f_i z^{-i} \tag{3.2}$$

3.3 Approach to Fay Identities

We begin to derive Fay identities by making substitutions in (2.13). We can do so since the equation is satisfied for any $n, s', s, \boldsymbol{t}, \boldsymbol{t}', \boldsymbol{a}, \boldsymbol{b}$ as described above. Therefore we set

$$n = 0, \quad s' - s = 1, \quad \boldsymbol{t}' = \boldsymbol{t} + [\lambda^{-1}] + [\mu^{-1}]$$

where $\lambda, \mu \in \mathbb{C}^*$. We will see below that this choice of substitutions gives us a nice cancellation in the exponential term.

Consider the easier case where $\boldsymbol{a} = \boldsymbol{b} = \boldsymbol{0}$ which was done in [1]. By substitution, we get

$$\oint \frac{dz}{2\pi i} z e^{\xi(\boldsymbol{t}'-\boldsymbol{t},z)/2} \tau(s+1, \boldsymbol{t}+[\lambda^{-1}]+[\mu^{-1}]-[z^{-1}], \boldsymbol{x})\tau(s, \boldsymbol{t}+[z^{-1}], \boldsymbol{x}) \tag{3.3}$$
$$= \oint \frac{dz}{2\pi i} z e^{\xi(\boldsymbol{t}-\boldsymbol{t}',z^{-1})/2} \tau(s+2, \boldsymbol{t}+[\lambda^{-1}]+[\mu^{-1}]-[z], \boldsymbol{x})\tau(s-1, \boldsymbol{t}-[z], \boldsymbol{x})$$

By Remark (2.1), we can multiply both sides of (3.3) by $f(z^{\pm 1}) = e^{\xi(\boldsymbol{t}'-\boldsymbol{t},z^{\pm 1})/2}$ to get

$$\oint \frac{dz}{2\pi i} z e^{\xi(\boldsymbol{t}'-\boldsymbol{t},z)} \tau(s+1, \boldsymbol{t}+[\lambda^{-1}]+[\mu^{-1}]-[z^{-1}], \boldsymbol{x})\tau(s, \boldsymbol{t}+[z^{-1}], \boldsymbol{x})$$
$$= \oint \frac{dz}{2\pi i} z \tau(s+2, \boldsymbol{t}+[\lambda^{-1}]+[\mu^{-1}]-[z], \boldsymbol{x})\tau(s-1, \boldsymbol{t}-[z], \boldsymbol{x})$$

We can simplify the expression $e^{\xi(\boldsymbol{t}'-\boldsymbol{t},z)}$ using the Taylor expansion

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots = -\sum_{n=1}^{\infty} \frac{x^n}{n}$$

Applying the above substitution $\boldsymbol{t}' = \boldsymbol{t} + [\lambda^{-1}] + [\mu^{-1}]$ we have

$$e^{\xi(\boldsymbol{t}'-\boldsymbol{t},z)} = \exp\left(\sum_{n=1}^{\infty} (\boldsymbol{t}'_n - \boldsymbol{t}_n) z^n\right) = \exp\left(\sum_{n=1}^{\infty} (\frac{\lambda^{-n}}{n} \cdot z^n)\right) \cdot \exp\left(\sum_{n=1}^{\infty} (\frac{\mu^{-n}}{n} \cdot z^n)\right)$$
$$= \exp\left(\ln(\frac{1}{1-z\lambda^{-1}})\right) \cdot \exp\left(\ln(\frac{1}{1-z\mu^{-1}})\right)$$
$$e^{\xi(\boldsymbol{t}'-\boldsymbol{t},z)} = \frac{1}{(1-z\lambda^{-1})(1-z\mu^{-1})} \tag{3.4}$$

Hence we can get the following equation:

$$\oint \frac{dz}{2\pi i} \frac{z}{(1-z\lambda^{-1})(1-z\mu^{-1})} \tau(s+1, \boldsymbol{t}+[\lambda^{-1}]+[\mu^{-1}]-[z^{-1}], \boldsymbol{x})\tau(s, \boldsymbol{t}+[z^{-1}], \boldsymbol{x}) \tag{3.5}$$
$$= \oint \frac{dz}{2\pi i} z \tau(s+2, \boldsymbol{t}+[\lambda^{-1}]+[\mu^{-1}]-[z], \boldsymbol{x})\tau(s-1, \boldsymbol{t}-[z], \boldsymbol{x})$$

Consider the right hand side of (3.5). The expansion of $z\tau(s+2, \boldsymbol{t}+[\lambda^{-1}]+[\mu^{-1}]-[z], \boldsymbol{x})\tau(s-1, \boldsymbol{t}-[z], \boldsymbol{x})$ consists of only positive powers of z, thus taking the residue will be 0. We can rewrite the left side using the partial fractions expansion

$$\frac{z}{(1-z\lambda^{-1})(1-z\mu^{-1})} = \frac{1}{\lambda^{-1}-\mu^{-1}} \left(\frac{1}{1-z\lambda^{-1}} - \frac{1}{1-z\mu^{-1}}\right). \tag{3.6}$$

Finally, we use (3.2) with $k=0$, to obtain

$$\oint \frac{dz}{2\pi i}\frac{1}{1-z\lambda^{-1}}f(z)=\lambda(f(\lambda)-f_0) \quad \text{if} \quad f(z)=\sum_{i=0}^{\infty}f_i z^{-i}$$

Here, $f(x)=\tau(s+1,\boldsymbol{t}+[\lambda^{-1}]+[\mu^{-1}]-[z^{-1}],\boldsymbol{x})\tau(s,\boldsymbol{t}+[z^{-1}],\boldsymbol{x})$. Since s is arbitrary, we obtain the following Fay identity

$$\begin{aligned}(\lambda-\mu)&\tau(s,\boldsymbol{t},\boldsymbol{x})\tau(s-1,\boldsymbol{t}-[\lambda^{-1}]-[\mu^{-1}],\boldsymbol{x})\\ &=\lambda\tau(s,\boldsymbol{t}-[\lambda^{-1}],\boldsymbol{x})\tau(s-1,\boldsymbol{t}-[\mu^{-1}],\boldsymbol{x})\\ &\quad-\mu\tau(s,\boldsymbol{t}-[\mu^{-1}],\boldsymbol{x})\tau(s-1,\boldsymbol{t}-[\lambda^{-1}],\boldsymbol{x})\end{aligned} \tag{3.7}$$

We can observe that there are τ-functions groups on both sides of the equation. In order to simplify the equation, we introduce the following notation as in [1]:

$$\begin{aligned}F_1(s',\boldsymbol{t}',\boldsymbol{x}';s,\boldsymbol{t},\boldsymbol{x})&=\tau(s',\boldsymbol{t}',\boldsymbol{x}')\tau(s-1,\boldsymbol{t}-[\lambda^{-1}]-[\mu^{-1}],\boldsymbol{x}),\\ F_2(s',\boldsymbol{t}',\boldsymbol{x}';s,\boldsymbol{t},\boldsymbol{x})&=\tau(s',\boldsymbol{t}'-[\lambda^{-1}],\boldsymbol{x}')\tau(s-1,\boldsymbol{t}-[\mu^{-1}],\boldsymbol{x}),\\ F_3(s',\boldsymbol{t}',\boldsymbol{x}';s,\boldsymbol{t},\boldsymbol{x})&=\tau(s',\boldsymbol{t}'-[\mu^{-1}],\boldsymbol{x}')\tau(s-1,\boldsymbol{t}-[\lambda^{-1}],\boldsymbol{x}).\end{aligned}$$

The Fay Identity becomes $((\lambda-\mu)F_1-\lambda F_2+\mu F_3)|_{s'=s,\boldsymbol{t}'=\boldsymbol{t},\boldsymbol{x}'=\boldsymbol{x}}=0$ with this notation. Now consider the case where $\boldsymbol{a}=(a_1,0,\cdots)$, $\boldsymbol{b}=0$, after substitution, we have

$$\begin{aligned}&\oint\frac{dz}{2\pi i}ze^{\xi(\boldsymbol{t}'-\boldsymbol{t},z)/2}\tau(s'-a_1z,\boldsymbol{t}+[\lambda^{-1}]+[\mu^{-1}]-[z^{-1}],x_1+a_1,x_2,x_3,\ldots)\tau(s,\boldsymbol{t}+[z^{-1}],\boldsymbol{x})\\ =&\oint\frac{dz}{2\pi i}ze^{\xi(\boldsymbol{t}-\boldsymbol{t}',z^{-1})/2}\tau(s'+1-a_1z^{-1},\boldsymbol{t}+[\lambda^{-1}]+[\mu^{-1}]-[z],x_1+a_1,x_2,x_3,\ldots)\tau(s-1,\boldsymbol{t}-[z],\boldsymbol{x})\end{aligned}$$

Note that the variables s' and x_1 in τ-function on the left hand side are shifted by $-a_1z$ and a_1 correspondingly. On the right hand side, s' and x_1 are shifted by $-a_1z^{-1}$ and a_1 correspondingly. Thus we can represent such shift using shift operators.

$$\begin{aligned}&\oint\frac{dz}{2\pi i}ze^{\xi(\boldsymbol{t}'-\boldsymbol{t},z)/2}e^{a_1(\partial_{x_1}-z\partial_{s'})}\tau(s',\boldsymbol{t}+[\lambda^{-1}]+[\mu^{-1}]-[z^{-1}],\boldsymbol{x})\tau(s,\boldsymbol{t}+[z^{-1}],\boldsymbol{x})\\ =&\oint\frac{dz}{2\pi i}ze^{\xi(\boldsymbol{t}-\boldsymbol{t}',z^{-1})/2}e^{a_1(\partial_{x_1}-z^{-1}\partial_{s'})}\tau(s'+1,\boldsymbol{t}+[\lambda^{-1}]+[\mu^{-1}]-[z],\boldsymbol{x})\tau(s-1,\boldsymbol{t}-[z],\boldsymbol{x})\end{aligned} \tag{3.8}$$

Using (2.1), we can multiply both sides by $f(z^{\pm1})=e^{\xi(\boldsymbol{t}'-\boldsymbol{t},z^{\pm1})/2}$.

$$\begin{aligned}&\oint\frac{dz}{2\pi i}ze^{\xi(\boldsymbol{t}'-\boldsymbol{t},z)}e^{a_1(\partial_{x_1}-z\partial_{s'})}\tau(s',\boldsymbol{t}+[\lambda^{-1}]+[\mu^{-1}]-[z^{-1}],\boldsymbol{x})\tau(s,\boldsymbol{t}+[z^{-1}],\boldsymbol{x})\\ =&\oint\frac{dz}{2\pi i}ze^{a_1(\partial_{x_1}-z^{-1}\partial_{s'})}\tau(s'+1,\boldsymbol{t}+[\lambda^{-1}]+[\mu^{-1}]-[z],\boldsymbol{x})\tau(s-1,\boldsymbol{t}-[z],\boldsymbol{x})\end{aligned}$$

By (3.4), this is simply

$$\oint \frac{dz}{2\pi i}\frac{z}{(1-z\lambda^{-1})(1-z\mu^{-1})}e^{a_1(\partial_{x_1}-z\partial_{s'})}\tau(s',\boldsymbol{t}+[\lambda^{-1}]+[\mu^{-1}]-[z^{-1}],\boldsymbol{x})\tau(s,\boldsymbol{t}+[z^{-1}],\boldsymbol{x}) \tag{3.9}$$

$$=\oint \frac{dz}{2\pi i} z e^{a_1(\partial_{x_1}-z^{-1}\partial_{s'})}\tau(s'+1,\boldsymbol{t}+[\lambda^{-1}]+[\mu^{-1}]-[z],\boldsymbol{x})\tau(s-1,\boldsymbol{t}-[z],\boldsymbol{x})$$

We then expand $e^{a_1(\partial_{x_1}-z\partial_{s'})}$ using Taylor expansion as in (3.1), we have

$$\begin{aligned} e^{a_1\partial_{x_1}}e^{-a_1 z\partial_{s'}} &= (1+a_1\partial_{x_1}+\frac{(a_1\partial_{x_1})^2}{2!}+\cdots)(1-a_1 z\partial_{s'}+\cdots) \\ &= 1+(\partial_{x_1}-z\partial_{s'})a_1+(\frac{\partial_{x_1}^2}{2}+\frac{z^2\partial_{s'}^2}{2}-z\partial_{x_1}\partial_{s'})a_1^2+\cdots \end{aligned} \tag{3.10}$$

The expansion for $e^{a_1(\partial_{x_1}-z^{-1}\partial_{s'})}$ is similar. To obtain a Fay Identity with respect to the logarithmic variables, we can compare the coefficients of a_1 and obtain:

$$\oint \frac{dz}{2\pi i}\frac{z}{(1-z\lambda^{-1})(1-z\mu^{-1})}(\partial_{x_1}-z\partial_{s'})\tau(s',\boldsymbol{t}+[\lambda^{-1}]+[\mu^{-1}]-[z^{-1}],\boldsymbol{x})\tau(s,\boldsymbol{t}+[z^{-1}],\boldsymbol{x})$$

$$=\oint \frac{dz}{2\pi i} z(\partial_{x_1}-z^{-1}\partial_{s'})\tau(s'+1,\boldsymbol{t}+[\lambda^{-1}]+[\mu^{-1}]-[z],\boldsymbol{x})\tau(s-1,\boldsymbol{t}-[z],\boldsymbol{x})$$

On the right hand side of the equation, there are no possible negative powers of z. Thus taking residue of the right hand side will give us 0, so that

$$\oint \frac{dz}{2\pi i}\frac{z}{(1-z\lambda^{-1})(1-z\mu^{-1})}(\partial_{x_1}-z\partial_{s'})\tau(s',\boldsymbol{t}+[\lambda^{-1}]+[\mu^{-1}]-[z^{-1}],\boldsymbol{x})\tau(s,\boldsymbol{t}+[z^{-1}],\boldsymbol{x})=0$$

Using (3.6) and (3.2), we are able to get a Fay Identity involving the $\boldsymbol{x}$ variables:

$$\begin{aligned} \big((\lambda-\mu)((\lambda+\mu)\partial_{s'}-\partial_{x_1'}+\partial_{s'}(\partial_{t_1}-\partial_{t_1'}))\mathrm{F}_1 \\ +\lambda(\partial_{x_1'}-\lambda\partial_{s'})\mathrm{F}_1-\mu(\partial_{x_1'}-\mu\partial_{s'})\mathrm{F}_3\big)|_{s'=s,\boldsymbol{t}'=\boldsymbol{t},\boldsymbol{x}'=\boldsymbol{x}}=0 \end{aligned} \tag{3.11}$$

4 Results: New Fay Identities for the Extended Toda Hierarchy

In this paper, we are going to derive further the Fay identities for the ETH following what was done in [1]. For the following cases, we are going to set

$$n=0,\quad s'-s=1,\quad \boldsymbol{t}'=\boldsymbol{t}+[\lambda^{-1}]+[\mu^{-1}]$$

and derive Fay identities from the bilinear equation (2.13).

4.1 Case 1

First, we wish to explore the case where we have non-zero $\boldsymbol{b}$ instead of $\boldsymbol{a}$. Letting $\boldsymbol{a}=\boldsymbol{0}$, $\boldsymbol{b}=(b_1,0,\cdots)$, we get

$$\begin{aligned}&\oint \frac{dz}{2\pi i} z e^{\xi(\boldsymbol{t}'-\boldsymbol{t},z)/2}\tau(s',\boldsymbol{t}+[\lambda^{-1}]+[\mu^{-1}]-[z^{-1}],\boldsymbol{x})\\ &\qquad\qquad \times\tau(s-b_1 z,\boldsymbol{t}+[z^{-1}],x_1+b_1,x_2,x_3,\ldots)\\ =&\oint \frac{dz}{2\pi i} z e^{\xi(\boldsymbol{t}-\boldsymbol{t}',z^{-1})/2}\tau(s'+1,\boldsymbol{t}+[\lambda^{-1}]+[\mu^{-1}]-[z],\boldsymbol{x})\\ &\qquad\qquad \times\tau(s-1-b_1 z^{-1},\boldsymbol{t}-[z],x_1+b_1,x_2,x_3,\ldots)\end{aligned}$$

By (3.1), we use shift operators to represent the shifting. The equation then becomes

$$\begin{aligned}&\oint \frac{dz}{2\pi i} z e^{\xi(\boldsymbol{t}'-\boldsymbol{t},z)/2}\tau(s',\boldsymbol{t}+[\lambda^{-1}]+[\mu^{-1}]-[z^{-1}],\boldsymbol{x})\\ &\qquad\qquad \times e^{b_1(\partial_{x_1}-z\partial s)}\tau(s,\boldsymbol{t}+[z^{-1}],\boldsymbol{x})\\ =&\oint \frac{dz}{2\pi i} z e^{\xi(\boldsymbol{t}-\boldsymbol{t}',z^{-1})/2}\tau(s'+1,\boldsymbol{t}+[\lambda^{-1}]+[\mu^{-1}]-[z],\boldsymbol{x})\\ &\qquad\qquad \times e^{b_1(\partial_{x_1}-z^{-1}\partial s)}\tau(s-1,\boldsymbol{t}-[z],\boldsymbol{x})\end{aligned}$$

Using (2.1), we multiply both sides by $f(z^{\pm1})=e^{\xi(\boldsymbol{t}'-\boldsymbol{t},z^{\pm1})/2}$, and simplify using (3.4)

$$\begin{aligned}&\oint \frac{dz}{2\pi i}\frac{z}{(1-z\lambda^{-1})(1-z\mu^{-1})}\tau(s',\boldsymbol{t}+[\lambda^{-1}]+[\mu^{-1}]-[z^{-1}],\boldsymbol{x})e^{b_1(\partial_{x_1}-z\partial s)}\tau(s,\boldsymbol{t}+[z^{-1}],\boldsymbol{x})\\ =&\oint \frac{dz}{2\pi i} z\tau(s'+1,\boldsymbol{t}+[\lambda^{-1}]+[\mu^{-1}]-[z],\boldsymbol{x})e^{b_1(\partial_{x_1}-z^{-1}\partial s)}\tau(s-1,\boldsymbol{t}-[z],\boldsymbol{x})\end{aligned}$$

The Taylor expansion of $e^{b_1(\partial_{x_1}-z\partial s)}$ and $e^{b_1(\partial_{x_1}-z^{-1}\partial s)}$ are similar to (3.10) replacing a_1 by b_1. By comparing the coefficients of b_1, we see that the right hand side also gives us 0. We get

$$\oint \frac{dz}{2\pi i}\frac{z}{(1-z\lambda^{-1})(1-z\mu^{-1})}\tau(s',\boldsymbol{t}+[\lambda^{-1}]+[\mu^{-1}]-[z^{-1}],\boldsymbol{x})(\partial_{x_1}-z\partial s)\tau(s,\boldsymbol{t}+[z^{-1}],\boldsymbol{x})=0$$

Using (3.6) and (3.2), we are able to get

$$\begin{aligned}&\big((\lambda-\mu)((\lambda+\mu)\partial_s-\partial_{x_1}+\partial_s(\partial_{t_1}-\partial_{t_1'}))\mathrm{F}_1\\ &\qquad +\lambda(\partial_{x_1}-\lambda\partial_s)\mathrm{F}_1-\mu(\partial_{x_1}-\mu\partial_s)\mathrm{F}_3\big)|_{s'=s,\boldsymbol{t}'=\boldsymbol{t},\boldsymbol{x}'=\boldsymbol{x}}=0\end{aligned}\tag{4.1}$$

This Fay identity is similar to (3.11). The difference between (4.1) and (3.11) is that we are shifting different terms within the product of tau functions. It is easy to see that one term on the right hand side of (4.1) is $-(\lambda-\mu)\partial_{x_1}F_1$, while the corresponding term in (3.11) is $-(\lambda-\mu)\partial_{x_1'}F_1$. Even though we set $x=x'$, note that this occurs *after* shifting; in other words, ∂_{x_1}, corresponds to shifting the first term in the product, whereas $\partial_{x_1'}$ corresponds to shifting the second part of τ-functions. Thus (3.11) and (4.1) are different.

4.2 Case 2

Next, we will observe what happens when both $\boldsymbol{a}$ and $\boldsymbol{b}$ have non-zero terms. Setting $\boldsymbol{a}=(a_1,0,\cdots)$ and $\boldsymbol{b}=(b_1,0,\cdots)$, we get

$$\begin{aligned}&\oint\frac{dz}{2\pi i}ze^{\xi(\boldsymbol{t'}-\boldsymbol{t},z)/2}e^{b_1(\partial_{x_1}-z\partial_s)}\tau(s'-a_1z,\boldsymbol{t'}-[z^{-1}],x_1+a_1,x_2,x_3)\\&\qquad\times\tau(s-b_1z,\boldsymbol{t}+[z^{-1}],x_1+b_1,x_2,x_3)\\&=\oint\frac{dz}{2\pi i}ze^{\xi(\boldsymbol{t'}-\boldsymbol{t},z^{-1})/2}e^{b_1(\partial_{x_1}-z^{-1}\partial_s)}\tau(s'+1-a_1z^{-1},\boldsymbol{t'}-[z],x_1+a_1,x_2,x_3)\\&\qquad\times\tau(s-1-b_1z^{-1},\boldsymbol{t}-[z],x_1+b_1,x_2,x_3)\end{aligned}$$

Using shift operators again as in (3.1), the equation becomes

$$\begin{aligned}&\oint\frac{dz}{2\pi i}ze^{\xi(\boldsymbol{t'}-\boldsymbol{t},z)/2}e^{a_1(\partial_{x_1}-z\partial_{s'})}\tau(s',\boldsymbol{t}+[\lambda^{-1}]+[\mu^{-1}]-[z^{-1}],\boldsymbol{x})\\&\qquad\times e^{b_1(\partial_{x_1}-z^{-1}\partial_s)}\tau(s,\boldsymbol{t}+[z^{-1}],\boldsymbol{x})\\&=\oint\frac{dz}{2\pi i}ze^{\xi(\boldsymbol{t}-\boldsymbol{t'},z^{-1})/2}e^{a_1(\partial_{x_1}-z^{-1}\partial_{s'})}\tau(s'+1,\boldsymbol{t}+[\lambda^{-1}]+[\mu^{-1}]-[z],\boldsymbol{x})\\&\qquad\times e^{b_1(\partial_{x_1}-z^{-1}\partial_s)}\tau(s-1,\boldsymbol{t}-[z],\boldsymbol{x})\end{aligned}$$

Using (2.1), we multiply both sides by $f(z^{\pm1})=e^{\xi(\boldsymbol{t'}-\boldsymbol{t},z^{\pm1})/2}$, and simplify using (3.4)

$$\begin{aligned}&\oint\frac{dz}{2\pi i}\frac{z}{(1-z\lambda^{-1})(1-z\mu^{-1})}\\&\qquad\times e^{a_1(\partial_{x_1}-z\partial_{s'})}\tau(s',\boldsymbol{t}+[\lambda^{-1}]+[\mu^{-1}]-[z^{-1}],\boldsymbol{x})e^{b_1(\partial_{x_1}-z^{-1}\partial s)}\tau(s,\boldsymbol{t}+[z^{-1}],\boldsymbol{x})\\&=\oint\frac{dz}{2\pi i}ze^{a_1(\partial_{x_1}-z\partial_{s'})}\tau(s'+1,\boldsymbol{t}+[\lambda^{-1}]+[\mu^{-1}]-[z],\boldsymbol{x})e^{b_1(\partial_{x_1}-z^{-1}\partial s)}\tau(s-1,\boldsymbol{t}-[z],\boldsymbol{x})\end{aligned}$$

We expand $e^{a_1(\partial_{x_1}-z^{\pm1}\partial_{s'})}$ and $e^{b_1(\partial_{x_1}-z^{\pm1}\partial s)}$ as in (3.1). Then we can compare the coefficients of a_1b_1. Note that the right hand side is still 0, we can write the equation as following

$$\begin{aligned}&\oint\frac{dz}{2\pi i}\frac{z}{(1-z\lambda^{-1})(1-z\mu^{-1})}(\partial_{x_1}-z\partial_{s'})\tau(s',\boldsymbol{t}+[\lambda^{-1}]+[\mu^{-1}]-[z^{-1}],\boldsymbol{x})\\&\qquad\qquad\times(\partial_{x_1}-z\partial s)\tau(s,\boldsymbol{t}+[z^{-1}],\boldsymbol{x})=0\end{aligned}$$

Using (3.6) and (3.2), we are able to get

$$\begin{aligned}
\big((\mu-\lambda)[&\partial_{x_1'}\partial_{x_1}+(\lambda+\mu)\partial_{x_1'}\partial_s(1+\partial_{t_1'}-\partial_{t_1})+(\lambda+\mu)\partial_{s'}\partial_{x_1}(1+\partial_{t_1'}-\partial_{t_1})\\
&+(\lambda^2+\lambda\mu+\mu^2)\partial_{s'}\partial_s-(\lambda+\mu)\partial_{s'}\partial_s(\partial_{t_1'}-\partial_{t_1})\\
&+\partial_{s'}\partial_s(\frac{1}{2}\partial^2_{t_1'}-\frac{1}{2}\partial_{t_2'}-\partial_{t_1'}\partial_{t_1}+\frac{1}{2}\partial^2_{t_1}+\frac{1}{2}\partial_{t_2})]\mathrm{F}_1\\
+\lambda(&\partial_{x_1'}\partial_{x_1}-\lambda(\partial_{x_1'}\partial_s+\partial_{s'}\partial_{x_1})+\lambda^2\partial_{s'}\partial_s)\mathrm{F}_2\\
-\mu(&\partial_{x_1'}\partial_{x_1}-\mu(\partial_{x_1'}\partial_s+\partial_{s'}\partial_{x_1})+\mu^2\partial_{s'}\partial_s)\mathrm{F}_3\big)|_{s'=s,\boldsymbol{t}'=\boldsymbol{t},\boldsymbol{x}'=\boldsymbol{x}}=0
\end{aligned}$$

Even though we have simplified the equation, the result we get is still very long and hard to read. We can observe there are some patterns for the coefficients of F_1, thus one may consider deriving a general formula for this equation.

4.3 Case 3

For the previous case where $\boldsymbol{a}=(a_1,0,\cdots)$, $\boldsymbol{b}=\mathbf{0}$, we wish to explore further what would happen if we compare the coefficients of $a_1^2,a_1^3,\ldots$

We resume the work from (3.9) which is

$$\begin{aligned}
&\oint\frac{dz}{2\pi i}\frac{z}{(1-z\lambda^{-1})(1-z\mu^{-1})}e^{a_1(\partial_{x_1}-z\partial_{s'})}\tau(s',\boldsymbol{t}+[\lambda^{-1}]+[\mu^{-1}]-[z^{-1}],\boldsymbol{x})\tau(s,\boldsymbol{t}+[z^{-1}],\boldsymbol{x})\\
=&\oint\frac{dz}{2\pi i}ze^{a_1(\partial_{x_1}-z^{-1}\partial_{s'})}\tau(s'+1,\boldsymbol{t}+[\lambda^{-1}]+[\mu^{-1}]-[z],\boldsymbol{x})\tau(s-1,\boldsymbol{t}-[z],\boldsymbol{x})
\end{aligned}$$

Using the Taylor expansion as in (3.1), we have:

$$\begin{aligned}
e^{a_1\partial_{x_1}}e^{-a_1z\partial_{s'}}=&(1+a_1\partial_{x_1}+\frac{a_1^2\partial^2_{x_1}}{2!}+\cdots+\frac{a_1^n\partial^n_{x_1}}{n!}+\cdots)\\
&(1-a_1z\partial_{s'}+\frac{a_1^2z^2\partial^2_{s'}}{2!}+\cdots+\frac{(-1)^na_1^nz^n\partial^n_{s'}}{n!}+\cdots)
\end{aligned}$$

This time we wish to compare the coefficients of a_1 with exponents greater than 1. Define $\mathrm{C}o(a_1^k)$ to be the coefficients of a_1^k. From a combination of terms, we have:

$$\begin{aligned}
\mathrm{C}o(a_1^2)&=\frac{\partial^2_{x_1}}{2!}-z\partial_{x_1}\partial_{s'}+\frac{z^2\partial^2_{s'}}{2!}\\
\mathrm{C}o(a_1^3)&=\frac{\partial^3_{x_1}}{3!}-\frac{z\partial^2_{x_1}\partial_{s'}}{2!}+\frac{z^2\partial_{x_1}\partial^2_{s'}}{2!}-\frac{z^3\partial^3_{s'}}{3!}\\
&\vdots\\
\mathrm{C}o(a_1^n)&=\sum_{i=0}^{n}\frac{(-1)^i}{(n-1)!i!}z^i\partial^{n-i}_{x_1}\partial^i_{s'} \qquad (4.2)\\
&=\frac{1}{n!}(\partial_{x_1}-z\partial_{s'})^n \qquad (4.3)
\end{aligned}$$

This is expected since $e^{a_1\partial_{x_1}}e^{-a_1 z\partial_{s'}} = e^{a_1(\partial_{x_1}-z\partial_{s'})}$. We will use (4.2) in the following calculation.

The expansion for $e^{a_1(\partial_{x_1}-z^{-1}\partial_{s'})}$ on the right hand side is similar. Note that unlike all the previous cases, the coefficients for z^{-1} on the right hand side is not 0. Using (3.6) as well, the equation becomes

$$\begin{aligned}&\frac{1}{\lambda^{-1}-\mu^{-1}}\oint\frac{dz}{2\pi i}\left(\frac{1}{1-z\lambda^{-1}}-\frac{1}{1-z\mu^{-1}}\right)\left(\sum_{i=0}^{n}\frac{(-1)^i}{(n-1)!i!}z^i\partial_{x_1}^{n-i}\partial_{s'}^{i}\right)\\&\quad\times\tau(s',\boldsymbol{t'}-[z^{-1}],\boldsymbol{x})\tau(s,\boldsymbol{t}+[z^{-1}],\boldsymbol{x})\\&=\oint\frac{dz}{2\pi i}\left(\sum_{i=0}^{n}\frac{(-1)^i}{(n-1)!i!}z^{1-i}\partial_{x_1}^{n-i}\partial_{s'}^{i}\right)\tau(s'+1,\boldsymbol{t'}-[z],\boldsymbol{x})\tau(s-1,\boldsymbol{t}+[z],\boldsymbol{x}))\end{aligned}\tag{4.4}$$

In the following calculation we will temporary ignore the coefficient $\frac{1}{\lambda^{-1}-\mu^{-1}}$. for convenience. We can expand the left hand side of the equation above into $2\cdot(n+1)$ terms as following:

$$\begin{aligned}&\oint\frac{dz}{2\pi i}\left(\frac{1}{1-z\lambda^{-1}}-\frac{1}{1-z\mu^{-1}}\right)\left(\sum_{i=0}^{n}\frac{(-1)^i}{(n-1)!i!}z^i\partial_{x_1}^{n-i}\partial_{s'}^{i}\right)\tau(s',\boldsymbol{t'}-[z^{-1}],\boldsymbol{x})\tau(s,\boldsymbol{t}+[z^{-1}],\boldsymbol{x})\\&=\sum_{i=0}^{n}\oint\frac{dz}{2\pi i}\frac{1}{1-z\lambda^{-1}}\left(\frac{(-1)^i}{(n-1)!i!}z^i\partial_{x_1}^{n-i}\partial_{s'}^{i}\right)\tau(s',\boldsymbol{t'}-[z^{-1}],\boldsymbol{x})\tau(s,\boldsymbol{t}+[z^{-1}],\boldsymbol{x}) \qquad (\mathbf{1})\\&-\sum_{i=0}^{n}\oint\frac{dz}{2\pi i}\frac{1}{1-z\mu^{-1}}\left(\frac{(-1)^i}{(n-1)!i!}z^i\partial_{x_1}^{n-i}\partial_{s'}^{i}\right)\tau(s',\boldsymbol{t'}-[z^{-1}],\boldsymbol{x})\tau(s,\boldsymbol{t}+[z^{-1}],\boldsymbol{x}) \qquad (\mathbf{2})\end{aligned}$$

Consider the term in (**1**) with index m, by (3.2), we obtain:

$$\begin{aligned}&\oint\frac{dz}{2\pi i}\frac{1}{1-z\lambda^{-1}}\frac{(-1)^m}{(n-m)!m!}z^m\partial_{x_1}^{n-m}\partial_{s'}^{m}\tau(s',\boldsymbol{t'}-[z^{-1}],\boldsymbol{x})\tau(s,\boldsymbol{t}+[z^{-1}],\boldsymbol{x})\\&=\frac{(-1)^m}{(n-m)!m!}\lambda^{m+1}(f(\lambda)-\sum_{i=0}^{m}f_i\lambda^{-i})\end{aligned}$$

where

$$\begin{aligned}f(\lambda)&=\partial_{x_1}^{n-m}\partial_{s'}^{m}\tau(s',\boldsymbol{t}+[\mu^{-1}],\boldsymbol{x})\tau(s,\boldsymbol{t}+[\lambda^{-1}],\boldsymbol{x})\\f(z)&=\partial_{x_1}^{n-m}\partial_{s'}^{m}\tau(s',\boldsymbol{t}+[\lambda^{-1}]+[\mu^{-1}]-[z^{-1}],\boldsymbol{x})\tau(s,\boldsymbol{t}+[z^{-1}],\boldsymbol{x})\end{aligned}$$

Using the shifting operator of τ as in (3.1), we can rewrite $f(z)$ as:

$$e^{\xi(-\tilde{\partial}_{\boldsymbol{t}},z^{-1})}\partial_{x_1}^{n-m}\partial_{s'}^{m}\tau(s',\boldsymbol{t}+[\lambda^{-1}]+[\mu^{-1}],\boldsymbol{x})\cdot e^{\xi(\tilde{\partial}_{\boldsymbol{t}},z^{-1})}\tau(s,\boldsymbol{t},\boldsymbol{x})$$

where

$$\tilde{\partial}_{\boldsymbol{t}}=(\partial_{t_1},\frac{1}{2}\partial_{t_2},\frac{1}{3}\partial_{t_3},\ldots)$$

For readability, we continue to use the notation F_1, so that

$$f(z)=e^{\xi(-\tilde{\partial}_{\boldsymbol{t}'},z^{-1})}e^{\xi(\tilde{\partial}_{\boldsymbol{t}},z^{-1})}\partial_{x_1'}^{n-m}\partial_{s'}^{m}\mathrm{F}_1|_{s'=s,\boldsymbol{t}'=\boldsymbol{t},\boldsymbol{x}'=\boldsymbol{x}} \tag{4.5}$$

We wish to have a simplified notation in order to expand $e^{\xi(-\tilde{\partial}_{\boldsymbol{t}'},z^{-1})}e^{\xi(\tilde{\partial}_{\boldsymbol{t}},z^{-1})}$. Inspired by Schur's polynomial, we define

$$\widetilde{\mathrm{S}}_n(\boldsymbol{t},\boldsymbol{t'})=\begin{cases}\sum\limits_{\substack{n_1,n_1',n_2,n_2',\dots\geq 0\\ n_1+n_1'+2n_2+2n_2'+\cdots=n}}\left(\prod\limits_{i=1}^{\infty}t_i^{(n_i)}\,{t'}_i^{(n_i')}\right) & ,\ n>0\\ 1 & ,\ n=0\\ 0 & ,\ n<0\end{cases}$$

where $x^{(n)}=x^n\cdot(n!)^{-1}$, so that we have

$$e^{\xi(\boldsymbol{t'},z^{-1})}e^{\xi(\boldsymbol{t},z^{-1})}=\sum_{i=0}^{\infty}\widetilde{\mathrm{S}}_i(\boldsymbol{t},\boldsymbol{t'})z^{-i} \tag{4.6}$$

The proof of this can be found in appendix.A.

From (4.5), using the symbol just defined, we have:

$$f_i=\partial_{x_1'}^{n-m}\partial_{s'}^{m}\widetilde{\mathrm{S}}_i(\partial_t,\partial_{t'})\mathrm{F}_1|_{s'=s,\boldsymbol{t}'=\boldsymbol{t},\boldsymbol{x}'=\boldsymbol{x}}$$

Thus we can write the terms with index m in (**1**) and (**2**) as:

$$\frac{(-1)^m}{(n-m)!m!}\lambda^{m+1}(\partial_{x_1'}^{n-m}\partial_{s'}^{m})(\mathrm{F}_2-\sum_{i=0}^{m}\widetilde{\mathrm{S}}_i(\partial_t,\partial_{t'})\lambda^{-i}\mathrm{F}_1)\qquad\text{for term in }(\mathbf{1})$$

$$\text{and}\quad\frac{(-1)^m}{(n-m)!m!}\mu^{m+1}(\partial_{x_1'}^{n-m}\partial_{s'}^{m})(\mathrm{F}_3-\sum_{i=0}^{m}\widetilde{\mathrm{S}}_i(\partial_t,\partial_{t'})\mu^{-i}\mathrm{F}_1)\qquad\text{for term in }(\mathbf{2})$$

Thus, relabeling m by k, the left hand side of equation (4.4) becomes:

$$\begin{aligned}\text{LHS }&=\frac{1}{\lambda^{-1}-\mu^{-1}}\cdot\sum_{k=0}^{n}\frac{(-1)^k}{(n-k)!k!}\lambda^{k+1}(\partial_{x_1'}^{n-k}\partial_{s'}^{k})\mathrm{F}_2\\ &+\frac{1}{\lambda^{-1}-\mu^{-1}}\cdot\sum_{k=0}^{n}\frac{(-1)^k}{(n-k)!k!}\mu^{k+1}(\partial_{x_1'}^{n-k}\partial_{s'}^{k})\mathrm{F}_3\\ &+\frac{1}{\lambda^{-1}-\mu^{-1}}\cdot\sum_{k=0}^{n}\frac{(-1)^k}{(n-k)!k!}(\partial_{x_1'}^{n-k}\partial_{s'}^{k})\left(\sum_{i=0}^{k}\widetilde{\mathrm{S}}_i(\partial_{\boldsymbol{t}},-\partial_{\boldsymbol{t'}})(\mu^{k+1-i}-\lambda^{k+1-i})\mathrm{F}_1\right)\end{aligned}$$

where $s'=s,\boldsymbol{t}'=\boldsymbol{t},\boldsymbol{x}'=\boldsymbol{x}$

$$\partial_{\boldsymbol{t}}=(\partial_{t_1},\partial_{t_2},\partial_{t_3},\cdots,\partial_{t_n},\cdots)$$
$$-\partial_{\boldsymbol{t'}}=(-\partial_{t_1'},-\partial_{t_2'},-\partial_{t_3'},\cdots,-\partial_{t_n'},\cdots)$$

Similarly, we expand the right hand side of the equation into $(n+1)$ terms, Consider the term with index m, using (4.6) we have:

$$\oint \frac{dz}{2\pi i} \frac{(-1)^m}{(n-m)!m!} z^{1-m} \partial_{x_1'}^{n-m} \partial_{s'}^{m} \tau(s'+1, \boldsymbol{t}+[\lambda^{-1}]+[\mu^{-1}]-[z], \boldsymbol{x})\tau(s-1, \boldsymbol{t}-[z], \boldsymbol{x})$$
$$= \frac{(-1)^m}{(n-m)!m!} \partial_{x_1'}^{n-m} \partial_{s'}^{m} \widetilde{S}_{m-2}(-\partial_{\boldsymbol{t}}, -\partial_{\boldsymbol{t'}}) F_1' |_{s'=s, \boldsymbol{t'}=\boldsymbol{t}, \boldsymbol{x'}=\boldsymbol{x}}$$

where $-\partial_{\boldsymbol{t}} = (-\partial_{t_1}, -\partial_{t_2}, -\partial_{t_3}, \cdots, -\partial_{t_n}, \cdots)$

$$-\partial_{\boldsymbol{t'}} = (-\partial_{t_1'}, -\partial_{t_2'}, -\partial_{t_3'}, \cdots, -\partial_{t_n'}, \cdots)$$
$$F_1'(s', \boldsymbol{t'}, \boldsymbol{x'}; s, \boldsymbol{t}, \boldsymbol{x}) = \tau(s', \boldsymbol{t'}, \boldsymbol{x'})\tau(s-3, \boldsymbol{t}-[\lambda^{-1}]-[\mu^{-1}], \boldsymbol{x})$$

Using the shifting operator to shift F_1' back to F_1, and relabeling m by k, we have:

$$\sum_{m=0}^{n} \frac{(-1)^m}{(n-m)!m!} \partial_{x_1'}^{n-m} \partial_{s'}^{m} \widetilde{S}_{m-2}(\tilde{\partial}_{\boldsymbol{t}}, \partial_{\boldsymbol{t'}}) F_1' = \sum_{k=0}^{n} \frac{(-1)^k}{(n-k)!k!} \partial_{x_1'}^{n-k} \partial_{s'}^{k} \widetilde{S}_{k-2}(\tilde{\partial}_{\boldsymbol{t}}, \partial_{\boldsymbol{t'}}) e^{-2\partial_s} F_1$$

Finally, we get the general Fay identity for $k=m=1, n=0, s'-s=1, \boldsymbol{t'}=\boldsymbol{t}+[\lambda^{-1}]+[\mu^{-1}]$, and $\boldsymbol{a}=(a_1,0,0,...)$ and $\boldsymbol{b}=0$:

$$\frac{1}{\lambda^{-1}-\mu^{-1}} \cdot \sum_{k=0}^{n} \frac{(-1)^k}{(n-k)!k!} \lambda^{k+1} (\partial_{x_1'}^{n-k} \partial_{s'}^{k}) F_2$$
$$+ \frac{1}{\lambda^{-1}-\mu^{-1}} \cdot \sum_{k=0}^{n} \frac{(-1)^k}{(n-k)!k!} \mu^{k+1} (\partial_{x_1'}^{n-k} \partial_{s'}^{k}) F_3$$
$$+ \frac{1}{\lambda^{-1}-\mu^{-1}} \cdot \sum_{k=0}^{n} \frac{(-1)^k}{(n-k)!k!} (\partial_{x_1'}^{n-k} \partial_{s'}^{k}) \left(-\widetilde{S}_{k-2}(\tilde{\partial}_{\boldsymbol{t}}, \partial_{\boldsymbol{t'}}) e^{-2\partial_s} + \sum_{i=0}^{k} \widetilde{S}_i(\partial_{\boldsymbol{t}}, \partial_{\boldsymbol{t'}})(\mu^{k+1-i}-\lambda^{k+1-i}) F_1 \right)$$
$$= 0$$

We expect the general formula of the Fay identity for $\boldsymbol{a}=0, \boldsymbol{b}=(b_1,0,...)$ to be similar in the sense that (3.11) and (4.1) are similar. Changing a_1 into b_1 makes the term that the shifting occurs different within the product of tau function. Thus, a possible guess for the b_1 Fay identity is replacing every x_1', s' by x_1, s in the formula. We left the proof detail to the reader.

5 Conclusion and Future work

In this paper, we first reviewed the extended Toda hierarchy and its Fay identities studied in [1]. Then, we obtained further Fay identities using a similar technique. Further, we tried to generalize a set of Fay identities for the substitution $\boldsymbol{a}=(a_1,0,0,\ldots)$. For future work, one may try to experiment with the cases where $\boldsymbol{a}$ and $\boldsymbol{b}$ have more non-zero terms or be some series. More advanced follow-up may be trying to work on Fay identities for the extended bilinear Toda hierarchy and trying to prove the equivalence between ETH and EBTH and their corresponding Fay identities.

A Proof of the validity of $\widetilde{\mathrm{S}}$

In this section, we are going to prove that

$$e^{\xi(\boldsymbol{t'},z^{-1})} \times e^{\xi(\boldsymbol{t},z^{-1})} = \sum_{i=0}^{\infty} \widetilde{\mathrm{S}}_i(\boldsymbol{t},\boldsymbol{t'})z^{-i} \tag{A.1}$$

We start by expanding the exponential function using regular Schur polynomial

$$\begin{aligned}
&e^{\xi(\boldsymbol{t'},z^{-1})} \times e^{\xi(\boldsymbol{t},z^{-1})} \\
&= \left(\sum_{i=0}^{\infty} \mathrm{S}_i(\boldsymbol{t'})z^{-i}\right) \times \left(\sum_{i=0}^{\infty} \mathrm{S}_i(\boldsymbol{t})z^{-i}\right) \\
&= \sum_{i=0}^{\infty} \left(\sum_{j=0}^{i} \mathrm{S}_j(\boldsymbol{t'})\mathrm{S}_{i-j}(\boldsymbol{t})\right) z^{-i}
\end{aligned}$$

Now we only need to proof that $\sum_{j=0}^{i} \mathrm{S}_j(\boldsymbol{t'})\mathrm{S}_{i-j}(\boldsymbol{t}) = \widetilde{\mathrm{S}}_i(\boldsymbol{t},\boldsymbol{t'})$. We expanding the Schur polynomial:

$$\begin{aligned}
\sum_{j=0}^{i} \mathrm{S}_j(\boldsymbol{t'})\mathrm{S}_{i-j}(\boldsymbol{t}) &= \sum_{j=0}^{i} \left(\left(\sum_{\substack{n'_1,n'_2,\ldots\geq 0 \\ 1n'_1+2n'_2+\ldots=j}} \prod_{k=1}^{\infty} t'^{(n'_k)}_k \right) \times \left(\sum_{\substack{n_1,n_2,\ldots\geq 0 \\ 1n_1+2n_2+\ldots=i-j}} \prod_{k=1}^{\infty} t^{(n_k)}_k \right) \right) \\
&= \sum_{j=0}^{i} \left(\sum_{\substack{n_1,n'_1,n_2,n'_2,\ldots\geq 0 \\ 1n'_1+2n'_2+\ldots=j \\ 1n_1+2n_2+\ldots=i-j}} \left(\prod_{k=1}^{\infty} t^{(n_k)}_k \, t'^{(n'_k)}_k \right) \right)
\end{aligned}$$

Fix some $i \geq 0$, Let

$$\begin{aligned}
&\mathbf{S}_1 = \{(n_1, n_2, \ldots, n'_1, n'_2) : 1n_1 + 1n'_1 + 2n_2 + 2n'_2 + \ldots = i\} \\
&\mathbf{S}_2 = \bigcup_{j=0}^{i} \{(n_1, n_2, \ldots, n'_1, n'_2) : 1n_1 + 2n_2 + \ldots = j, 1n'_1 + 2n'_2 + \ldots = i - j\} \\
&\quad \text{where } n_1, n'_1, n_2, n'_2, \ldots \geq 0
\end{aligned}$$

We claim that $\mathbf{S}_1 = \mathbf{S}_2$.

First we prove that $\mathbf{S}_1 \subseteq \mathbf{S}_2$. For any $e \in \mathbf{S}_1$, Since $n_1, n'_1, n_2, n'_2, \ldots \geq 0$, we have $1n_1 + 2n_2 + \ldots \geq 0$, and $1n'_1 + 2n'_2 + \ldots \geq 0$. Let $1n_1 + 2n_2 + \ldots = m$, $1n'_1 + 2n'_2 + \ldots = n$. Since $1n_1 + 1n'_1 + 2n_2 + 2n'_2 + \ldots = i$, we have $m \leq i$, $n = i - m$. Thus $e \in \mathbf{S}_2$.

We also have $\mathbf{S}_2 \subseteq \mathbf{S}_1$. For any $e \in \mathbf{S}_2$, Since $n_1, n'_1, n_2, n'_2, ... \geq 0$, $1n_1 + 2n_2 + ... = j$, $1n'_1 + 2n'_2 + ... = i - j$. We have $1n_1 + 1n'_1 + 2n_2 + 2n'_2 + ... = i$. Thus $e \in \mathbf{S}_1$.

Hence, $\mathbf{S}_1 = \mathbf{S}_2$. We are able to combine the product of two regular Schur polynomial into what we have defined:

$$\sum_{j=0}^{i} \left(\sum_{\substack{n_1, n'_1, n_2, n'_2, ... \geq 0 \\ 1n'_1 + 2n'_2 + ... = j \\ 1n_1 + 2n_2 + ... = i - j}} \left(\prod_{k=1}^{\infty} t_k^{(n_k)} {t'}_k^{(n'_k)} \right) \right) = \sum_{\substack{n_1, n'_1, n_2, n'_2, \cdots \geq 0 \\ n_1 + n'_1 + 2n_2 + 2n'_2 + \cdots = i}} \left(\prod_{k=1}^{\infty} t_k^{(n_k)} {t'}_k^{(n'_k)} \right) = \widetilde{\mathrm{S}}_i(\boldsymbol{t}, \boldsymbol{t'})$$

Therefore, (4.6) is proved.

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