Rose-Hulman Undergraduate Mathematics Journal

Volume 22
Issue 2

Article 5

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Cover Page Footnote

The author would like to thank Dr. Guy Biyogmam for his patience and support throughout the process of this undergraduate research project.

ℝose-Hulman Undergraduate Mathematics Journal
Volume 22, Issue 2, 2021

Lie-Derivations of Three-Dimensional Non-Lie Leibniz Algebras

By *Emily H. Belanger*

Abstract. The concept of Lie-derivation was recently introduced as a generalization of the notion of derivations for non-Lie Leibniz algebras. In this project, we determine the Lie algebras of Lie-derivations of all three-dimensional non-Lie Leibniz algebras. As a result of our calculations, we make conjectures on the basis of the Lie algebra of derivations of Lie-solvable non-Lie Leibniz algebras.

1 Introduction and Preliminaries

The concept of Leibniz algebra first appeared in works published in the sixties by Bloh [1], and were popularized by Jean Louis Loday [8] in the early nineties. Leibniz algebras have been studied in many fields of mathematics and mathematical physics. Essentially, Leibniz algebras are a generalization of Lie algebras, and are usually considered as noncommutative Lie algebras. For that reason, extending properties of Lie algebras to Leibniz algebras have been a main focus of research. In section 2, we will define Lie algebras and provide other definitions essential to understanding this paper.

Derivations of Leibniz algebras are important in understanding their structure, and have been intensively investigated by many authors, with the essential goal of extending results known in the case of Lie algebras (see some of the most cited results in [7, 9, 10, 11, 12]). The concept of Lie-derivations recently introduced in [2] by Biyogmam and his collaborators, relying on the fact that the quotient space $\frac{\mathfrak{g}}{\mathsf{Leib}(\mathfrak{g})}$ of a Leibniz algebra $\mathfrak{g}$ by the two-sided ideal $\mathsf{Leib}(\mathfrak{g})$ is a Lie algebra, where $\mathsf{Leib}(\mathfrak{g}) := \langle\{[x,x],\ x \in \mathfrak{g}\}\rangle$ is referred to as the Leibniz kernel of $\mathfrak{g}$. In this paper, we discuss this new concept of derivation of Leibniz algebras and completely determine in section 3 the Lie algebra of Lie-derivations of three dimensional non-Lie Leibniz algebras, identifying the inner and outer derivations among the basis element. Finally, in section 4, we conjecture that a basis of the Lie algebra of Lie-derivations of a solvable Leibniz algebra $\mathfrak{g}$ such that $\mathsf{Leib}(\mathfrak{g}) = [\mathfrak{g},\mathfrak{g}]_{\mathsf{Lie}}$ admits a non-special inner derivation, and the basis of the Lie algebra of Lie-derivations of a solvable Leibniz algebra $\mathfrak{g}$ such that $\mathsf{Leib}(\mathfrak{g}) \neq [\mathfrak{g},\mathfrak{g}]_{\mathsf{Lie}}$ admits no non-special inner derivation.

Mathematics Subject Classification. 17A32
Keywords. Leibniz algebras, Lie-derivations

2 Lie-derivations of Leibniz Algebras

In this section, we recall some definitions and background results needed in these calculations.

Definition 2.1. A (left)*Leibniz algebra* [8] is a vector space $\mathfrak{g}$ equipped with a bilinear map $[-,-] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$, usually called the Leibniz bracket of $\mathfrak{g}$, satisfying the *Leibniz identity:*

$$[x,[y,z]] = [[x,y],z] + [y,[x,z]], x, y, z \in \mathfrak{g}.$$

Remark 2.2. Notice that the definition of a Leibniz algebra is very similar to the definition of a Lie algebra, but Lie algebras have an extra condition: skew-symmetry, i.e. $[x,y] = -[y,x]$ for all $x, y \in \mathfrak{g}$. Thus, every Lie algebra is a Leibniz algebra but not every Leibniz algebra is a Lie algebra.

Definition 2.3. A subalgebra $\mathfrak{h}$ of a Leibniz algebra $\mathfrak{g}$ is said to be left (resp. right) ideal of $\mathfrak{g}$ if $[h,g] \in \mathfrak{h}$ (resp. $[g,h] \in \mathfrak{h}$), for all $h \in \mathfrak{h}, g \in \mathfrak{g}$. If $\mathfrak{h}$ is both left and right ideal, then $\mathfrak{h}$ is called two-sided ideal of $\mathfrak{g}$.

Definition 2.4. A linear map $d : \mathfrak{g} \to \mathfrak{g}$ of a Leibniz algebra $\mathfrak{g}$ is said to be an absolute derivation if for all $x, y \in \mathfrak{g}$,

$$d([x,y]) = [d(x), y] + [x, d(y)]$$

Definition 2.5. A linear map $d : \mathfrak{g} \to \mathfrak{g}$ of a Leibniz algebra $\mathfrak{g}$ is said to be a Lie-derivation if for all $x, y \in \mathfrak{g}$, the following condition holds:

$$d([x,y]_{lie}) = [d(x), y]_{lie} + [x, d(y)]_{lie}$$

Remark 2.6. The absolute derivations are also Lie-derivations since, for all $x, y \in \mathfrak{g}$,

$$\begin{aligned} d([x,y]_{lie}) &= d([x,y] + [y,x]) \\ &= [d(x), y] + [x, d(y)] + [d(y), x] + [y, d(x)] \\ &= [d(x), y]_{lie} + [x, d(y)]_{lie}. \end{aligned}$$

However, the converse is not true. For instance, every linear map $d \colon \mathfrak{g} \to \mathfrak{g}$ is a Lie-derivation for any Lie algebra $\mathfrak{g}$, but it is not a derivation in general.

The set of all Lie-derivations of a Leibniz algebra $\mathfrak{g}$ is denoted $\mathsf{Der}^{\mathsf{Lie}}(\mathfrak{g})$, and can be equipped with a structure of Lie algebra given by the bracket $[d_1, d_2] = d_1 \circ d_2 - d_2 \circ d_1$, for all $d_1, d_2 \in \mathsf{Der}(\mathfrak{g})$.

Definition 2.7. [4] A derivation $d \colon \mathfrak{g} \to \mathfrak{g}$ is said to be an inner derivation if

$$d(\mathfrak{g}) - \mathrm{L}_x(\mathfrak{g}) \subseteq \mathsf{Leib}(\mathfrak{g})$$

where $\mathrm{L}_x \colon \mathfrak{g} \to \mathfrak{g}$, is defined by $\mathrm{L}_x(y) = [x,y]$, for all $y \in \mathfrak{g}$.

Derivations of a Leibniz algebra $\mathfrak{g}$ of the form $\mathrm{L}_x, x \in \mathfrak{g}$ are referred to as special inner derivations [3] of $\mathfrak{g}$.

3 Determination of Lie-derivations of three-dimensional non-Lie Leibniz algebras

In this section, we will calculate all Lie-derivations of three-dimensional non-Lie Leibniz algebras.

Theorem 3.1. *[6] Let $\mathfrak{g}$ be a non-Lie Leibniz algebra with $\dim(\mathfrak{g}) = 3$. Then $\mathfrak{g}$ is isomorphic to a Leibniz algebra spanned by $\{x, y, z\}$ whose nonzero products are given by one of the following:*

1.) $[x, x] = y, [x, y] = z$

2.) $[x, x] = z$

3.) $[x, y] = z, [y, z] = z$

4.) $[x, y] = z, [y, x] = -z, [y, y] = z$

5.) $[x, y] = z, [y, x] = kz$ *where* $k \in \mathbb{R} - \{1, -1\}$

6.) $[z, x] = x$

7.) $[z, x] = kx$ *where* $k \in \mathbb{R} - \{0\}, [z, y] = y, [y, z] = -y$

8.) $[z, y] = y, [y, z] = -y, [z, z] = x$

9.) $[z, x] = 2x, [y, y] = x, [z, y] = y, [y, z] = -y, [z, z] = x$

10.) $[z, y] = y, [z, x] = kx$ *where* $k \in \mathbb{R} - \{0\}$

11.) $[z, x] = x + y, [z, y] = y$

12.) $[z, x] = y, [z, y] = y, [z, z] = x$

Proposition 3.2. *Let $\mathfrak{g}$ be the Leibniz algebra spanned by $\{x, y, z\}$, whose nonzero brackets are given by $[x, x] = y$ and $[x, y] = z$. Then the set $\mathrm{Der}^{\mathrm{Lie}}(\mathfrak{g})$ of Lie-derivations of $\mathfrak{g}$ is a three-dimensional Lie algebra spanned by the set $\{\alpha_1, \alpha_2, \alpha_3\}$, where*

$$\alpha_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \quad \alpha_2 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad \text{and} \quad \alpha_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Moreover, α_1 and α_3 are outer derivations, and α_2 is a special inner derivation.

Proof. Let $\alpha \in \mathsf{Der}^{\mathsf{Lie}}(\mathfrak{g})$ whose matrix M in the basis $\{x, y, z\}$ is given by

$$\mathrm{M} = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}.$$

This implies that $\alpha(x) = a_1 x + a_2 y + a_3 z$, $\alpha(y) = b_1 x + b_2 y + b_3 z$ and $\alpha(z) = c_1 x + c_2 y + c_3 z$. Since $\alpha \in \mathsf{Der}^{\mathsf{Lie}}(\mathfrak{g})$, we must have $\alpha([u, v]_{lie}) = [u, \alpha(v)]_{lie} + [\alpha(u), v]_{lie}$ for $u, v \in \mathfrak{g}$. It follows that:

$$\begin{aligned}
\alpha([x,x]_{lie}) &= 2[x,\alpha(x)]_{lie} \\
\alpha(2[x,x]) &= 2[x, a_1 x + a_2 y + a_3 z]_{lie} \\
\alpha(2y) &= 2a_1[x,x]_{lie} + 2a_2[x,y]_{lie} + 2a_3[x,z]_{lie} \\
2b_1 x + 2b_2 y + 2b_3 z &= 2a_1(2y) + 2a_2(z) \\
2b_1 x + 2b_2 y + 2b_3 z &= 4a_1 y + 2a_2 z \\
b_1 x + b_2 y + b_3 z &= 2a_1 y + a_2 z
\end{aligned}$$

$$\begin{aligned}
&\implies b_1 x + b_2 y + b_3 z - 2a_1 y - a_2 z = 0 \\
&\implies b_1 x + (b_2 - 2a_1) y + (b_3 - a_2) z = 0 \\
&\implies b_1 = 0, b_2 - 2a_1 = 0, b_3 - a_2 = 0
\end{aligned}$$

since $x, y,$ and z are linearly independent, and thus $b_1 = 0, b_2 = 2a_1$, and $b_3 = a_2$.

$$\begin{aligned}
\text{Also, } \alpha([x,y]_{lie}) &= [x,\alpha(y)]_{lie} + [\alpha(x), y]_{lie} \\
\alpha([x,y] + [y,x]) &= [x, b_1 x + b_2 y + b_3 z]_{lie} + [a_1 x + a_2 y + a_3 z, y]_{lie} \\
\alpha(z + 0) &= b_1[x,x]_{lie} + b_2[x,y]_{lie} + b_3[x,z]_{lie} + a_1[x,y]_{lie} + a_2[y,y]_{lie} \\
&\quad + a_3[z,y]_{lie} \\
\alpha(z) &= b_1(2y) + b_2(z) + 0 + a_1(z) + 0 + 0 \\
c_1 x + c_2 y + c_3 z &= 2b_1 y + b_2 z + a_1(z)
\end{aligned}$$

$$\begin{aligned}
&\implies c_1 x + c_2 y + c_3 z - 2b_1 y - b_2 z - a_1 z = 0 \\
&\implies c_1 x + (c_2 - 2b_1) y + (c_3 - b_2 - a_1) z = 0 \\
&\implies c_1 = 0, c_2 - 2b_1 = 0, c_3 - b_2 - a_1 = 0
\end{aligned}$$

since $x, y,$ and z are linearly independent, and thus $c_1 = 0, c_2 = 2b_1$, and $c_3 = b_2 + a_1$.

$$\begin{aligned}
\text{Also, } \alpha([x,z]_{lie}) &= [x,\alpha(z)]_{lie} + [\alpha(x), z]_{lie} \\
\alpha([x,z] + [z,x]) &= [x, c_1 x + c_2 y + c_3 z]_{lie} + [a_1 x + a_2 y + a_3 z, z]_{lie} \\
\alpha(0 + 0) &= c_1[x,x]_{lie} + c_2[x,y]_{lie} + c_3[x,z]_{lie} + a_1[x,z]_{lie} + a_2[y,z]_{lie} \\
&\quad + a_3[z,z]_{lie} \\
0 &= c_1(2y) + c_2(z) \\
0 &= 2c_1 y + c_2 z.
\end{aligned}$$

This implies that $c_1 = 0$ and $c_2 = 0$ since y and z are linearly independent.

$$\begin{aligned}
\text{Also, }\ \alpha([y,y]_{lie}) &= 2[y,\alpha(y)]_{lie}\\
\alpha(2[y,y]) &= 2[y, b_1x + b_2y + b_3z]_{lie}\\
\alpha(0) &= 2b_1[y,x]_{lie} + 2b_2[y,y]_{lie} + 2b_3[y,z]_{lie}\\
0 &= 2b_1(z),\ \text{ implying that } b_1 = 0.
\end{aligned}$$

$$\begin{aligned}
\text{Also, }\ \alpha([y,z]_{lie}) &= [y,\alpha(z)]_{lie} + [\alpha(y),z]_{lie}\\
\alpha([y,z]+[z,y]) &= [y, c_1x + c_2y + c_3z]_{lie} + [b_1x + b_2y + b_3z, z]_{lie}\\
\alpha(0+0) &= c_1[y,x]_{lie} + c_2[y,y]_{lie} + c_3[y,z]_{lie} + b_1[x,z]_{lie} + b_2[y,z]_{lie}\\
&\quad + b_3[z,z]_{lie}\\
0 &= c_1(z),\ \text{ implying that } c_1 = 0.
\end{aligned}$$

Note that the identity $\alpha([z,z]_{lie}) = 2[z,\alpha(z)]_{lie}$ yields $0=0$.
In summary, $b_1 = 0, b_2 = 2a_1, b_3 = a_2, c_1 = 0, c_2 = 0$, and $c_3 = 3a_1$.

Therefore,

$$\alpha = \begin{bmatrix} a_1 & 0 & 0\\ a_2 & 2a_1 & 0\\ a_3 & a_2 & 3a_1 \end{bmatrix} = a_1 \underbrace{\begin{bmatrix} 1 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 3 \end{bmatrix}}_{\alpha_1} + a_2 \underbrace{\begin{bmatrix} 0 & 0 & 0\\ 1 & 0 & 0\\ 0 & 1 & 0 \end{bmatrix}}_{\alpha_2} + a_3 \underbrace{\begin{bmatrix} 0 & 0 & 0\\ 0 & 0 & 0\\ 1 & 0 & 0 \end{bmatrix}}_{\alpha_3}.$$

It is straightforward to show that the vectors $\{\alpha_1, \alpha_2, \alpha_3\}$ are linearly independent.

Now, since $\mathsf{Leib}(\mathfrak{g}) = \langle y \rangle$, if $\alpha \in \{\alpha_1, \alpha_2, \alpha_3\}$ is an inner derivation, then $(\alpha - \mathrm{L}_x)(\mathfrak{g}) \subseteq \langle y \rangle$. Note that $\mathrm{L}_x(x) = y$, $\mathrm{L}_x(y) = z$ and $\mathrm{L}_x(z) = 0$. In this case, α_2 is a special inner derivation because $(\alpha_2 - \mathrm{L}_x)(\mathfrak{g}) = 0$. α_1 is an outer derivation because $(\alpha_1 - \mathrm{L}_x)(x) = x - y \notin \langle y \rangle$. Similarly, α_3 is an outer derivation because$(\alpha_3 - \mathrm{L}_x)(x) = z - y \notin \langle y \rangle$. □

Proposition 3.3. *Let $\mathfrak{g}$ be the Leibniz algebra spanned by $\{x, y, z\}$, whose nonzero brackets are given by $[x,x] = z$. Then the set $\mathsf{Der}^{\mathsf{Lie}}(\mathfrak{g})$ of Lie-derivations of $\mathfrak{g}$ is a five-dimensional Lie algebra spanned by the matrices*

$$\alpha_1 = \begin{bmatrix} 1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 2 \end{bmatrix},\ \alpha_2 = \begin{bmatrix} 0 & 0 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0 \end{bmatrix},\ \alpha_3 = \begin{bmatrix} 0 & 0 & 0\\ 0 & 0 & 0\\ 1 & 0 & 0 \end{bmatrix},\ \alpha_4 = \begin{bmatrix} 0 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0 \end{bmatrix},\ \alpha_5 = \begin{bmatrix} 0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 1 & 0 \end{bmatrix}.$$

Moreover, α_1, α_2, α_4 are outer derivations, α_5 is an inner derivation and α_3 is a special inner derivation.

Proof. Let $\alpha \in \mathsf{Der}^{\mathsf{Lie}}(\mathfrak{g})$ whose matrix M in the basis $\{x, y, z\}$ is given by

$$\alpha = \begin{bmatrix} a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\\ a_3 & b_3 & c_3 \end{bmatrix}.$$

This implies that $\alpha(x) = a_1x + a_2y + a_3z$, $\alpha(y) = b_1x + b_2y + b_3z$, and $\alpha(z) = c_1x + c_2y + c_3z$. Since $\alpha \in \mathsf{Der}^{\mathsf{Lie}}(\mathfrak{g})$, we must have $\alpha([u,v]_{lie}) = [u,\alpha(v)]_{lie} + [\alpha(u),v]_{lie}$ for $u, v \in \mathfrak{g}$. It follows that

$$\begin{aligned}
\alpha([x,x]_{lie}) &= 2[x,\alpha(x)]_{lie}\\
\alpha(2[x,x]) &= 2[x, a_1x + a_2y + a_3z]_{lie}\\
\alpha(2z) &= 2a_1[x,x]_{lie} + 2a_2[x,y]_{lie} + 2a_3[x,z]_{lie}\\
2c_1x + 2c_2y + 2c_3z &= 2a_1(2z)\\
2c_1x + 2c_2y + 2c_3z &= 4a_1z\\
c_1x + c_2y + c_3z &= 2a_1z
\end{aligned}$$

This implies that $c_1x + c_2y + (c_3 - 2a_1)z = 0$, i.e. $c_1 = 0, c_2 = 0, c_3 - 2a_1 = 0$ since $x, y,$ and z are linearly independent, and thus $c_1 = 0, c_2 = 0$, and $c_3 = 2a_1$.

$$\begin{aligned}
\text{Also, }\ \alpha([x,y]_{lie}) &= [x,\alpha(y)]_{lie} + [\alpha(x),y]_{lie}\\
\alpha([x,y]+[y,x]) &= [x, b_1x + b_2y + b_3z]_{lie} + [a_1x + a_2y + a_3z, y]_{lie}\\
\alpha(0+0) &= b_1[x,x]_{lie} + b_2[x,y]_{lie} + b_3[x,z]_{lie} + a_1[x,y]_{lie} + a_2[y,y]_{lie}\\
&\quad + a_3[z,y]_{lie}\\
0 &= b_1(2z), \ \text{ implying that } b_1 = 0.
\end{aligned}$$

$$\begin{aligned}
\text{Also, }\ \alpha([x,z]_{lie}) &= [x,\alpha(z)]_{lie} + [\alpha(x),z]_{lie}\\
\alpha([x,z]+[z,x]) &= [x, c_1x + c_2y + c_3z]_{lie} + [a_1x + a_2y + a_3z, z]_{lie}\\
\alpha(0+0) &= c_1[x,x]_{lie} + c_2[x,y]_{lie} + c_3[x,z]_{lie} + a_1[x,z]_{lie} + a_2[y,z]_{lie}\\
&\quad + a_3[z,z]_{lie}\\
0 &= 2c_1z, \ \text{ implying that } c_1 = 0.
\end{aligned}$$

Note that all the identities $\alpha([y,y]_{lie}) = 2[y,\alpha(y)]_{lie}$, $\alpha([y,z]_{lie}) = [y,\alpha(z)]_{lie} + [\alpha(y),z]_{lie}$ and $\alpha([z,z]_{lie}) = 2[z,\alpha(z)]_{lie}$ yield $0 = 0$.

In summary, $b_1 = 0, c_1 = 0, c_2 = 0$, and $c_3 = 2a_1$. Therefore,

$$\begin{aligned}
\alpha = \begin{bmatrix} a_1 & 0 & 0\\ a_2 & b_2 & 0\\ a_3 & b_3 & 2a_1 \end{bmatrix} &= a_1 \underbrace{\begin{bmatrix} 1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 2 \end{bmatrix}}_{\alpha_1} + a_2 \underbrace{\begin{bmatrix} 0 & 0 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0 \end{bmatrix}}_{\alpha_2} + a_3 \underbrace{\begin{bmatrix} 0 & 0 & 0\\ 0 & 0 & 0\\ 1 & 0 & 0 \end{bmatrix}}_{\alpha_3}\\
&\quad + b_2 \underbrace{\begin{bmatrix} 0 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0 \end{bmatrix}}_{\alpha_4} + b_3 \underbrace{\begin{bmatrix} 0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 1 & 0 \end{bmatrix}}_{\alpha_5}.
\end{aligned}$$

Now, since $\mathsf{Leib}(\mathfrak{g}) = \langle z \rangle$, if $\alpha \in \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\}$ is an inner derivation, then $(\alpha - \mathrm{L}_x)(\mathfrak{g}) \subseteq \langle z \rangle$. Note that $\mathrm{L}_x(x) = z$, $\mathrm{L}_x(y) = 0$ and $\mathrm{L}_x(z) = 0$. In this case, α_3 is a special inner derivation because $(\alpha_3 - L_x)(\mathfrak{g}) = 0$. α_5 is an inner derivation. α_1 is an outer derivation because $(\alpha_1 - L_x)(x) = x - z \notin \langle z \rangle$. Similarly, α_2 is an outer derivation because $(\alpha_2 - L_x)(x) = y - z \notin \langle z \rangle$. Finally, α_4 is an outer derivation because $(\alpha_4 - L_x)(y) = y \notin \langle z \rangle$. $\square$

Proposition 3.4. *Let $\mathfrak{g}$ be the Leibniz algebra spanned by $\{x, y, z\}$, whose nonzero brackets are given by $[x, y] = z$ and $[y, z] = z$. Then the set $\mathsf{Der}^{\mathsf{Lie}}(\mathfrak{g})$ of Lie-derivations of $\mathfrak{g}$ is a three-dimensional Lie algebra spanned by the set $\{\alpha_1, \alpha_2, \alpha_3\}$, where*

$$\alpha_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \alpha_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad \textit{and} \quad \alpha_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}.$$

Moreover, α_1, α_2 and α_3 are outer derivations.

Proof. Let $\alpha \in \mathsf{Der}^{\mathsf{Lie}}(\mathfrak{g})$ whose matrix M in the basis $\{x, y, z\}$ is given by

$$\alpha = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}.$$

This implies that $\alpha(x) = a_1 x + a_2 y + a_3 z$, $\alpha(y) = b_1 x + b_2 y + b_3 z$ and $\alpha(z) = c_1 x + c_2 y + c_3 z$. Since $\alpha \in \mathsf{Der}^{\mathsf{Lie}}(\mathfrak{g})$, we must have $\alpha([u, v]_{lie}) = [u, \alpha(v)]_{lie} + [\alpha(u), v]_{lie}$ for $u, v \in \mathfrak{g}$. It follows that

$$\begin{aligned} \alpha([x,x]_{lie}) &= 2[x, \alpha(x)]_{lie} \\ \alpha(2[x,x]) &= 2[x, a_1 x + a_2 y + a_3 z]_{lie} \\ \alpha(0) &= 2a_1[x,x]_{lie} + 2a_2[x,y]_{lie} + 2a_3[x,z]_{lie} \\ 0 &= 2a_2(z), \text{ implying that } a_2 = 0. \end{aligned}$$

$$\begin{aligned} \text{Also, } \alpha([x,y]_{lie}) &= [x, \alpha(y)]_{lie} + [\alpha(x), y]_{lie} \\ \alpha([x,y] + [y,x]) &= [x, b_1 x + b_2 y + b_3 z]_{lie} + [a_1 x + a_2 y + a_3 z, y]_{lie} \\ \alpha(z + 0) &= b_1[x,x]_{lie} + b_2[x,y]_{lie} + b_3[x,z]_{lie} + a_1[x,y]_{lie} + a_2[y,y]_{lie} \\ &\quad + a_3[z,y]_{lie} \\ c_1 x + c_2 y + c_3 z &= b_2(z) + a_1(z) + a_3(z) \end{aligned}$$

This implies that $c_1 x + c_2 y + (c_3 - b_2 - a_1 - a_3)z = 0$ i.e. $c_1 = 0, c_2 = 0, c_3 - b_2 - a_1 - a_3 = 0$ since $x, y,$ and z are linearly independent, and thus $c_1 = 0, c_2 = 0$, and $c_3 = b_2 + a_1 + a_3$.

$$\begin{aligned} \text{Also, } \alpha([x,z]_{lie}) &= [x, \alpha(z)]_{lie} + [\alpha(x), z]_{lie} \\ \alpha([x,z] + [z,x]) &= [x, c_1 x + c_2 y + c_3 z]_{lie} + [a_1 x + a_2 y + a_3 z, z]_{lie} \\ \alpha(0 + 0) &= c_1[x,x]_{lie} + c_2[x,y]_{lie} + c_3[x,z]_{lie} + a_1[x,z]_{lie} + a_2[y,z]_{lie} \\ &\quad + a_3[z,z]_{lie} \\ 0 &= c_2(z) + a_2(z) \\ 0 &= (c_2 + a_2)z, \text{ which implies } c_2 = -a_2. \end{aligned}$$

$$\begin{aligned}
\text{Also,}\ \alpha([y,y]_{lie}) &= 2[y,\alpha(y)]_{lie}\\
\alpha(2[y,y]) &= 2[x, b_1x+b_2y+b_3z]_{lie}\\
\alpha(0) &= 2b_1[y,x]_{lie}+2b_2[y,y]_{lie}+2b_3[y,z]_{lie}\\
0 &= 2b_1(z)+2b_3(z)\\
0 &= (b_1+b_3)z,\ \text{which implies } b_1=-b_3.
\end{aligned}$$

$$\begin{aligned}
\text{Also,}\ \alpha([y,z]_{lie}) &= [y,\alpha(z)]_{lie}+[\alpha(y),z]_{lie}\\
\alpha([y,z]+[z,y]) &= [y, c_1x+c_2y+c_3z]_{lie}+[b_1x+b_2y+b_3z,z]_{lie}\\
\alpha(z+0) &= c_1[y,x]_{lie}+c_2[y,y]_{lie}+c_3[y,z]_{lie}+b_1[x,z]_{lie}+b_2[y,z]_{lie}\\
&\quad + b_3[z,z]_{lie}\\
c_1x+c_2y+c_3z &= c_1(z)+c_3(z)+b_2(z)\\
c_1x+c_2y &= (c_1+b_2)z.
\end{aligned}$$

This implies $c_1=0, c_2=0, c_1+b_2=0$ since $x, y,$ and z are linearly independent, and thus $c_1=0, c_2=0$, and $c_1=-b_2$.

$$\begin{aligned}
\text{Finally,}\ \alpha([z,z]_{lie}) &= 2[z,\alpha(z)]_{lie}\\
\alpha(2[z,z]) &= 2[z, c_1x+c_2y+c_3z]_{lie}\\
\alpha(0) &= 2c_1[z,x]_{lie}+2c_2[z,y]_{lie}+2c_3[z,z]_{lie}\\
0 &= 2c_2(z),\ \text{implying that } c_2=0.
\end{aligned}$$

In summary, $a_2=0, b_2=0, b_3=-b_1, c_1=0, c_2=0$ and $c_3=b_2+a_1+a_3=a_1+a_3$. Therefore,

$$\alpha=\begin{bmatrix} a_1 & b_1 & 0\\ 0 & 0 & 0\\ a_3 & -b_1 & a_1+a_3\end{bmatrix} = a_1\underbrace{\begin{bmatrix}1&0&0\\0&0&0\\0&0&1\end{bmatrix}}_{\alpha_1}+a_3\underbrace{\begin{bmatrix}0&0&0\\0&0&0\\1&0&1\end{bmatrix}}_{\alpha_2}+b_1\underbrace{\begin{bmatrix}0&1&0\\0&0&0\\0&-1&0\end{bmatrix}}_{\alpha_3}.$$

Now, since $\mathsf{Leib}(\mathfrak{g})=0$, if $\alpha\in\{\alpha_1,\alpha_2,\alpha_3\}$ is an inner derivation, then either $\alpha=\mathrm{L}_x$ or $\alpha=\mathrm{L}_y$. Note that $\mathrm{L}_x(x)=\mathrm{L}_x(z)=0$ and $\mathrm{L}_x(y)=z$, and $\mathrm{L}_y(x)=\mathrm{L}_y(y)=0$ and $\mathrm{L}_y(z)=z$. In this case, α_1 is an outer derivation because $(\alpha_1-\mathrm{L}_x)(x)=(\alpha_1-\mathrm{L}_y)(x)=x\neq 0$. Similarly, α_2 is an outer derivation because $(\alpha_2-\mathrm{L}_x)(x)=(\alpha_2-\mathrm{L}_y)(x)=z\neq 0$. Finally, α_3 is an outer derivation because $(\alpha_3-\mathrm{L}_x)(y)=x-2z\neq 0$ and $(\alpha_3-\mathrm{L}_y)(y)=x-z\neq 0$. □

Proposition 3.5. *Let $\mathfrak{g}$ be the Leibniz algebra spanned by $\{x,y,z\}$, whose nonzero brackets are given by $[x,y]=z$, $[y,x]=-z$, and $[y,y]=z$. Then the set $\mathsf{Der}^{\mathsf{Lie}}(\mathfrak{g})$ of Lie-derivations of $\mathfrak{g}$ is a five-dimensional Lie algebra spanned by the set $\{\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5\}$ where*

$$\alpha_1=\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix},\ \alpha_2=\begin{bmatrix}0&0&0\\0&0&0\\1&0&0\end{bmatrix},\ \alpha_3=\begin{bmatrix}0&1&0\\0&0&0\\0&0&0\end{bmatrix},\ \alpha_4=\begin{bmatrix}0&0&0\\0&1&0\\0&0&2\end{bmatrix},\ \alpha_5=\begin{bmatrix}0&0&0\\0&0&0\\0&1&0\end{bmatrix}.$$

Moreover, α_1, α_3 *and* α_4 *are outer derivations, and* α_2 *is an inner derivation and* α_5 *is a special inner derivation.*

Proof. Let $\alpha \in \mathsf{Der}^{\mathsf{Lie}}(\mathfrak{g})$ whose matrix M in the basis $\{x, y, z\}$ is given by

$$\alpha = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}.$$

This implies that $\alpha(x) = a_1x + a_2y + a_3z$, $\alpha(y) = b_1x + b_2y + b_3z$ and $\alpha(z) = c_1x + c_2y + c_3z$. Since $\alpha \in \mathsf{Der}^{\mathsf{Lie}}(\mathfrak{g})$, we must have $\alpha([u,v]_{lie}) = [u,\alpha(v)]_{lie} + [\alpha(u),v]_{lie}$ for $u, v \in \mathfrak{g}$. It follows that

$$\begin{aligned}
\alpha([x,y]_{lie}) &= [x,\alpha(y)]_{lie} + [\alpha(x),y]_{lie} \\
\alpha([x,y]+[y,x]) &= [x, b_1x + b_2y + b_3z]_{lie} + [a_1x + a_2y + a_3z, y]_{lie} \\
\alpha(z-z) &= b_1[x,x]_{lie} + b_2[x,y]_{lie} + b_3[x,z]_{lie} + a_1[x,y]_{lie} + a_2[y,y]_{lie} \\
&\quad + a_3[z,y]_{lie} \\
0 &= b_2(z-z) + a_1(z-z) + a_2(2z) \\
0 &= 2a_2z, \;\text{ implying that } a_2 = 0.
\end{aligned}$$

$$\begin{aligned}
\text{Also, }\; \alpha([y,y]_{lie}) &= 2[y,\alpha(y)]_{lie} \\
\alpha(2[y,y]) &= 2[y, b_1x + b_2y + b_3z]_{lie} \\
\alpha(2z) &= 2b_1[y,x]_{lie} + 2b_2[y,y]_{lie} + 2b_3[y,z]_{lie} \\
2c_1x + 2c_2y + 2c_3z &= 2b_1(z-z) + 2b_2(2z) \\
2c_1x + 2c_2y + 2c_3z &= 4b_2z \\
c_1x + c_2y + c_3z &= 2b_2z.
\end{aligned}$$

This implies that $c_1 = 0, c_2 = 0$, and $c_3 = 2b_2$ since $x, y,$ and z are linearly independent.

$$\begin{aligned}
\text{Also, }\; \alpha([y,z]_{lie}) &= [y,\alpha(z)]_{lie} + [\alpha(y),z]_{lie} \\
\alpha([y,z]+[z,y]) &= [y, c_1x + c_2y + c_3z]_{lie} + [b_1x + b_2y + b_3z, z]_{lie} \\
\alpha(0+0) &= c_1[y,x]_{lie} + c_2[y,y]_{lie} + c_3[y,z]_{lie} + b_1[x,z]_{lie} + b_2[y,z]_{lie} \\
&\quad + b_3[z,z]_{lie} \\
0 &= c_1(z-z) + c_2(2z) \\
0 &= 2c_2z, \;\text{ implying that } c_2 = 0.
\end{aligned}$$

Note that all the identities $\alpha([x,x]_{lie}) = 2[x,\alpha(x)]_{lie}$, $\alpha([x,z]_{lie}) = [x,\alpha(z)]_{lie} + [\alpha(x),z]_{lie}$ and $\alpha([z,z]_{lie}) = 2[z,\alpha(z)]_{lie}$ yield $0 = 0$.

In summary, $a_2=0, c_1=0, c_2=0$, and $c_3=2b_2$. Therefore,

$$\alpha=\begin{bmatrix} a_1 & b_1 & 0\\ 0 & b_2 & 0\\ a_3 & b_3 & 2b_2\end{bmatrix}=a_1\underbrace{\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}}_{\alpha_1}+a_3\underbrace{\begin{bmatrix}0&0&0\\0&0&0\\1&0&0\end{bmatrix}}_{\alpha_2}+b_1\underbrace{\begin{bmatrix}0&1&0\\0&0&0\\0&0&0\end{bmatrix}}_{\alpha_3}$$
$$+b_2\underbrace{\begin{bmatrix}0&0&0\\0&1&0\\0&0&2\end{bmatrix}}_{\alpha_4}+b_3\underbrace{\begin{bmatrix}0&0&0\\0&0&0\\0&1&0\end{bmatrix}}_{\alpha_5}.$$

Now, since $\mathsf{Leib}(\mathfrak{g})=\langle z\rangle$, if $\alpha\in\{\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5\}$ is an inner derivation, then either $(\alpha-\mathrm{L}_x)(\mathfrak{g})\subseteq\langle z\rangle$ or $(\alpha-\mathrm{L}_y)(\mathfrak{g})\subseteq\langle z\rangle$. Note that $\mathrm{L}_x(x)=\mathrm{L}_x(z)=0$ and $\mathrm{L}_x(y)=z$, and $\mathrm{L}_y(x)=-\mathrm{L}_y(y)=-z$ and $\mathrm{L}_y(z)=0$. In this case, α_2 is an inner derivation because $(\alpha_2-\mathrm{L}_x)(\mathfrak{g})=\langle z\rangle$. We also have $(\alpha_2-\mathrm{L}_y)(\mathfrak{g})=\langle z\rangle$. α_5 is a special inner derivation since $(\alpha_5-\mathrm{L}_x)(\mathfrak{g})=0$. However, α_1 is an outer derivation because $(\alpha_1-\mathrm{L}_x)(x)=x\notin\langle z\rangle$ and $(\alpha_1-\mathrm{L}_y)(x)=x+z\notin\langle z\rangle$. Similarly, α_3 is an outer derivation because $(\alpha_3-\mathrm{L}_x)(y)=x-z\notin\langle z\rangle$ and $(\alpha_3-\mathrm{L}_y)(y)=x-z\notin\langle z\rangle$. Finally, α_4 is an outer derivation because $(\alpha_4-\mathrm{L}_x)(y)=y-z\notin\langle z\rangle$ and $(\alpha_4-\mathrm{L}_y)(y)=y-z\notin\langle z\rangle$. □

Proposition 3.6. *Let $\mathfrak{g}$ be the Leibniz algebra spanned by $\{x,y,z\}$, whose nonzero brackets are given by $[x,y]=z$ and $[y,x]=kz$ where $k\in\mathbb{R}-\{1,-1\}$. Then the set $\mathsf{Der}^{\mathsf{Lie}}(\mathfrak{g})$ of Lie-derivations of $\mathfrak{g}$ is a four-dimensional Lie algebra spanned by the set $\{\alpha_1,\alpha_2,\alpha_3,\alpha_4\}$, where*

$$\alpha_1=\begin{bmatrix}1&0&0\\0&0&0\\0&0&1\end{bmatrix},\quad \alpha_2=\begin{bmatrix}0&0&0\\0&1&0\\0&0&1\end{bmatrix},\quad \alpha_3=\begin{bmatrix}0&0&0\\0&0&0\\1&0&0\end{bmatrix},\quad \alpha_4=\begin{bmatrix}0&0&0\\0&0&0\\0&1&0\end{bmatrix}.$$

Moreover, α_1, α_2 and α_3 are outer derivations and α_4 is a special inner derivation.

Proof. Let $\alpha\in\mathsf{Der}^{\mathsf{Lie}}(\mathfrak{g})$ whose matrix M in the basis $\{x,y,z\}$ is given by

$$\alpha=\begin{bmatrix}a_1&b_1&c_1\\a_2&b_2&c_2\\a_3&b_3&c_3\end{bmatrix}.$$

This implies that $\alpha(x)=a_1x+a_2y+a_3z$, $\alpha(y)=b_1x+b_2y+b_3z$ and $\alpha(z)=c_1x+c_2y+c_3z$. Since $\alpha\in\mathsf{Der}^{\mathsf{Lie}}(\mathfrak{g})$, we must have $\alpha([u,v]_{lie})=[u,\alpha(v)]_{lie}+[\alpha(u),v]_{lie}$ for $u,v\in\mathfrak{g}$. It follows that

$$\begin{aligned}
\alpha([x,x]_{lie})&=2[x,\alpha(x)]_{lie}\\
\alpha(2[x,x])&=2[x,a_1x+a_2y+a_3z]_{lie}\\
\alpha(0)&=2a_1[x,x]_{lie}+2a_2[x,y]_{lie}+2a_3[x,z]_{lie}\\
0&=2a_2(z+kz)\\
0&=2a_2(1+k)z,\ \text{ implying that } a_2=0 \text{ since } k\neq-1.
\end{aligned}$$

$$\begin{aligned}
\text{Also, }\ \alpha([x,y]_{lie}) &= [x,\alpha(y)]_{lie} + [\alpha(x),y]_{lie}\\
\alpha([x,y]+[y,x]) &= [x, b_1x+b_2y+b_3z]_{lie} + [a_1x+a_2y+a_3z, y]_{lie}\\
\alpha(z+kz) &= b_1[x,x]_{lie} + b_2[x,y]_{lie} + b_3[x,z]_{lie} + a_1[x,y]_{lie}\\
&\quad + a_2[y,y]_{lie} + a_3[z,y]_{lie}\\
\alpha((1+k)z) &= b_2(z+kz) + a_1(z+kz).
\end{aligned}$$

So, $c_1(1+k)x + c_2(1+k)y + c_3(1+k)z = b_2(1+k)z + a_1(1+k)z$, and thus $c_1x + c_2y + (c_3 - b_2 - a_1)z = 0$. This implies $c_1 = 0, c_2 = 0$ and $c_3 - b_2 - a_1 = 0$ since x, y, and z are linearly independent. Therefore $c_1 = 0, c_2 = 0$ and $c_3 = a_1 + b_2$.

$$\begin{aligned}
\text{Also, }\ \alpha([x,z]_{lie}) &= [x,\alpha(z)]_{lie} + [\alpha(x),z]_{lie}\\
\alpha([x,z]+[z,x]) &= [x, c_1x+c_2y+c_3z]_{lie} + [a_1x+a_2y+a_3z, z]_{lie}\\
\alpha(0+0) &= c_1[x,x]_{lie} + c_2[x,y]_{lie} + c_3[x,z]_{lie} + a_1[x,z]_{lie} + a_2[y,z]_{lie}\\
&\quad + a_3[z,z]_{lie}\\
0 &= c_2(z+kz), \ \text{ implying that } c_2 = 0.
\end{aligned}$$

$$\begin{aligned}
\text{Also, }\ \alpha([y,y]_{lie}) &= 2[y,\alpha(y)]_{lie}\\
\alpha(2[y,y]) &= 2[y, b_1x+b_2y+b_3z]_{lie}\\
\alpha(0) &= 2b_1[y,x]_{lie} + 2b_2[y,y]_{lie} + 2b_3[y,z]_{lie}\\
0 &= 2b_1(z+kz)\\
0 &= 2b_1(1+k)z, \ \text{ implying that } b_1 = 0.
\end{aligned}$$

$$\begin{aligned}
\text{Also, }\ \alpha([y,z]_{lie}) &= [y,\alpha(z)]_{lie} + [\alpha(y),z]_{lie}\\
\alpha([y,z]+[z,y]) &= [y, c_1x+c_2y+c_3z]_{lie} + [b_1x+b_2y+b_3z, z]_{lie}\\
\alpha(0+0) &= c_1[y,x]_{lie} + c_2[y,y]_{lie} + c_3[y,z]_{lie} + b_1[x,z]_{lie} + b_2[y,z]_{lie}\\
&\quad + b_3[z,z]_{lie}\\
0 &= c_1(z+kz)\\
0 &= c_1(1+k)z, \ \text{ implying that } c_1 = 0 \text{ since } k \neq -1.
\end{aligned}$$

Note that the identity $\alpha([z,z]_{lie}) = 2[z,\alpha(z)]_{lie}$ yields $0 = 0$.
In summary, $a_2 = 0, b_1 = 0, c_1 = 0, c_2 = 0$, and $c_3 = a_1 + b_2$. Therefore,

$$\alpha = \begin{bmatrix} a_1 & 0 & 0 \\ 0 & b_2 & 0 \\ a_3 & b_3 & a_1+b_2 \end{bmatrix} = a_1 \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\alpha_1} + b_2 \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\alpha_2} + a_3 \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}}_{\alpha_3} + b_3 \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}}_{\alpha_4}.$$

It is straightforward to show that the vectors $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ are linearly independent.

Now, since $\mathsf{Leib}(\mathfrak{g}) = 0$, if $\alpha \in \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ is an inner derivation, then either $\alpha = \mathrm{L}_x$ or $\alpha = \mathrm{L}_y$. Note that $\mathrm{L}_x(x) = \mathrm{L}_x(z) = 0$ and $\mathrm{L}_x(y) = z$, and $\mathrm{L}_y(y) = \mathrm{L}_y(z) = 0$ and $\mathrm{L}_y(x) =$

$kz,\ k \neq \pm 1$. In this case, α_4 is a special inner derivation since $\alpha_4 = \mathrm{L}_x$. α_1 is an outer derivation because $(\alpha_1 - \mathrm{L}_x)(z) = (\alpha_1 - \mathrm{L}_y)(z) = z \neq 0$. Similarly, α_2 is an outer derivation because $(\alpha_2 - \mathrm{L}_x)(z) = (\alpha_2 - \mathrm{L}_y)(z) = z \neq 0$. Finally, α_3 is an outer derivation because $(\alpha_3 - \mathrm{L}_x)(x) = z \neq 0$ and $(\alpha_3 - \mathrm{L}_y)(x) = (1-k)z \neq 0$ since $k \neq 1$. □

Proposition 3.7. *Let $\mathfrak{g}$ be the Leibniz algebra spanned by $\{x, y, z\}$, whose nonzero brackets are given by $[z, x] = x$. Then the set $\mathsf{Der}^{\mathsf{Lie}}(\mathfrak{g})$ of Lie-derivations of $\mathfrak{g}$ is a three-dimensional Lie algebra spanned by the set $\{\alpha_1, \alpha_2, \alpha_3\}$, where*

$$\alpha_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \alpha_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \textit{and} \quad \alpha_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Moreover, α_2 and α_3 are outer derivations and α_1 is a special inner derivation.

Proof. Let $\alpha \in \mathsf{Der}^{\mathsf{Lie}}(\mathfrak{g})$ whose matrix M in the basis $\{x, y, z\}$ is given by

$$\alpha = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}.$$

This implies that $\alpha(x) = a_1x + a_2y + a_3z$, $\alpha(y) = b_1x + b_2y + b_3z$ and $\alpha(z) = c_1x + c_2y + c_3z$. Since $\alpha \in \mathsf{Der}^{\mathsf{Lie}}(\mathfrak{g})$, we must have $\alpha([u, v]_{lie}) = [u, \alpha(v)]_{lie} + [\alpha(u), v]_{lie}$ for $u, v \in \mathfrak{g}$. It follows that

$$\begin{aligned} \alpha([x,x]_{lie}) &= 2[x, \alpha(x)]_{lie} \\ \alpha(2[x,x]) &= 2[x, a_1x + a_2y + a_3z]_{lie} \\ \alpha(0) &= 2a_1[x,x]_{lie} + 2a_2[x,y]_{lie} + 2a_3[x,z]_{lie} \\ 0 &= 2a_3(x), \ \text{ implying that } a_3 = 0. \end{aligned}$$

$$\begin{aligned} \text{Also,}\ \ \alpha([x,y]_{lie}) &= [x, \alpha(y)]_{lie} + [\alpha(x), y]_{lie} \\ \alpha([x,y] + [y,x]) &= [x, b_1x + b_2y + b_3z]_{lie} + [a_1x + a_2y + a_3z, y]_{lie} \\ \alpha(0+0) &= b_1[x,x]_{lie} + b_2[x,y]_{lie} + b_3[x,z]_{lie} + a_1[x,y]_{lie} + a_2[y,y]_{lie} \\ &\quad + a_3[z,y]_{lie} \\ 0 &= b_3(x), \ \text{ implying that } b_3 = 0. \end{aligned}$$

$$\begin{aligned} \text{Also,}\ \ \alpha([x,z]_{lie}) &= [x, \alpha(z)]_{lie} + [\alpha(x), z]_{lie} \\ \alpha([x,z] + [z,x]) &= [x, c_1x + c_2y + c_3z]_{lie} + [a_1x + a_2y + a_3z, z]_{lie} \\ \alpha(0+x) &= c_1[x,x]_{lie} + c_2[x,y]_{lie} + c_3[x,z]_{lie} + a_1[x,z]_{lie} \\ &\quad + a_2[y,z]_{lie} + a_3[z,z]_{lie} \\ a_1x + a_2y + a_3z &= c_3(x) + a_1(x) \end{aligned}$$

which implies that $-c_3x+a_2y+a_3z=0$, and thus $c_3=0, a_2=0$ and $a_3=0$ since x, y, and z are linearly independent.

$$\begin{aligned}
\alpha([y,z]_{lie}) &= [y,\alpha(z)]_{lie}+[\alpha(y),z]_{lie}\\
\alpha([y,z]+[z,y]) &= [y,c_1x+c_2y+c_3z]_{lie}+[b_1x+b_2y+b_3z,z]_{lie}\\
\alpha(0+0) &= c_1[y,x]_{lie}+c_2[y,y]_{lie}+c_3[y,z]_{lie}+b_1[x,z]_{lie}+b_2[y,z]_{lie}\\
&\quad + b_3[z,z]_{lie}\\
0 &= b_1(x),\ \text{ implying that } b_1=0.
\end{aligned}$$

$$\begin{aligned}
\text{Also, }\ \alpha([z,z]_{lie}) &= 2[z,\alpha(z)]_{lie}\\
\alpha(2[z,z]) &= 2[z,c_1x+c_2y+c_3z]_{lie}\\
\alpha(0) &= 2c_1[z,x]_{lie}+2c_2[z,y]_{lie}+2c_3[z,z]_{lie}\\
0 &= 2c_1(x),\ \text{ implying that } c_1=0.
\end{aligned}$$

Note that the identity $\alpha([y,y]_{lie})=2[y,\alpha(y)]_{lie}$ yields $0=0$. In summary, $a_2=0, a_3=0, b_1=0, b_3=0, c_1=0$, and $c_3=0$.
Therefore,

$$\alpha=\begin{bmatrix} a_1 & 0 & 0\\ 0 & b_1 & c_2\\ 0 & 0 & 0\end{bmatrix}=a_1\underbrace{\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}}_{\alpha_1}+b_2\underbrace{\begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix}}_{\alpha_2}+c_2\underbrace{\begin{bmatrix}0&0&0\\0&0&1\\0&0&0\end{bmatrix}}_{\alpha_3}.$$

Now, since $\mathsf{Leib}(\mathfrak{g})=0$, if $\alpha\in\{\alpha_1,\alpha_2,\alpha_3\}$ is an inner derivation, then $\alpha=\mathrm{L}_z$ is a special inner derivation, in which case $\alpha(x)=x$ and $\alpha(y)=\alpha(z)=0$. In this case, α_2 is an outer derivation because $\alpha_2(x)=0\neq x$, and α_3 is an outer derivation because $\alpha_3(x)=0\neq x$. However, $\alpha_1(x)=x$, $\alpha_1(y)=0$, and $\alpha_1(z)=0$. Therefore, α_1 is an inner derivation. □

Proposition 3.8. *Let $\mathfrak{g}$ be the Leibniz algebra spanned by $\{x,y,z\}$, whose nonzero brackets are given by $[z,x]=kx$ where $k\in\mathbb{R}-\{0\}$, $[z,y]=y$, and $[y,z]=-y$. Then the set $\mathsf{Der}^{\mathsf{Lie}}(\mathfrak{g})$ of Lie-derivations of $\mathfrak{g}$ is a three-dimensional Lie algebra spanned by the set $\{\alpha_1,\alpha_2,\alpha_3\}$, where*

$$\alpha_1=\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix},\quad \alpha_2=\begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix},\quad \textit{and}\quad \alpha_3=\begin{bmatrix}0&0&0\\0&0&1\\0&0&0\end{bmatrix}.$$

Moreover, α_1,α_2 and α_3 are outer derivations.

Proof. Let $\alpha\in\mathsf{Der}^{\mathsf{Lie}}(\mathfrak{g})$ whose matrix M in the basis $\{x,y,z\}$ is given by

$$\mathrm{M}=\begin{bmatrix}a_1&b_1&c_1\\a_2&b_2&c_2\\a_3&b_3&c_3\end{bmatrix}.$$

This implies that $\alpha(x) = a_1x + a_2y + a_3z$, $\alpha(y) = b_1x + b_2y + b_3z$ and $\alpha(z) = c_1x + c_2y + c_3z$. Since $\alpha \in \mathrm{Der}^{\mathrm{Lie}}(\mathfrak{g})$, we must have $\alpha([u,v]_{lie}) = [u,\alpha(v)]_{lie} + [\alpha(u),v]_{lie}$ for $u, v \in \mathfrak{g}$. It follows that

$$\begin{aligned}
\alpha([x,x]_{lie}) &= 2[x,\alpha(x)]_{lie}\\
\alpha(2[x,x]) &= 2[x, a_1x + a_2y + a_3z]_{lie}\\
\alpha(0) &= 2a_1[x,x]_{lie} + 2a_2[x,y]_{lie} + 2a_3[x,z]_{lie}\\
0 &= 2a_3(kx), \text{ implying that } a_3 = 0 \text{ since } k \neq 0.
\end{aligned}$$

$$\begin{aligned}
\text{Also, } \alpha([x,y]_{lie}) &= [x,\alpha(y)]_{lie} + [\alpha(x),y]_{lie}\\
\alpha([x,y]+[y,x]) &= [x, b_1x + b_2y + b_3z]_{lie} + [a_1x + a_2y + a_3z, y]_{lie}\\
\alpha(0+0) &= b_1[x,x]_{lie} + b_2[x,y]_{lie} + b_3[x,z]_{lie} + a_1[x,y]_{lie} + a_2[y,y]_{lie}\\
&\quad + a_3[z,y]_{lie}\\
0 &= b_3(kx) + a_3(y-y)\\
0 &= b_3kx, \text{ implying that } b_3 = 0 \text{ since } k \neq 0.
\end{aligned}$$

$$\begin{aligned}
\text{Also, } \alpha([x,z]_{lie}) &= [x,\alpha(z)]_{lie} + [\alpha(x),z]_{lie}\\
\alpha([x,z]+[z,x]) &= [x, c_1x + c_2y + c_3z]_{lie} + [a_1x + a_2y + a_3z, z]_{lie}\\
\alpha(0+kx) &= c_1[x,x]_{lie} + c_2[x,y]_{lie} + c_3[x,z]_{lie} + a_1[x,z]_{lie} + a_2[y,z]_{lie}\\
&\quad + a_3[z,z]_{lie}\\
k(a_1x + a_2y + a_3z) &= c_3(kx) + a_1(kx) + a_2(y-y)\\
k(a_1x + a_2y + a_3z) &= k(c_3x + a_1x)\\
a_1x + a_2y + a_3z &= c_3x + a_1x
\end{aligned}$$

which implies that $a_2y + a_3z - c_3x = 0$, and thus $a_2 = 0, a_3 = 0$ and $c_3 = 0$ since x, y, and z are linearly independent.

$$\begin{aligned}
\text{Also, } \alpha([y,z]_{lie}) &= [y,\alpha(z)]_{lie} + [\alpha(y),z]_{lie}\\
\alpha([y,z]+[z,y]) &= [y, c_1x + c_2y + c_3z]_{lie} + [b_1x + b_2y + b_3z, z]_{lie}\\
\alpha(-y+y) &= c_1[y,x]_{lie} + c_2[y,y]_{lie} + c_3[y,z]_{lie} + b_1[x,z]_{lie} + b_2[y,z]_{lie}\\
&\quad + b_3[z,z]_{lie}\\
0 &= c_3(y-y) + b_1(kx) + b_2(y-y)\\
0 &= kb_1x, \text{ implying that } b_1 = 0 \text{ since } k \neq 0.
\end{aligned}$$

$$\begin{aligned}
\text{Finally, } \alpha([z,z]_{lie}) &= 2[z,\alpha(z)]_{lie}\\
\alpha(2[z,z]) &= 2[z, c_1x + c_2y + c_3z]_{lie}\\
\alpha(0) &= 2c_1[z,x]_{lie} + 2c_2[z,y]_{lie} + 2c_3[z,z]_{lie}\\
0 &= 2c_1(kx) + 2c_2(y-y)\\
0 &= 2kc_1x, \text{ implying that } c_1 = 0 \text{ since } k \neq 0.
\end{aligned}$$

Note that the identity $\alpha([y,y]_{lie}) = 2[y,\alpha(y)]_{lie}$ yields $0 = 0$. In summary, $a_2 = 0, a_3 = 0, b_1 = 0, b_3 = 0, c_1 = 0$, and $c_3 = 0$.
Therefore,

$$\alpha = \begin{bmatrix} a_1 & 0 & 0 \\ 0 & b_2 & c_2 \\ 0 & 0 & 0 \end{bmatrix} = a_1 \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\alpha_1} + b_2 \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\alpha_2} + c_2 \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}}_{\alpha_3}.$$

It is straightforward to show that the vectors $\{\alpha_1, \alpha_2, \alpha_3\}$ are linearly independent.

Now, since $\mathsf{Leib}(\mathfrak{g}) = 0$, if $\alpha \in \{\alpha_1, \alpha_2, \alpha_3\}$ is an inner derivation, then either $\alpha = \mathrm{L}_y$ or $\alpha = \mathrm{L}_z$, in which case $\alpha(x) = \alpha(y) = 0$ and $\alpha(z) = -y$ or $\alpha(x) = kx, k \neq 0$, $\alpha(y) = y$ and $\alpha(z) = 0$ respectively. In this case, α_1 is an outer derivation because $\alpha_1(x) = x \neq 0$ and $\alpha_1(y) = 0 \neq y$. Similarly, α_2 is an outer derivation because $\alpha_2(y) = y \neq 0$ and $\alpha_2(x) = 0 \neq kx$ since $k \neq 0$. Finally, α_3 is an outer derivation because $\alpha_3(z) = y$, which does not equal $-y$ or 0. □

Proposition 3.9. *Let $\mathfrak{g}$ be the Leibniz algebra spanned by $\{x, y, z\}$, whose nonzero brackets are given by $[z, y] = y$, $[y, z] = -y$, $[z, z] = x$. Then the set $\mathsf{Der}^{\mathsf{Lie}}(\mathfrak{g})$ of Lie-derivations of $\mathfrak{g}$ is a five-dimensional Lie algebra spanned by the set $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\}$, where*

$$\alpha_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \alpha_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \alpha_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \alpha_4 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \alpha_5 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Moreover, α_2 is an inner (non-special) derivation, and α_1, α_3, α_4 and α_5 are outer derivations.

Proof. Let $\alpha \in \mathsf{Der}^{\mathsf{Lie}}(\mathfrak{g})$ whose matrix M in the basis $\{x, y, z\}$ is given by

$$\alpha = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}.$$

This implies that $\alpha(x) = a_1x + a_2y + a_3z$, $\alpha(y) = b_1x + b_2y + b_3z$ and $\alpha(z) = c_1x + c_2y + c_3z$. Since $\alpha \in \mathsf{Der}^{\mathsf{Lie}}(\mathfrak{g})$, we must have $\alpha([u, v]_{lie}) = [u, \alpha(v)]_{lie} + [\alpha(u), v]_{lie}$ for $u, v \in \mathfrak{g}$. It follows that

$$\begin{aligned}
\text{Also,}\ \ \alpha([x,z]_{lie}) &= [x,\alpha(z)]_{lie} + [\alpha(x), z]_{lie} \\
\alpha([x,z]+[z,x]) &= [x, c_1x + c_2y + c_3z]_{lie} + [a_1x + a_2y + a_3z, z]_{lie} \\
\alpha(0+0) &= c_1[x,x]_{lie} + c_2[x,y]_{lie} + c_3[x,z]_{lie} + a_1[x,z]_{lie} + a_2[y,z]_{lie} \\
&\quad + a_3[z,z]_{lie} \\
0 &= a_2(y-y) + a_3(2x) \\
0 &= 2a_3x, \ \ \text{implying that } a_3 = 0.
\end{aligned}$$

$$\begin{aligned}
\text{Also, }\ \alpha([y,z]_{lie}) &= [y,\alpha(z)]_{lie} + [\alpha(y),z]_{lie}\\
\alpha([y,z]+[z,y]) &= [y, c_1x+c_2y+c_3z]_{lie} + [b_1x+b_2y+b_3z, z]_{lie}\\
\alpha(-y+y) &= c_1[y,x]_{lie} + c_2[y,y]_{lie} + c_3[y,z]_{lie} + b_1[x,z]_{lie} + b_2[y,z]_{lie}\\
&\quad + b_3[z,z]_{lie}\\
0 &= c_3(y-y) + b_2(y-y) + b_3(2x)\\
0 &= 2b_3x, \ \text{ implying that } b_3 = 0.
\end{aligned}$$

$$\begin{aligned}
\text{Finally, }\ \alpha([z,z]_{lie}) &= 2[z,\alpha(z)]_{lie}\\
\alpha(2[z,z]) &= 2[z, c_1x+c_2y+c_3z]_{lie}\\
\alpha(2x) &= 2c_1[z,x]_{lie} + 2c_2[z,y]_{lie} + 2c_3[z,z]_{lie}\\
2(a_1x+a_2y+a_3z) &= 4c_3x\\
a_1x+a_2y+a_3z &= 2c_3x
\end{aligned}$$

which implies $(a_1-2c_3)x + a_2y + a_3z = 0$, and thus $a_1 - 2c_3 = 0, a_2 = 0$, and $a_3 = 0$ since x, y, and z are linearly independent. Therefore $a_1 = 2c_3, a_2 = 0$, and $a_3 = 0$.
Note that the identities $\alpha([x,x]_{lie}) = 2[x,\alpha(x)]_{lie}$, $\alpha([x,y]_{lie}) = [x,\alpha(y)]_{lie} + [\alpha(x),y]_{lie}$ and $\alpha([y,y]_{lie}) = 2[y,\alpha(y)]_{lie}$ yield $0 = 0$. In summary, $a_1 = 2c_3, a_2 = 0, a_3 = 0$, and $b_3 = 0$. Therefore,

$$\alpha = \begin{bmatrix} 2c_3 & b_1 & c_1\\ 0 & b_2 & c_2\\ 0 & 0 & c_3 \end{bmatrix} = b_1 \underbrace{\begin{bmatrix} 0&1&0\\0&0&0\\0&0&0 \end{bmatrix}}_{\alpha_1} + b_2 \underbrace{\begin{bmatrix} 0&0&0\\0&1&0\\0&0&0 \end{bmatrix}}_{\alpha_2} + c_1 \underbrace{\begin{bmatrix} 0&0&1\\0&0&0\\0&0&0 \end{bmatrix}}_{\alpha_3}$$
$$+ c_2 \underbrace{\begin{bmatrix} 0&0&0\\0&0&1\\0&0&0 \end{bmatrix}}_{\alpha_4} + c_3 \underbrace{\begin{bmatrix} 2&0&0\\0&0&0\\0&0&1 \end{bmatrix}}_{\alpha_5}.$$

Now, since $\mathsf{Leib}(\mathfrak{g}) = \langle x\rangle$, if $\alpha \in \{\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5\}$ is an inner derivation, then either $(\alpha - \mathrm{L}_y)(\mathfrak{g}) \subseteq \langle x\rangle$ or $(\alpha - \mathrm{L}_z)(\mathfrak{g}) \subseteq \langle x\rangle$. Note that $\mathrm{L}_y(x) = \mathrm{L}_y(y) = 0$ and $\mathrm{L}_y(z) = -y$, and $\mathrm{L}_z(x) = 0$, $\mathrm{L}_z(y) = y$ and $\mathrm{L}_z(z) = x$. In this case, α_2 is an inner derivation because $(\alpha_2 - \mathrm{L}_z)(\mathfrak{g}) = \langle x\rangle$. α_1 is an outer derivation because $(\alpha_1 - \mathrm{L}_y)(z) = y \notin \langle x\rangle$ and $(\alpha_1 - \mathrm{L}_z)(y) = x - y \notin \langle x\rangle$. Similarly, α_3 is an outer derivation because$(\alpha_3 - \mathrm{L}_y)(z) = x + y \notin \langle x\rangle$ and $(\alpha_3 - \mathrm{L}_z)(y) = -y \notin \langle x\rangle$. Also, α_4 is an outer derivation because$(\alpha_4 - \mathrm{L}_y)(z) = 2y \notin \langle x\rangle$ and $(\alpha_4 - \mathrm{L}_z)(z) = -y \notin \langle x\rangle$. Finally, α_5 is an outer derivation because $(\alpha_5 - \mathrm{L}_y)(z) = z - y \notin \langle x\rangle$ and $(\alpha_5 - \mathrm{L}_z)(y) = z - x \notin \langle x\rangle$. □

Proposition 3.10. *Let $\mathfrak{g}$ be the Leibniz algebra spanned by $\{x,y,z\}$, whose nonzero brackets are given by $[z,x] = 2x$, $[y,y] = x$, $[z,y] = y$, $[y,z] = -y$, and $[z,z] = x$. Then the set $\mathsf{Der}^{\mathsf{Lie}}(\mathfrak{g})$ of Lie-derivations of $\mathfrak{g}$ is a two-dimensional Lie algebra spanned by the set*

$\{\alpha_1, \alpha_2\}$, *where*

$$\alpha_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \alpha_2 = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Moreover, α_1 *is an outer derivation and* α_2 *is a special inner derivation.*

Proof. Let $\alpha \in \mathsf{Der}^{\mathsf{Lie}}(\mathfrak{g})$ whose matrix M in the basis $\{x, y, z\}$ is given by

$$\alpha = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}.$$

This implies that $\alpha(x) = a_1x + a_2y + a_3z$, $\alpha(y) = b_1x + b_2y + b_3z$ and $\alpha(z) = c_1x + c_2y + c_3z$. Since $\alpha \in \mathsf{Der}^{\mathsf{Lie}}(\mathfrak{g})$, we must have $\alpha([u, v]_{lie}) = [u, \alpha(v)]_{lie} + [\alpha(u), v]_{lie}$ for $u, v \in \mathfrak{g}$. It follows that

$$\begin{aligned} \alpha([x,x]_{lie}) &= 2[x, \alpha(x)]_{lie} \\ \alpha(2[x,x]) &= 2[x, a_1x + a_2y + a_3z]_{lie} \\ \alpha(0) &= 2a_1[x,x]_{lie} + 2a_2[x,y]_{lie} + 2a_3[x,z]_{lie} \\ 0 &= 2a_3(2x) \\ 0 &= 4a_3x \text{ which implies } a_3 = 0. \end{aligned}$$

$$\begin{aligned} \text{Also, } \alpha([x,y]_{lie}) &= [x, \alpha(y)]_{lie} + [\alpha(x), y]_{lie} \\ \alpha([x,y] + [y,x]) &= [x, b_1x + b_2y + b_3z]_{lie} + [a_1x + a_2y + a_3z, y]_{lie} \\ \alpha(0+0) &= b_1[x,x]_{lie} + b_2[x,y]_{lie} + b_3[x,z]_{lie} + a_1[x,y]_{lie} + a_2[y,y]_{lie} \\ &\quad + a_3[z,y]_{lie} \\ 0 &= b_3(2x) + a_2(2x) + a_3(y - y) \\ 0 &= 2(b_3 + a_2)x, \text{ which implies } b_3 = -a_2. \end{aligned}$$

$$\begin{aligned} \text{Also, } \alpha([x,z]_{lie}) &= [x, \alpha(z)]_{lie} + [\alpha(x), z]_{lie} \\ \alpha([x,z] + [z,x]) &= [x, c_1x + c_2y + c_3z]_{lie} + [a_1x + a_2y + a_3z, z]_{lie} \\ \alpha(0 + 2x) &= c_1[x,x]_{lie} + c_2[x,y]_{lie} + c_3[x,z]_{lie} + a_1[x,z]_{lie} \\ &\quad + a_2[y,z]_{lie} + a_3[z,z]_{lie} \\ 2a_1x + 2a_2y + 2a_3z &= c_3(2x) + a_1(2x) + a_2(y - y) + a_3(2x) \\ 2a_1x + 2a_2y + 2a_3z &= 2c_3x + 2a_1x + 2a_3x \\ a_1x + a_2y + a_3z &= c_3x + a_1x + a_3x \end{aligned}$$

which implies $(-c_3 - a_3)x + a_2y + a_3z = 0$, and thus $c_3 + a_3 = 0$, $a_2 = 0$ and $a_3 = 0$ since x,

y, and z are linearly independent. Therefore $c_3 = 0$ since $a_3 = 0$.

$$\begin{aligned}
\text{Also, } \alpha([y,y]_{lie}) &= 2[y,\alpha(y)]_{lie}\\
\alpha(2[y,y]) &= 2[y, b_1x + b_2y + b_3z]_{lie}\\
\alpha(2x) &= 2b_1[y,x]_{lie} + 2b_2[y,y]_{lie} + 2b_3[y,z]_{lie}\\
2a_1x + 2a_2y + 2a_3z &= 2b_2(2x) + 2b_3(y-y)\\
a_1x + a_2y + a_3z &= 2b_2x
\end{aligned}$$

which implies $(a_1 - 2b_2)x + a_2y + a_3z = 0$, and thus $a_1 - 2b_2 = 0, a_2 = 0$, and $a_3 = 0$ since x, y, and z are linearly independent. Therefore $a_1 = 2b_2$.

$$\begin{aligned}
\text{Also, } \alpha([y,z]_{lie}) &= [y,\alpha(z)]_{lie} + [\alpha(y),z]_{lie}\\
\alpha([y,z]+[z,y]) &= [y, c_1x + c_2y + c_3z]_{lie} + [b_1x + b_2y + b_3z, z]_{lie}\\
\alpha(-y+y) &= c_1[y,x]_{lie} + c_2[y,y]_{lie} + c_3[y,z]_{lie} + b_1[x,z]_{lie} + b_2[y,z]_{lie}\\
&\quad + b_3[z,z]_{lie}\\
0 &= c_2(2x) + c_3(y-y) + b_1(2x) + b_2(y-y) + b_3(2x)\\
0 &= 2c_2x + 2b_1x + 2b_3x\\
0 &= (c_2 + b_1 + b_3)x.
\end{aligned}$$

This implies $c_2 - b_1 - b_3 = 0$, and thus $c_2 = b_1$ since $b_3 = 0$.

$$\begin{aligned}
\text{Finally, } \alpha([z,z]_{lie}) &= 2[z,\alpha(z)]_{lie}\\
\alpha(2[z,z]) &= 2[z, c_1x + c_2y + c_3z]_{lie}\\
\alpha(2x) &= 2c_1[z,x]_{lie} + 2c_2[z,y]_{lie} + 2c_3[z,z]_{lie}\\
2a_1x + 2a_2y + 2a_3z &= 2c_1(2x) + 2c_2(y-y) + 2c_3(2x)\\
a_1x + a_2y + a_3z &= 2c_1x + 2c_3x
\end{aligned}$$

which implies $(a_1 - 2c_1 - 2c_3)x + a_2y + a_3z = 0$, and thus $a_1 - 2c_1 - 2c_3 = 0, a_2 = 0$, and $a_3 = 0$. So, $a_1 = 2c_1$ since $c_3 = 0$. Thus, $c_1 = b_2$ since $a_1 = 2b_2$.
In summary, $a_1 = 2b_2, a_2 = 0, a_3 = 0, b_3 = 0, c_1 = b_2, c_2 = b_1$, and $c_3 = 0$.
Thus,

$$\alpha = \begin{bmatrix} 2b_2 & b_1 & b_2\\ 0 & b_2 & b_1\\ 0 & 0 & 0 \end{bmatrix} = b_1 \underbrace{\begin{bmatrix} 0 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0 \end{bmatrix}}_{\alpha_1} + b_2 \underbrace{\begin{bmatrix} 2 & 0 & 1\\ 0 & 1 & 0\\ 0 & 0 & 0 \end{bmatrix}}_{\alpha_2}.$$

Now, since $\mathsf{Leib}(\mathfrak{g}) = \langle x \rangle$, if $\alpha \in \{\alpha_1, \alpha_2\}$ is an inner derivation, then either $(\alpha - \mathrm{L}_y)(\mathfrak{g}) \subseteq \langle x \rangle$ or $(\alpha - \mathrm{L}_z)(\mathfrak{g}) \subseteq \langle x \rangle$. Note that $\mathrm{L}_y(x) = 0$, $\mathrm{L}_y(y) = x$ and $\mathrm{L}_y(z) = -y$, and $\mathrm{L}_z(x) = 2x$, $\mathrm{L}_z(y) = y$ and $\mathrm{L}_z(z) = x$. In this case, α_2 is a special inner derivation because $(\alpha_2 - \mathrm{L}_z)(\mathfrak{g}) = 0$. However, α_1 is an outer derivation because $(\alpha_1 - \mathrm{L}_y)(z) = 2y \notin \langle x \rangle$ and $(\alpha_1 - \mathrm{L}_z)(y) = x - y \notin \langle x \rangle$. □

Proposition 3.11. *Let $\mathfrak{g}$ be the Leibniz algebra spanned by $\{x, y, z\}$, whose nonzero brackets are given by $[z, y] = y$ and $[z, x] = kx$ where $k \in \mathbb{R} - \{0\}$ Then the set $\mathsf{Der}^{\mathsf{Lie}}(\mathfrak{g})$ of Lie-derivations of $\mathfrak{g}$ is a two-dimensional Lie algebra spanned by the set $\{\alpha_1, \alpha_2\}$, where*

$$\alpha_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \alpha_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \textit{if} \quad k \neq 1.$$

And the set $\mathsf{Der}^{\mathsf{Lie}}(\mathfrak{g})$ of Lie-derivations of $\mathfrak{g}$ is a four-dimensional Lie algebra spanned by the set $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$, where

$$\alpha_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \alpha_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \alpha_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \alpha_4 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \textit{if} \quad k = 1.$$

Moreover, α_1, α_2, α_3 and α_4 are outer derivations.

Proof. Let $\alpha \in \mathsf{Der}^{\mathsf{Lie}}(\mathfrak{g})$ whose matrix M in the basis $\{x, y, z\}$ is given by

$$\alpha = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}.$$

This implies that $\alpha(x) = a_1x + a_2y + a_3z$, $\alpha(y) = b_1x + b_2y + b_3z$ and $\alpha(z) = c_1x + c_2y + c_3z$. Since $\alpha \in \mathsf{Der}^{\mathsf{Lie}}(\mathfrak{g})$, we must have $\alpha([u, v]_{lie}) = [u, \alpha(v)]_{lie} + [\alpha(u), v]_{lie}$ for $u, v \in \mathfrak{g}$. It follows that

$$\begin{aligned} \alpha([x, x]_{lie}) &= 2[x, \alpha(x)]_{lie} \\ \alpha(2[x, x]) &= 2[x, a_1x + a_2y + a_3z]_{lie} \\ \alpha(0) &= 2a_1[x, x]_{lie} + 2a_2[x, y]_{lie} + 2a_3[x, z]_{lie} \\ 0 &= 2a_3(kx), \text{ which implies } \ a_3 = 0 \text{ since } k \neq 0. \end{aligned}$$

$$\begin{aligned} \text{Also, } \ \alpha([x, y]_{lie}) &= [x, \alpha(y)]_{lie} + [\alpha(x), y]_{lie} \\ \alpha([x, y] + [y, x]) &= [x, b_1x + b_2y + b_3z]_{lie} + [a_1x + a_2y + a_3z, y]_{lie} \\ \alpha(0 + 0) &= b_1[x, x]_{lie} + b_2[x, y]_{lie} + b_3[x, z]_{lie} + a_1[x, y]_{lie} + a_2[y, y]_{lie} \\ &\quad + a_3[z, y]_{lie} \\ 0 &= b_3(kx) + a_3(y), \text{ which implies } \ a_3 = b_3 = 0 \text{ since } k \neq 0. \end{aligned}$$

$$\begin{aligned} \text{Also, } \ \alpha([x, z]_{lie}) &= [x, \alpha(z)]_{lie} + [\alpha(x), z]_{lie} \\ \alpha([x, z] + [z, x]) &= [x, c_1x + c_2y + c_3z]_{lie} + [a_1x + a_2y + a_3z, z]_{lie} \\ \alpha(0 + kx) &= c_1[x, x]_{lie} + c_2[x, y]_{lie} + c_3[x, z]_{lie} + a_1[x, z]_{lie} \\ &\quad + a_2[y, z]_{lie} + a_3[z, z]_{lie} \\ ka_1x + ka_2y + ka_3z &= c_3(kx) + a_1(kx) + a_2(y). \end{aligned}$$

This implies that $-kc_3x + a_2(k-1)y + ka_3z = 0$, and thus $c_3 = 0$ and $a_2 = 0$ if $k \neq 1$, and $a_3 = 0$ since x, y, and z are linearly independent.

$$\begin{aligned}
\text{Also, } \alpha([y,y]_{lie}) &= 2[y,\alpha(y)]_{lie}\\
\alpha(2[y,y]) &= 2[x, b_1x + b_2y + b_3z]_{lie}\\
\alpha(0) &= 2b_1[y,x]_{lie} + 2b_2[y,y]_{lie} + 2b_3[y,z]_{lie}\\
0 &= 2b_3(y), \text{ which implies } b_3 = 0.
\end{aligned}$$

$$\begin{aligned}
\text{Also, } \alpha([y,z]_{lie}) &= [y,\alpha(z)]_{lie} + [\alpha(y),z]_{lie}\\
\alpha([y,z]+[z,y]) &= [y, c_1x + c_2y + c_3z]_{lie} + [b_1x + b_2y + b_3z, z]_{lie}\\
\alpha(0+y) &= c_1[y,x]_{lie} + c_2[y,y]_{lie} + c_3[y,z]_{lie} + b_1[x,z]_{lie} + b_2[y,z]_{lie}\\
&\quad + b_3[z,z]_{lie}\\
b_1x + b_2y + b_3z &= c_3(y) + b_1(kx) + b_2(y).
\end{aligned}$$

This implies that $b_1(1-k)x - c_3y - b_3z = 0$, and thus $b_1 = 0$ if $k \neq 1$, $c_3 = 0$, and $b_3 = 0$ since x, y, and z are linearly independent.

$$\begin{aligned}
\text{Finally, } \alpha([z,z]_{lie}) &= 2[z,\alpha(z)]_{lie}\\
\alpha(2[z,z]) &= 2[z, c_1x + c_2y + c_3z]_{lie}\\
\alpha(0) &= 2c_1[z,x]_{lie} + 2c_2[z,y]_{lie} + 2c_3[z,z]_{lie}\\
0 &= 2c_1(kx) + 2c_2(y).
\end{aligned}$$

This implies $c_1 = 0$ and $c_2 = 0$ since $k \neq 0$.

In summary, $a_2 = 0$ if $k \neq 1$, $a_3 = 0$, $b_1 = 0$ if $k \neq 1$, $b_3 = 0, c_1 = 0, c_2 = 0$, and $c_3 = 0$. Thus, if $k \neq 1$, then

$$\alpha = \begin{bmatrix} a_1 & 0 & 0\\ 0 & b_2 & 0\\ 0 & 0 & 0 \end{bmatrix} = a_1 \underbrace{\begin{bmatrix} 1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0 \end{bmatrix}}_{\alpha_1} + b_2 \underbrace{\begin{bmatrix} 0 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0 \end{bmatrix}}_{\alpha_2}.$$

However, if $k = 1$, then

$$\begin{aligned}
\alpha = \begin{bmatrix} a_1 & b_1 & 0\\ a_2 & b_2 & 0\\ 0 & 0 & 0 \end{bmatrix} &= a_1 \underbrace{\begin{bmatrix} 1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0 \end{bmatrix}}_{\alpha_1} + b_2 \underbrace{\begin{bmatrix} 0 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0 \end{bmatrix}}_{\alpha_2} + b_1 \underbrace{\begin{bmatrix} 0 & 1 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0 \end{bmatrix}}_{\alpha_3}\\
&\quad + a_2 \underbrace{\begin{bmatrix} 0 & 0 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0 \end{bmatrix}}_{\alpha_4}.
\end{aligned}$$

Notice that α_1 and α_2 are basis elements in both cases. Also, note that $\mathsf{Leib}(\mathfrak{g}) = 0$. So, if $\alpha \in \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ is an inner derivation, then $\alpha = \mathrm{L}_z$ is a special inner derivation, in which case $\alpha(x) = kx$ where $k \neq 0$, $\alpha(y) = y$, and $\alpha(z) = 0$. In this case, α_1 is an outer derivation because $\alpha_1(y) = 0 \neq y$ and α_2 is an outer derivation because $\alpha_2(x) = 0 \neq kx$ since $k \neq 0$. Similarly, α_3 is an outer derivation because $\alpha_3(x) = 0 \neq kx$ since $k \neq 0$ and α_4 is an outer derivation because $\alpha_4(x) = y \neq kx$. □

Proposition 3.12. *Let $\mathfrak{g}$ be the Leibniz algebra spanned by $\{x, y, z\}$, whose nonzero brackets are given by $[z, x] = x + y$ and $[z, y] = y$. Then the set $\mathsf{Der}^{\mathsf{Lie}}(\mathfrak{g})$ of Lie-derivations of $\mathfrak{g}$ is a two-dimensional Lie algebra spanned by the set $\{\alpha_1, \alpha_2\}$, where*

$$\alpha_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \alpha_2 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Moreover, α_1 and α_2 are outer derivations.

Proof. Let $\alpha \in \mathsf{Der}^{\mathsf{Lie}}(\mathfrak{g})$ whose matrix M in the basis $\{x, y, z\}$ is given by

$$\alpha = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}.$$

This implies that $\alpha(x) = a_1x + a_2y + a_3z$, $\alpha(y) = b_1x + b_2y + b_3z$ and $\alpha(z) = c_1x + c_2y + c_3z$. Since $\alpha \in \mathsf{Der}^{\mathsf{Lie}}(\mathfrak{g})$, we must have $\alpha([u, v]_{lie}) = [u, \alpha(v)]_{lie} + [\alpha(u), v]_{lie}$ for $u, v \in \mathfrak{g}$. It follows that

$$\begin{aligned}
\alpha([x,x]_{lie}) &= 2[x, \alpha(x)]_{lie} \\
\alpha(2[x,x]) &= 2[x, a_1x + a_2y + a_3z]_{lie} \\
\alpha(0) &= 2a_1[x,x]_{lie} + 2a_2[x,y]_{lie} + 2a_3[x,z]_{lie} \\
0 &= 2a_3(x+y) \\
0 &= 2a_3x + 2a_3y, \text{ which implies } \ a_3 = 0.
\end{aligned}$$

$$\begin{aligned}
\text{Also, } \ \alpha([x,y]_{lie}) &= [x, \alpha(y)]_{lie} + [\alpha(x), y]_{lie} \\
\alpha([x,y] + [y,x]) &= [x, b_1x + b_2y + b_3z]_{lie} + [a_1x + a_2y + a_3z, y]_{lie} \\
\alpha(0+0) &= b_1[x,x]_{lie} + b_2[x,y]_{lie} + b_3[x,z]_{lie} + a_1[x,y]_{lie} + a_2[y,y]_{lie} \\
&\quad + a_3[z,y]_{lie} \\
0 &= b_3(x+y) + a_3(y) \\
0 &= b_3x + (b_3 + a_3)y, \text{ which implies } \ a_3 = b_3 = 0.
\end{aligned}$$

$$\begin{aligned}
\text{Also, } \ \alpha([x,z]_{lie}) &= [x, \alpha(z)]_{lie} + [\alpha(x), z]_{lie} \\
\alpha([x,z] + [z,x]) &= [x, c_1x + c_2y + c_3z]_{lie} + [a_1x + a_2y + a_3z, z]_{lie} \\
\alpha(0 + x + y) &= c_1[x,x]_{lie} + c_2[x,y]_{lie} + c_3[x,z]_{lie} \\
&\quad + a_1[x,z]_{lie} + a_2[y,z]_{lie} + a_3[z,z]_{lie}.
\end{aligned}$$

This yields

$$(a_1+b_1)x+(a_2+b_2)y+(a_3+b_3)z = c_3(x+y)+a_1(x+y)+a_2(y)$$

i.e. $(b_1-c_3)x+(b_2-c_3-a_1)y+(a_3+b_3)z=0$. So $b_1-c_3=0$, $b_2-c_3-a_1=0$, and $a_3+b_3=0$ since x, y, and z are linearly independent. Thus $b_1=c_3$ and $b_2=c_3+a_1$.

$$\begin{aligned}\text{Also, }\ \alpha([y,y]_{lie}) &= 2[y,\alpha(y)]_{lie}\\ \alpha(2[y,y]) &= 2[y,b_1x+b_2y+b_3z]_{lie}\\ \alpha(0) &= 2b_1[y,x]_{lie}+2b_2[y,y]_{lie}+2b_3[y,z]_{lie}\\ 0 &= 2b_3(y),\ \text{which implies }\ b_3=0.\end{aligned}$$

$$\begin{aligned}\text{Also, }\ \alpha([y,z]_{lie}) &= [y,\alpha(z)]_{lie}+[\alpha(y),z]_{lie}\\ \alpha([y,z]+[z,y]) &= [y,c_1x+c_2y+c_3z]_{lie}+[b_1x+b_2y+b_3z,z]_{lie}\\ \alpha(0+y) &= c_1[y,x]_{lie}+c_2[y,y]_{lie}+c_3[y,z]_{lie}+b_1[x,z]_{lie}+b_2[y,z]_{lie}\\ &\quad +b_3[z,z]_{lie}\\ b_1x+b_2y+b_3z &= c_3(y)+b_1(x+y)+b_2(y)\\ b_3z-c_3y-b_1y &= 0\\ (-c_3-b_1)y+b_3z &= 0,\ \text{which implies }\ c_3=-b_1 \text{ and } b_3=0.\end{aligned}$$

$$\begin{aligned}\text{Finally, }\ \alpha([z,z]_{lie}) &= 2[z,\alpha(z)]_{lie}\\ \alpha(2[z,z]) &= 2[z,c_1x+c_2y+c_3z]_{lie}\\ \alpha(0) &= 2c_1[z,x]_{lie}+2c_2[z,y]_{lie}+2c_3[z,z]_{lie}\\ 0 &= 2c_1(x+y)+2c_2(y)\\ 0 &= c_1x+(c_1+c_2)y,\ \text{which implies }\ c_1=c_2=0.\end{aligned}$$

In summary,
$a_3=0, b_1=c_3, b_2=c_3+a_1, b_3=0, c_1=0, c_2=0$, and $c_3=-b_1$, and thus $b_1=0, c_3=0$, and $b_2=a_1$. Thus,

$$\alpha = \begin{bmatrix} a_1 & 0 & 0\\ a_2 & a_1 & 0\\ 0 & 0 & 0\end{bmatrix} = a_1\underbrace{\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}}_{\alpha_1} + a_2\underbrace{\begin{bmatrix}0&0&0\\1&0&0\\0&0&0\end{bmatrix}}_{\alpha_2}.$$

Now, since $\mathsf{Leib}(\mathfrak{g})=0$, if $\alpha\in\{\alpha_1,\alpha_2\}$ is an inner derivation, then $\alpha=\mathrm{L}_z$ is a special inner derivation, in which case $\alpha(x)=x+y$, $\alpha(y)=y$, and $\alpha(z)=0$. In this case, α_1 is an outer derivation because $\alpha_1(x)=x\neq x+y$ and α_2 is an outer derivation because $\alpha_2(x)=y\neq x+y$. $\square$

Proposition 3.13. *Let $\mathfrak{g}$ be the Leibniz algebra spanned by $\{x, y, z\}$, whose nonzero brackets are given by $[z, x] = y$, $[z, y] = y$, and $[z, z] = x$. Then the set $\mathsf{Der}^{\mathsf{Lie}}(\mathfrak{g})$ of Lie-derivations of $\mathfrak{g}$ is a two-dimensional Lie algebra spanned by the set $\{\alpha_1, \alpha_2\}$, where*

$$\alpha_1 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \alpha_2 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Moreover, α_1 is a special inner derivation and α_2 is an outer derivation.

Proof. Let $\alpha \in \mathsf{Der}^{\mathsf{Lie}}(\mathfrak{g})$ whose matrix M in the basis $\{x, y, z\}$ is given by

$$\alpha = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}.$$

This implies that $\alpha(x) = a_1x + a_2y + a_3z$, $\alpha(y) = b_1x + b_2y + b_3z$ and $\alpha(z) = c_1x + c_2y + c_3z$. Since $\alpha \in \mathsf{Der}^{\mathsf{Lie}}(\mathfrak{g})$, we must have $\alpha([u, v]_{lie}) = [u, \alpha(v)]_{lie} + [\alpha(u), v]_{lie}$ for $u, v \in \mathfrak{g}$. It follows that

$$\begin{aligned} \alpha([x, x]_{lie}) &= 2[x, \alpha(x)]_{lie} \\ \alpha(2[x, x]) &= 2[x, a_1x + a_2y + a_3z]_{lie} \\ \alpha(0) &= 2a_1[x, x]_{lie} + 2a_2[x, y]_{lie} + 2a_3[x, z]_{lie} \\ 0 &= 2a_3(y), \text{ which implies } \ a_3 = 0. \end{aligned}$$

$$\begin{aligned} \text{Also,}\ \ \alpha([x, y]_{lie}) &= [x, \alpha(y)]_{lie} + [\alpha(x), y]_{lie} \\ \alpha([x, y] + [y, x]) &= [x, b_1x + b_2y + b_3z]_{lie} + [a_1x + a_2y + a_3z, y]_{lie} \\ \alpha(0 + 0) &= b_1[x, x]_{lie} + b_2[x, y]_{lie} + b_3[x, z]_{lie} + a_1[x, y]_{lie} + a_2[y, y]_{lie} \\ &\quad + a_3[z, y]_{lie} \\ 0 &= b_3(y) + a_3(y) = (b_3 + a_3)y. \end{aligned}$$

This implies $b_3 + a_3 = 0$, which implies $b_3 = 0$ since $a_3 = 0$.

$$\begin{aligned} \text{Also,}\ \ \alpha([x, z]_{lie}) &= [x, \alpha(z)]_{lie} + [\alpha(x), z]_{lie} \\ \alpha([x, z] + [z, x]) &= [x, c_1x + c_2y + c_3z]_{lie} + [a_1x + a_2y + a_3z, z]_{lie} \\ \alpha(0 + y) &= c_1[x, x]_{lie} + c_2[x, y]_{lie} + c_3[x, z]_{lie} + a_1[x, z]_{lie} \\ &\quad + a_2[y, z]_{lie} + a_3[z, z]_{lie} \\ b_1x + b_2y + b_3z &= c_3(y) + a_1(y) + a_2(y) + a_3(x). \end{aligned}$$

This implies $(b_1 - a_3)x + (b_2 - c_3 - a_1 - a_2)y + b_3z = 0$, and thus $b_1 - a_3 = 0$, $b_2 - c_3 - a_1 - a_2 = 0$, and $b_3 = 0$ since x, y, and z are linearly independent. Therefore, $b_1 = 0$ since $a_3 = 0$,

$b_2 = c_3 + a_1 + a_2$, and $b_3 = 0$.

$$\begin{aligned} \text{Also, }\ \alpha([y,y]_{lie}) &= 2[y,\alpha(y)]_{lie} \\ \alpha(2[y,y]) &= 2[y, b_1x + b_2y + b_3z]_{lie} \\ \alpha(0) &= 2b_1[y,x]_{lie} + 2b_2[y,y]_{lie} + 2b_3[y,z]_{lie} \\ 0 &= 2b_3(y),\ \text{which implies }\ b_3 = 0. \end{aligned}$$

$$\begin{aligned} \text{Also, }\ \alpha([y,z]_{lie}) &= [y,\alpha(z)]_{lie} + [\alpha(y),z]_{lie} \\ \alpha([y,z]+[z,y]) &= [y, c_1x + c_2y + c_3z]_{lie} + [b_1x + b_2y + b_3z, z]_{lie} \\ \alpha(0+y) &= c_1[y,x]_{lie} + c_2[y,y]_{lie} + c_3[y,z]_{lie} + b_1[x,z]_{lie} + b_2[y,z]_{lie} \\ &\quad + b_3[z,z]_{lie} \\ b_1x + b_2y + b_3z &= c_3(y) + b_1(y) + b_2(y) + b_3(x) \end{aligned}$$

This implies that $(b_1 - b_3)x + (-c_3 - b_1)y + b_3z = 0$, and thus $b_1 - b_3 = 0$, $c_3 + b_1 = 0$, and $b_3 = 0$. Therefore $c_3 = 0$ since $b_1 = 0$.

$$\begin{aligned} \text{Finally, }\ \alpha([z,z]_{lie}) &= 2[z,\alpha(z)]_{lie} \\ \alpha(2[z,z]) &= 2[z, c_1x + c_2y + c_3z]_{lie} \\ \alpha(2x) &= 2c_1[z,x]_{lie} + 2c_2[z,y]_{lie} + 2c_3[z,z]_{lie} \\ 2a_1x + 2a_2y + 2a_3z &= 2c_1(y) + 2c_2(y) + 2c_3(2x) \\ a_1x + a_2y + a_3z &= c_1y + c_2y + 2c_3x \end{aligned}$$

This implies that $(a_1-2c_3)x+(a_2-c_1-c_2)y+a_3z = 0$, and thus $a_1-2c_3 = 0$, $a_2-c_1-c_2 = 0$, and $a_3 = 0$ since x, y, and z are linearly independent. Therefore $a_1 = 0$ since $c_3 = 0$, and $a_2 = c_1 + c_2$.

In summary, $a_1 = 0, a_2 = c_1 + c_2, a_3 = 0, b_1 = 0, b_2 = c_3 + a_1 + a_2 = c_1 + c_2, b_3 = 0$, and $c_3 = 0$. Thus,

$$\alpha = \begin{bmatrix} 0 & 0 & c_1 \\ c_1+c_2 & c_1+c_2 & c_2 \\ 0 & 0 & 0 \end{bmatrix} = c_1 \underbrace{\begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\alpha_1} + c_2 \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}}_{\alpha_2}.$$

Now, since $\mathsf{Leib}(\mathfrak{g}) = \langle x \rangle$, if $\alpha \in \{\alpha_1, \alpha_2, \alpha_3\}$ is an inner derivation, then $(\alpha - \mathrm{L}_z)(\mathfrak{g}) \subseteq \langle x \rangle$. Note that $\mathrm{L}_z(x) = y$, $\mathrm{L}_z(y) = y$, and $\mathrm{L}_z(z) = x$. In this case, α_1 is a special inner derivation because $(\alpha_1 - \mathrm{L}_z)(\mathfrak{g}) = 0$ and α_2 is an outer derivation because $(\alpha_2 - \mathrm{L}_z)(z) = y - x \notin \langle x \rangle$. $\square$

4 Conclusion

In this paper, we explicitly determine a basis for the Lie algebra $\mathsf{Der}^{\mathsf{Lie}}(\mathfrak{g})$ of every three-dimensional non-Lie Leibniz algebra. Recall from [5], the following two-sided ideal of

$\mathfrak{g}: [\mathfrak{g},\mathfrak{g}]_{\mathsf{Lie}} = \langle\{[x,y]_{lie}, x\in\mathfrak{g}, y\in\mathfrak{g}\}\rangle$. On one hand, one can easily verify that the Leibniz algebras 2), 4) and 8) of **Theorem 3.1** are the only ones in the classification satisfying the condition $\mathsf{Leib}(\mathfrak{g}) = [\mathfrak{g},\mathfrak{g}]_{\mathsf{Lie}}$, and we obtained in **Proposition 3.3**, **Proposition 3.5** and **Proposition 3.9** that the bases of their respective Lie algebras $\mathsf{Der}^{\mathsf{Lie}}(\mathfrak{g})$ of Lie-derivations have each an inner derivation that is not special. On the other hand, one verifies that all the other Leibniz algebras in the classification satisfy the condition $\mathsf{Leib}(\mathfrak{g}) \neq [\mathfrak{g},\mathfrak{g}]_{\mathsf{Lie}}$, and we obtained in **Proposition 3.2**, **Proposition 3.4**, **Proposition 3.6**, **Proposition 3.7**, **Proposition 3.10**, **Proposition 3.11**, **Proposition 3.12** and **Proposition 3.13**, that each basis element of the respective Lie algebras $\mathsf{Der}^{\mathsf{Lie}}(\mathfrak{g})$ of Lie-derivations that is not an outer derivation is a special inner derivation. Consequently we state the following conjectures:

Conjecture 1. Let $\mathfrak{g}$ be a solvable non-Lie Leibniz algebra satisfying the condition $\mathsf{Leib}(\mathfrak{g}) = [\mathfrak{g},\mathfrak{g}]_{\mathsf{Lie}}$. The basis of the Lie algebra $\mathsf{Der}^{\mathsf{Lie}}(\mathfrak{g})$ of Lie-derivations of $\mathfrak{g}$ admits a non-special inner derivation.

Conjecture 2. Let $\mathfrak{g}$ be a solvable non-Lie Leibniz algebra satisfying the condition $\mathsf{Leib}(\mathfrak{g}) \neq [\mathfrak{g},\mathfrak{g}]_{\mathsf{Lie}}$. The basis of the Lie algebra $\mathsf{Der}^{\mathsf{Lie}}(\mathfrak{g})$ of Lie-derivations of $\mathfrak{g}$ admits no non-special inner derivation.

Acknowledgements

The author would like to thank Dr. Guy Biyogmam for his patience and support throughout the process of this undergraduate research project.

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