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An Introduction to Fractal Analysis

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Cover Page Footnote

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An Introduction to Fractal Analysis

By *Lucas Yong*

Abstract. Classical analysis is not able to treat functions whose domain is fractal. We present an introduction to analysis on a particular class of fractals known as post-critically finite (PCF) self-similar sets that is suitable for the undergraduate reader. We develop discrete approximations of PCF self-similar sets, and construct discrete Dirichlet forms and corresponding discrete Laplacians that both preserve self-similarity and are compatible with a notion of harmonic functions that is analogous to a classical setting. By taking the limit of these discrete Laplacians, we construct continuous Laplacians on PCF self-similar sets. With respect to this continuous Laplacian, we also construct a Green's function that can be used to find solutions to the Dirichlet problem for Poisson's equation.

1 Motivation

The study of differential equations, which is concerned with the relationship between functions and their derivatives, is often motivated by its applications to physical phenomena such as vibration, flow, and heat distribution. As an example of a classical differential operator, let $\Omega \subseteq \mathbb{R}^d$, and let $u\colon \Omega \to \mathbb{R}$ be twice-differentiable. The *Laplacian* of u is defined by

$$(\Delta u)(x) := \sum_{i=1}^{d} \frac{\partial^2 u}{\partial x_i^2},$$

where the summands are the second-order partial derivatives of u. The values of these partial derivatives at $x \in \Omega$ depend on the behaviour of u near that point, so x must be an *interior point* for u to be differentiable at x. More precisely, u is differentiable at x if for some $r > 0$, the open ball

$$\mathrm{B}_r(x) := \left\{ y \in \Omega \;\middle|\; \|x - y\| < r \right\}$$

is contained in Ω. Functions that are differentiable (at least once) over all of Ω are sometimes called *smooth* functions.

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Since differential equations are associated with physical phenomena, it is reasonable to hope that such phenomena can studied on functions defined on "real-world domains" such as trees, mountains, and rivers. However, such domains do not contain interior points in the sense of open balls described above. We encourage the reader to contemplate this for a moment: observe a nearby tree or plant, and notice that they are characterized by a certain "roughness". They do not contain "interior points", and as such there cannot be any smooth functions defined on them. Since the classical notion of the derivative is not suitable for studying the dynamical properties of objects found in nature, an analysis of these objects would require a theory of differential operators that do not depend on the (classical) derivative.

Developing an analysis on *fractals* is a first step in this direction. In [3], Benoit Mandelbrot characterizes fractals as subsets of metric spaces whose Hausdorff dimension is noninteger. The Hausdorff dimension is, informally, a measure of roughness: "smooth" objects, like cubes and spheres, have integer Hausdorff dimension (the same value as their topological dimension), while fractals do not. The curious reader may refer to [1, Chapter 6] for more background on the Hausdorff dimension. Mandelbrot argues that the geometry of objects in nature possess a "fractal face". For example, he interprets coastlines as approximate fractal curves. If the study of fractal geometry is linked to the static properties of nature in this manner, then fractal analysis is analogously linked to its dynamical aspects.

This paper presents an introduction to analysis on a particular class of fractals known as *post critically finite* (PCF) *self-similar sets*. Informally, self-similar sets are fractals whose parts are comprised of smaller versions of the whole. The prefix "PCF" roughly means that the set has a boundary consisting of a finite number of points. A ubiquitous example (see [2, 4]) of a PCF self-similar set is the *Sierpiński gasket*, or SG for short. Referring to Figure 1, notice that if we appropriately zoom in to some portion of SG, we see a smaller version of its entirety.

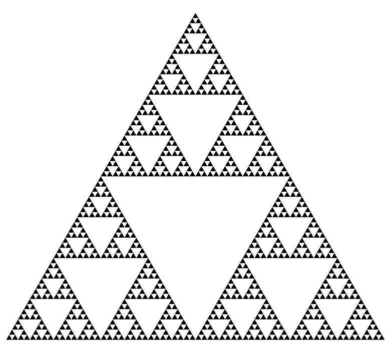

Figure 1: The Sierpiński gasket.

For $i \in \{1,2,3\}$, let $p_i \in \mathbb{C}$ be the vertices of an equilateral triangle, and define $f_i : \mathbb{C} \to \mathbb{C}$

by $f_i(z) = \frac{z+p_i}{2}$. Then $\mathrm{SG} \subseteq \mathbb{C}$ is the unique nonempty compact subset satisfying

$$\bigcup_{i=1}^{3} f_i(\mathrm{SG}) = \mathrm{SG}.$$

It is not trivial that such a nonempty compact subset exists, or that it is unique – this will be made rigorous (for self-similar sets in general) in the next section.

Since SG does not have interior points (with respect to the Euclidean metric), functions defined on SG are not differentiable in the usual way, so we are not able to define classical differential operators like the Laplacian on functions $u\colon \mathrm{SG} \to \mathbb{R}$. In this paper we will define, for general PCF self-similar sets K, a Laplacian Δ_μ with respect to a measure μ on functions on K that does not rely on the classical notion of the derivative. Broadly, the idea is to define finite approximations V_m of K, along with associated discrete Laplacians H_m. With respect to a measure μ, a Laplacian on K is the limit of the discrete operators H_m.

For concreteness, we briefly explain this process for the Sierpiński gasket SG. Let

$$V_0 = \{p_1, p_2, p_3\},$$

and recursively define the mth approximation of SG to be

$$V_m = \bigcup_{i=1}^{3} f_i(V_{m-1})$$

for $m \in \mathbb{N}$. See Figure 2 for a visualization of these approximations.

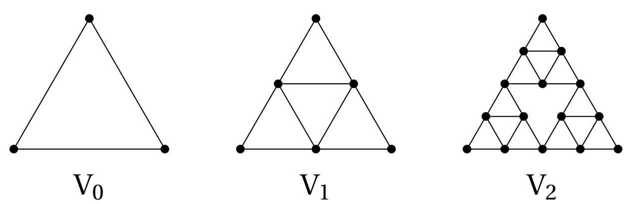

Figure 2: Approximations of SG for $m = 0, 1, 2$.

In Figure 2, points $p, q \in V_m$ that are joined by an edge satisfy an equivalence relation, $p \underset{m}{\sim} q$. The relation $\underset{m}{\sim}$, which is known as the *neighbor* relation, will be made precise in Section 3, but for now the reader may think of it as a description of "closeness" between points in the discrete approximations of SG. Next, define $H_m\colon \ell(V_m) \to \ell(V_m)$ by

$$(H_m u)(p) = \left(\frac{5}{3}\right)^m \sum_{p \underset{m}{\sim} q} \big(u(q) - u(p)\big),$$

where $\ell(V_m)$ is the set of all real-valued functions on V_m. It is not obvious why this definition includes a factor of $\left(\frac{5}{3}\right)^m$ on the right-hand side – this is known as a *renormalization factor,* and its neccesity will be explained in Section 3. We may then define a Laplacian Δ_μ on functions $u\colon \mathrm{SG} \to \mathbb{R}$ by taking the limit,

$$(\Delta_\mu u)(p) = \lim_{m\to\infty} \frac{3}{2} 5^m \sum_{p \underset{m}{\sim} q} \big(u(q) - u(p)\big).$$

In the literature, Δ_μ written above is known as the *standard Laplacian* on the SG. Note that it is not the only Laplacian that can be defined on SG. As we will see in the proceeding sections, the definition of Laplacians Δ_μ of this kind on general self-similar sets K depends on the underlying sequence of discrete Laplacians H_m, as well as the choice of measure μ on K. In Section 4, we will show that Δ_μ is deserving of the name "Laplacian" by giving an example of the unit interval $[0,1]$ defined as a self-similar set, and that for a suitable choice of discrete Laplacians and measure μ, Δ_μ on $[0,1]$ coincides with the classical Laplacian on $[0,1]$.

The structure of the paper is as follows. In Section 2, we prove the existence and uniqueness of a self-similar set K with respect to contraction mappings $f_1, \dots, f_N$, for some $N \in \mathbb{N}$. We also develop the notion of a boundary V_0 on K, and define a sequence V_m of discrete approximations of K. In Section 3, we develop a Dirichlet form for functions $V_m \to \mathbb{R}$. From this Dirichlet form we construct a discrete Laplacian H_m on V_m. In Section 4, we show that for certain functions one can take the limit of the discrete Laplacians H_m to define a continuous Laplacian Δ_μ on functions $K \to \mathbb{R}$. Finally, we consider the Dirichlet problem for Poisson's equation for functions $K \to \mathbb{R}$ in Section 5, and construct a Green's function that can provide a solution.

2 Self-Similar Structures and Shift Space

Throughout this section, let (M, d) be a complete metric space, and let $S := \{1, \dots, N\}$. We will prove the existence and uniqueness of self-similar sets K, which are defined with respect to contractions $f_i\colon M \to M$ for $i \in S$. We also define discrete approximations V_m of K for $m \in \mathbb{N}$, recursively defined via the *boundary* V_0 of K. We then give a precise definition of *post critically finite* (PCF) self-similar sets, which are, roughly speaking, self-similar sets whose boundary consists of a finite number of points. The exposition in this section is based on [2, Chapter 1].

Definition 2.1. A function $f\colon M \to M$ is **Lipschitz continuous** if

$$\mathrm{Lip}(f) := \sup_{x,y\in M, x\neq y} \frac{d(f(x), f(y))}{d(x,y)} < \infty.$$

If $\mathrm{Lip}(f) < 1$, f is called a **contraction** and $\mathrm{Lip}(f)$ is called its contraction ratio.

Given contractions $\{f_i\}_{i\in S}$, our first goal will be to show that there exists a unique nonempty compact set $K \subseteq M$ satisfying

$$F(K) := \bigcup_{i\in S} f_i(K) = K.$$

Theorem 2.2 (Banach–Caccioppoli)**.** *Let* (M, d) *be a complete metric space, and let* $f: M \to M$ *be a contraction.*

(i) There exists a unique $x_0 \in M$ *such that* $f(x_0) = x_0$*,*

(ii) If x_0 *is the fixed point of* f*, then the sequence* $\{f^n(x)\}$ *converges to* x_0 *for all* $x \in M$*, where* f^n *is the* n*th iteration of* f*.*

The proof of the above "fixed point theorem" is fun, and is left as an exercise to the reader.

Definition 2.3. Let $X, Y \subseteq M$ be compact. Define

$$\delta(X, Y) := \inf\left\{r > 0 \;\middle|\; U_r(X) \supseteq Y \text{ and } U_r(Y) \supseteq X\right\},$$

where

$$U_r(X) := \left\{x \in M \;\middle|\; d(x, y) \leq r \text{ for some } y \in X\right\} = \cup_{y\in X}\overline{B_r}(y).$$

The map δ is known as the **Hausdorff metric**.

Let $\mathscr{C}(M) := \left\{X \subseteq M \;\middle|\; X \text{ is nonempty and compact}\right\}$. We do not belabor the details here, but invite the reader to check that δ is a metric on $\mathscr{C}(M)$. To show the existence of K, we will show that $(\mathscr{C}(M), \delta)$ is a complete metric space, then apply **Theorem** 2.2 by viewing $F = \cup_{i\in S} f_i$ as a function on $\mathscr{C}(M)$.

Definition 2.4. Let $X \subseteq M$.

(i) A finite set $A \subseteq X$ is called a **finite** r**-net** of X for $r > 0$ if $U_r(A) = \cup_{x\in A}\overline{B_r}(x) \supseteq X$.

(ii) We say X is **totally bounded** if there exists a finite r-net of X for any $r > 0$.

Lemma 2.5. *Every compact set* $X \subseteq M$ *is totally bounded.*

Proof. Let $r > 0$, and consider the open cover $\left\{B_r(x) \;\middle|\; x \in X\right\}$. By compactness, there are finitely-many such balls that cover X, and the set A containing all points that are the centers of these balls is a finite r-net of X, so X is totally bounded. □

The following important proposition shows that $(\mathscr{C}(M), \delta)$ is itself a complete metric space.

Proposition 2.6. $(\mathscr{C}(M), \delta)$ *is a complete metric space.*

Proof. Let $X, Y, Z \in \mathscr{C}(M)$. The following properties show that δ is a metric on $\mathscr{C}(M)$.

(i) (Identity of indiscernibles) For any $r > 0$, $U_r(X) \supseteq X$, so we have $\delta(X, X) = 0$. Conversely, assume $\delta(X, Y) = 0$. Then for any $n \in \mathbb{N}$, $U_{1/n}(Y) \supseteq X$. For all $x \in X$, and we can choose x_n in Y such that $d(x, x_n) \leq \frac{1}{n}$. As $n \to \infty$, $d(x, x_n) \to 0$. Compactness of Y means that the sequence $\{x_n\}$ has a subsequence that converges in Y, so $x \in Y$, i.e. $X \subseteq Y$. By a similar argument, $Y \subseteq X$, so $X = Y$.

(ii) (Symmetry) $\delta(X, Y) = \inf\left\{r > 0 \,\middle|\, U_r(X) \supseteq Y \text{ and } U_r(Y) \supseteq X\right\} = \delta(Y, X)$.

(iii) (Triangle inequality) Let $r > \delta(X, Y)$ and $s > \delta(Y, Z)$. Then $U_{r+s}(X) \supseteq Z$ and $U_{r+s}(Z) \supseteq X$, so $r + s \geq \delta(X, Z)$, implying $\delta(X, Y) + \delta(Y, Z) \leq \delta(X, Z)$.

It remains to show that $(\mathscr{C}(M), \delta)$ is complete. Let $\{A_n\}_{n\geq 1}$ be a Cauchy sequence in $(\mathscr{C}(M), \delta)$. Define $B_n := \overline{\cup_{k\geq n} A_k}$. We will first show that B_n is compact for $n \geq 1$. Since $B_{n+1} \subseteq B_n$ and B_n is closed for $n \geq 1$ by definition, it is sufficient to show that B_1 is compact (since a closed subset of a compact set is itself compact). Let $r > 0$, and choose $m \in \mathbb{N}$ such that $U_{r/2}(A_m) \supseteq A_k$ for all $k \geq m$. Since $A_m \in \mathscr{C}(M)$, **Lemma 2.5** tells us that it is totally bounded, so there exists a finite $r/2$-net P of A_m. Then it follows that

$$U_r(P) \supseteq U_{r/2}(A_m) \supseteq \cup_{k\geq m} A_k.$$

Since $U_{r/2}(P)$ is the finite union of closed balls, it is a closed set, and we can see that P is an r-net of B_m. Adding r-nets of $A_1, \ldots, A_{m-1}$ to P, we obtain an r-net of B_1, so B_1 is totally bounded. Further, B_1 is complete because it is a closed subset of M, which is complete by assumption. Since B_1 is both totally bounded and complete, it is compact, and so B_n is compact.

Since $\{B_n\}$ is a monotonically decreasing sequence of nonempty compact sets, $A = \cap_{n\geq 1} B_n$ is compact and nonempty. For any $r > 0$, we can choose m so that $U_r(A_m) \supseteq A_k$ for all $k \geq m$. Then $U_r(A_m) \supseteq B_m \supseteq A$. On the other hand, $U_r(A) \supseteq B_m \supseteq A_m$ for sufficiently large m. Hence $A_m \to A$ as $m \to \infty$ (with respect to δ), so $(\mathscr{C}(M), \delta)$ is complete. □

Recall that $F(X) = f_1(X) \cup \cdots \cup f_N(X)$, where the maps f_i are contractions for $i \in S$. The following two lemmas show that F viewed as a function $\mathscr{C}(M) \to \mathscr{C}(M)$ is itself a contraction with respect to the Hausdorff metric δ.

Lemma 2.7. *For* $X_1, X_2, Y_1, Y_2 \in \mathscr{C}(M)$,

$$\delta(X_1 \cup X_2, Y_1 \cup Y_2) \leq \max\{\delta(X_1, Y_1), \delta(X_2, Y_2)\}.$$

Proof. Let $r = \max\{\delta(X_1, Y_1), \delta(X_2, Y_2)\}$. Then $U_r(X_1) \supseteq Y_1$ and $U_r(X_2) \supseteq Y_2$, so $U_r(X_1 \cup X_2) \supseteq Y_1 \cup Y_2$. By a similar argument, $U_r(Y_1 \cup Y_2) \supseteq X_1 \cup X_2$, and hence $r \geq \delta(X_1 \cup X_2, Y_1 \cup Y_2)$, as desired. □

Lemma 2.8. *Let* $f\colon \mathrm{M}\to\mathrm{M}$ *be a contraction with ratio* r*. For any* $\mathrm{X},\mathrm{Y}\in\mathscr{C}(\mathrm{M})$*,*

$$\delta(f(\mathrm{X}), f(\mathrm{Y})) \le r\delta(\mathrm{X},\mathrm{Y}).$$

Proof. Let $\delta(\mathrm{X},\mathrm{Y}) = s$. Then $\mathrm{U}_{rs}(f(\mathrm{X})) \supseteq f(\mathrm{U}_s(\mathrm{X})) \supseteq f(\mathrm{Y})$. Similarly,

$$\mathrm{U}_{rs}(f(\mathrm{Y})) \supseteq f(\mathrm{U}_s(\mathrm{Y})) \supseteq f(\mathrm{X}),$$

so $\delta(d(f(\mathrm{X}), f(\mathrm{Y})) \le rs$. □

We are now ready to show the existence and uniqueness of a self-similar set K with respect to contractions $\{f_i\colon \mathrm{M}\to\mathrm{M}\}_{i\in\mathrm{S}}$.

Theorem 2.9. *Let* (M, d) *be complete, and let* $\{f_i\colon \mathrm{M}\to\mathrm{M}\}_{i\in\mathrm{S}}$ *be contractions. Define*

$$\begin{aligned}\mathrm{F}\colon \mathscr{C}(\mathrm{M}) &\longrightarrow \mathscr{C}(\mathrm{M})\\ \mathrm{X} &\longmapsto \cup_{i\in\mathrm{S}} f_i(\mathrm{X})\end{aligned}$$

Then there exists a unique $\mathrm{K}\in\mathscr{C}(\mathrm{M})$ *such that* $\mathrm{F}(\mathrm{K}) = \mathrm{K}$*. Moreover, for any* $\mathrm{X}\in\mathscr{C}(\mathrm{M})$*, the sequence* $\{\mathrm{F}^n(\mathrm{X})\}$ *converges to* K *in* $(\mathscr{C}(\mathrm{M}),\delta)$ *as* $n\to\infty$*, where* $\mathrm{F}^n(\mathrm{X})$ *is the* n*th iteration of* F*.*

Proof. Using **Lemma 2.7** repeatedly, we have that for all X, Y in $\mathscr{C}(\mathrm{M})$,

$$\delta(\mathrm{F}(\mathrm{X}),\mathrm{F}(\mathrm{Y})) = \delta(\cup_{i\in\mathrm{S}} f_i(\mathrm{X}), \cup_{i\in\mathrm{S}} f_i(\mathrm{X})) \le \max_{i\in\mathrm{S}} \delta(f_i(\mathrm{X}), f_i(\mathrm{Y})).$$

By **Lemma 2.8**, $\delta(f_i(\mathrm{X}), f_i(\mathrm{Y})) \le r_i\delta(\mathrm{X},\mathrm{Y})$, where r_i is the contraction ratio of f_i. Let $r = \max_{i\in\mathrm{S}}\{r_i\}$, and note that $\delta(\mathrm{F}(\mathrm{X}),\mathrm{F}(\mathrm{Y})) \le r\delta(\mathrm{X},\mathrm{Y})$. As such, F is a contraction with ratio r (with respect to δ). By **Proposition 2.6**, $(\mathscr{C}(\mathrm{M}),\delta)$ is complete, so **Theorem 2.2** tells us that there exists a unique $\mathrm{K}\in\mathscr{C}(\mathrm{M})$ satisfying the desired conditions. □

Definition 2.10. We refer to the 3-tuple $\mathscr{L} := (\mathrm{K},\mathrm{S},\{f_i\}_{i\in\mathrm{S}})$ as a **self-similar structure**.

Definition 2.11. Recall that $\mathrm{S} = \{1,\ldots,\mathrm{N}\}$, where $\mathrm{N}\in\mathbb{N}$, and let $m\ge 1$. Define the following sets of **words with N symbols**.

(1) $\mathrm{W}_m := \left\{w_1\ldots w_m \,\middle|\, w_i\in\mathrm{S}\right\}$,

(2) $\mathrm{W}_* := \cup_{m\ge 0}\mathrm{W}_m$,

(3) $\Sigma := \left\{w_1w_2\ldots \,\middle|\, w_i\in\mathrm{S} \text{ for } i\in\mathbb{N}\right\}$,

(4) For any $w\in\Sigma$,

$$\Sigma_w := \left\{w' = w'_1w'_2\cdots\in\Sigma \,\middle|\, w'_1w'_2\ldots w'_m = w_1w_2\ldots w_m\right\}.$$

In [2], Σ is called the **shift space**.

Proposition 2.12. *The shift space Σ is a metric space, with respect to $\delta_r\colon \Sigma\times\Sigma\to\mathbb{R}$ defined by*

$$\delta_r(w,\tau)=\begin{cases}0, & \text{if } w=\tau;\\ r^{s(w,\tau)}, & \text{otherwise.}\end{cases}$$

where $0<r<1$, and

$$s(w,\tau)=\min\{m \mid w_m\neq\tau_m\}-1,$$

i.e. the last position where the two words agree.

The proof is left as an exercise for the reader. For the rest of this section, let $\mathscr{L}=(\mathrm{K},\mathrm{S},\{f_i\}_{i\in\mathrm{S}}$ be a self-similar structure, and let $w=w_1w_2\cdots\in\Sigma$. For brevity, we will write

$$f_w := f_{w_1}\circ f_{w_2}\circ\dots,$$

and

$$\mathrm{K}_w := f_w(\mathrm{K}).$$

Definition 2.13. Define the following maps on Σ.

$$\begin{aligned}\sigma_k\colon \Sigma &\longrightarrow \Sigma\\ w_1w_2\dots &\longmapsto kw_1w_2\dots\end{aligned}$$

and

$$\begin{aligned}\sigma\colon \Sigma &\longrightarrow \Sigma\\ w_1w_2\dots &\longmapsto w_2w_3\dots\end{aligned}$$

Proposition 2.14. *The map $\pi\colon\Sigma\to\mathrm{K}$ defined by*

$$\{\pi(w)\}=\cap_{m\geq1}\mathrm{K}_{w_1w_2\dots w_m}=\mathrm{K}_{w_1}\cap\mathrm{K}_{w_1w_2}\cap\cdots\cap\mathrm{K}_{w_1w_2w_3\dots}$$

is well-defined, i.e. the set on the left-hand side contains only one element. Further, π is a continuous surjection, and for any $i\in\mathrm{S}$, $\pi\circ\sigma_i=f_i\circ\pi$.

Proof. Define the *diameter* of $\mathrm{A}\subseteq\mathrm{M}$ to be

$$\operatorname{diam}(\mathrm{A}) := \sup_{x,y\in\mathrm{A}} d(x,y).$$

Set $\mathrm{R}:=\max_{1\leq i\leq\mathrm{N}}\operatorname{Lip} f_i$, and observe that

$$\operatorname{diam}(f_i(\mathrm{A}))\leq\mathrm{R}\operatorname{diam}(\mathrm{A}).$$

Also, for $w = w_1 w_2 \cdots \in \Sigma$ and $m \in \mathbb{N}$,

$$K_{w_1 \dots w_m w_{m+1}} \subseteq K_{w_1 \dots w_m}.$$

Consider the set $X := \cap_{m\geq 1} K_{w_1 \dots w_m}$. Since $K_{w_1 \dots w_m}$ is compact, X is a nonempty compact set. Further, we know that

$$\mathrm{diam}(K_{w_1 \dots w_m}) \leq R^m \mathrm{diam}(K).$$

Since $0 < R < 1$, the previous inequality implies that $\mathrm{diam}(X) = 0$, meaning X contains only one element.

Next, suppose $\delta_r(w, \tau) \leq r^m$, where $0 < r < 1$ (see the definition of δ_r in **Proposition 2.12**). Then,

$$\pi(w), \pi(\tau) \in K_{w_1 \dots w_m} = K_{\tau_1 \dots \tau_m}.$$

Thus, $d(\pi(w), \pi(\tau)) \leq R^m \,\mathrm{diam}(K) = R^m \sup_{x,y\in K} d(x, y)$, so π is (Lipschitz) continuous. Also,

$$\pi(\Sigma) = \pi(\sigma_1(\Sigma) \cup \cdots \cup \sigma_N(\Sigma)) = \pi(\sigma_1(\Sigma)) \cup \cdots \cup \pi(\sigma_N(\Sigma)) = f_1(\pi(\Sigma)) \cup \cdots \cup f_N(\pi(\Sigma)).$$

Since $\pi(\Sigma)$ is a nonempty compact set, **Theorem 2.9** implies that $\pi(\Sigma) = K$, i.e. π is surjective.

Finally, observe that

$$\{\pi(\sigma_i(w)\} = \cap_{m\geq 1} K_{i w_1 \dots w_m} = \cap_{m\geq 1} f_i(K_{w_1 \dots w_m}) = \{f_i(\pi(w))\}$$

and so $\pi \circ \sigma_i = f_i \circ \pi$. □

Proposition 2.15. *$f_w(\pi(\dot{w})) = \pi(\dot{w})$ for $\dot{w} = www\dots$, if $w \in W_*$ and $w \neq \emptyset$.*

Proof. Since f_w is a contraction, it has a unique fixed point by **Theorem 2.2**. By **Proposition 2.14**,

$$\pi(\dot{w}) = \pi(w\dot{w}) = f_w(\pi(\dot{w})).$$

As such, $\pi(\dot{w})$ is the unique fixed point of f_w. □

Definition 2.16. Given a self-similar structure $\mathscr{L}$, we define the following associated sets.

(i) $C_{\mathscr{L},K} = \cup_{i,j\in S, i\neq j} (f_i(K) \cap f_j(K))$,

(ii) $\mathscr{C} = \pi^{-1}(C_{\mathscr{L},K})$, the **critical set** of $\mathscr{L}$,

(iii) $\mathscr{P} = \cup_{n\geq 1} \sigma^n(\mathscr{C})$, the **post critical set** of $\mathscr{L}$,

(iv) $V_0 = \pi(\mathscr{P})$, the **boundary** of K.

Example 2.17 (Sierpiński gasket)**.** We present an example of the four sets in the previous definition for $\mathrm{K} = \mathrm{SG}$. Recall that in this case, $\mathrm{S} = \{1, 2, 3\}$, and $f_i(z) = \frac{z+p_i}{2}$, for $i \in \mathrm{S}$, where p_i are the vertices of an equilateral triangle. Then,

$$\mathrm{C}_{\mathscr{L},\mathrm{K}} = \big(f_1(\mathrm{K}) \cap f_2(\mathrm{K})\big) \cup \big(f_1(\mathrm{K}) \cap f_3(\mathrm{K})\big) \cup \big(f_2(\mathrm{K}) \cap f_3(\mathrm{K})\big) = \big\{q_1, q_2, q_3\big\},$$

where the q_i are depicted in Figure 3. Further,

$$\mathscr{C} = \pi^{-1}(\mathrm{C}_{\mathscr{L},\mathrm{K}}) = \big\{1\dot{2}, 2\dot{1}, 1\dot{3}, 3\dot{1}, 2\dot{3}, 3\dot{2}\big\} \quad \text{and} \quad \mathscr{P} = \cup_{n\geq 1}\sigma^n(\mathscr{C}) = \big\{\dot{1}, \dot{2}, \dot{3}\big\},$$

so $\mathrm{V}_0 = \pi(\mathscr{P}) = \big\{p_1, p_2, p_3\big\}$.

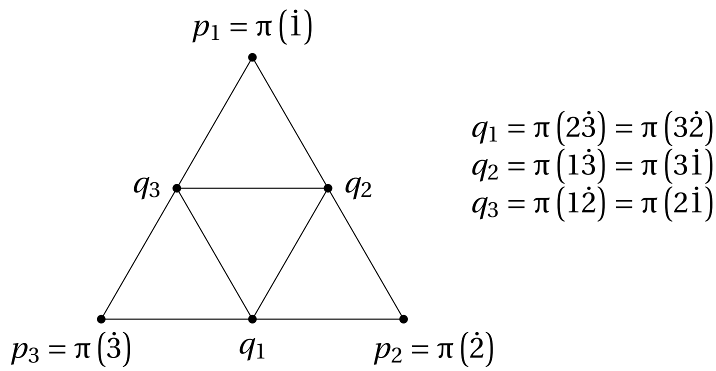

Figure 3: The approximation V_1 of $\mathrm{K} = \mathrm{SG}$. A similar figure can be found in [2, p. 16].

Definition 2.18. Let $\mathscr{L} = \big(\mathrm{K}, \mathrm{S}, \{f_i\}_{i\in\mathrm{S}}\big)$ be a self-similar structure. $\mathscr{L}$ is said to be **post critically finite**, or PCF for short, if the post critical set $\mathscr{P}$ is finite.

Definition 2.19. Define

$$\mathrm{V}_m = \bigcup_{w\in\mathrm{W}_m} f_w(\mathrm{V}_0).$$

Then $\mathrm{V}_m \subseteq \mathrm{V}_{m+1}$ and

$$\mathrm{V}_{m+1} = \bigcup_{i\in\mathrm{S}} f_i(\mathrm{V}_m).$$

3 Laplacians and Dirichlet Forms on Finite Sets

As in the previous section, let $\mathscr{L} = \big(\mathrm{K}, \mathrm{S}, \{f_i\}_{i\in\mathrm{S}}\big)$ be a PCF self-similar structure, and let V_m be an approximation of K for some $m \in \mathbb{N}$. Also denote by $\ell((\mathrm{V}_m)$ the collection of real-valued functions on V_m. Note that $\ell(\mathrm{V}_m)$ is an inner product space, with the standard inner product

$$\langle u, v\rangle = \sum_{p\in\mathrm{V}_m} u(p)v(p),$$

for $u, v \in \ell(\mathrm{V}_m)$.

Definition 3.1. The **characteristic function** χ_U of a subset $U \subseteq V_m$ is

$$\chi_U(q) = \begin{cases} 1, & \text{if } q \in U; \\ 0, & \text{otherwise.} \end{cases}$$

If $U = \{p\}$, we write χ_p (instead of $\chi_{\{p\}}$).

Definition 3.2. A symmetric bilinear form $\mathscr{E}_m : \ell(V_m) \times \ell(V_m) \to \mathbb{R}$ is called a **Dirichlet form** on $\ell(V_m)$ if it satisfies

(DF1) $\mathscr{E}_m(u,u) \geq 0$ for any $u \in \ell(V_m)$,

(DF2) $\mathscr{E}_m(u,u) = 0$ if and only if u is constant on V_m,

(DF3) For any $u \in \ell(V_m)$, $\mathscr{E}_m(u,u) \geq \mathscr{E}_m(\overline{u},\overline{u})$, where $\overline{u}$ is defined by

$$\overline{u}(p) = \begin{cases} 1, & \text{if } u(p) \geq 1; \\ u(p), & \text{if } 0 < u(p) < 1; \\ 0, & \text{if } u(p) \leq 0. \end{cases}$$

We denote by $\mathscr{DF}(V_m)$ the collection of Dirichlet forms on $\ell(V_m)$.

Definition 3.3. A symmetric linear operator $H_m : \ell(V_m) \to \ell(V_m)$ is called a **Laplacian** on $\ell(V_m)$ if it satisfies

(L1) H_m is non-positive definite,

(L2) $H_m u = 0$ if and only if u is constant on V_m,

(L3) $(H_m)_{pq} := \left(H_m \chi_q\right)(p) \geq 0$ for all $p \neq q \in V_m$.

We denote by $\mathscr{LA}(V_m)$ the collection of Laplacians on $\ell(V_m)$.

Lemma 3.4. *The collections $\mathscr{DF}(V_m)$ and $\mathscr{LA}(V_m)$ are in bijective correspondence, via the map*

$$q : \mathscr{DF}(V_m) \longrightarrow \mathscr{LA}(V_m)$$
$$\mathscr{E}_m \longmapsto H_m$$

where $\mathscr{E}_m(u,v) = -\langle u, H_m v\rangle$ for $u, v \in \ell(V_m)$.

The proof is left as an exercise for the reader.

Definition 3.5. Given a function $u \in \ell(V_{m-1})$, we define an **extension** of u to be any function $u' \in \ell(V_m)$ such that $u'|_{V_{m-1}} = u$.

Definition 3.6. Let $u \in \ell(V_{m-1})$, and let $u' \in \ell(V_m)$ be an extension of u. The **harmonic extension** $\tilde{u}$ of u is the one that minimizes $\mathscr{E}_m$, i.e.

$$\mathscr{E}_m(\tilde{u},\tilde{u}) \le \mathscr{E}_m(u',u') \quad \text{if } \tilde{u} = u' = u \text{ on } V_{m-1}.$$

For the sequence $\{V_m\}_{m\ge1}$ of finite approximations of K, we wish to construct a sequence of Dirichlet forms $\{\mathscr{E}_m\}_{m\ge0}$ (and corresponding Laplacians $\{H_m\}_{m\ge0}$). Such a sequence should preserve the self-similar structure, i.e. we require that, for $u, v \in \ell(V_m)$,

$$\mathscr{E}_m(u,v) = \sum_{i=1}^{N} \frac{1}{r_i}\mathscr{E}_{m-1}(u\circ F_i, v\circ F_i) \tag{1}$$

where r_i are known as *renormalization factors* for $i \in S$. We will further require that for $u \in \ell(V_{m-1})$ with harmonic extension $\tilde{u} \in \ell(V_m)$,

$$\mathscr{E}_m(\tilde{u},\tilde{u}) = \mathscr{E}_{m-1}(u,u), \tag{2}$$

i.e. the harmonic extension of a function leaves the value of the Dirichlet form unchanged.

Definition 3.7. For $m \in \mathbb{N}$, the restriction of $\mathscr{E}_m$ to V_{m-1} is

$$\widetilde{\mathscr{E}_m}(u,u) := \mathscr{E}_m(\tilde{u},\tilde{u}).$$

We also write

$$\widetilde{H_m} := q(\widetilde{\mathscr{E}_m})$$

where q is the map from **Lemma 3.4**.

Proposition 3.8. *Let* $u \in \ell(V_{m-1}), H_m \in \mathscr{LA}(V_m)$. *Then*

$$H_m = \begin{pmatrix} T & J^\top \\ J & X \end{pmatrix}, \tag{3}$$

where $T\colon \ell(V_{m-1}) \to \ell(V_{m-1})$, $J\colon \ell(V_{m-1}) \to \ell(V_m \setminus V_{m-1})$, *and* $X\colon \ell(V_m \setminus V_{m-1}) \to \ell(V_m \setminus V_{m-1})$. *Also,* $\widetilde{H_m} = T - J^\top X^{-1} J$.

Proof. We leave Equation (3) to the reader, and prove the second part. Since $\widetilde{\mathscr{E}_m}(u,u) = \mathscr{E}_m(\tilde{u},\tilde{u})$ by definition, we must have

$$-\left\langle u, \widetilde{H_m} u\right\rangle = -\langle \tilde{u}, H_m \tilde{u}\rangle.$$

Since $\tilde{u}$ is the harmonic extension of u, $(H_m\tilde{u})(p) = 0$ for all $p \in V_m \setminus V_{m-1}$. Also,

$$H_m\tilde{u} = \begin{pmatrix} T & J^\top \\ J & X \end{pmatrix}\begin{pmatrix} u \\ \tilde{u}|_{V_m\setminus V_{m-1}} \end{pmatrix},$$

so we must have $Ju + X\tilde{u}|_{V_m \setminus V_{m-1}} = 0$. This is true precisely when

$$\tilde{u}|_{V_m \setminus V_{m-1}} = -X^{-1}Ju,$$

meaning that

$$\widetilde{H_m}u = H_m\tilde{u} = \begin{pmatrix} T & J^\top \\ J & X \end{pmatrix}\begin{pmatrix} u \\ -X^{-1}Ju \end{pmatrix} = \begin{pmatrix} (T - J^\top X^{-1}J)u \\ 0 \end{pmatrix}.$$

□

Based on the proof of the above proposition, the harmonic extension $\tilde{u}$ of $u \in \ell(V_{m-1})$ is the unique element of $\ell(V_m)$ that satisfies

$$\begin{cases} \tilde{u}|_{V_{m-1}} & = u, \\ \tilde{u}|_{V_m \setminus V_{m-1}} & = -X^{-1}Ju. \end{cases}$$

We now describe the construction of a sequence of Dirichlet forms $\{\mathscr{E}_m\}_{m\geq 0}$ on the discrete approximations $\{V_m\}_{m\geq 0}$ of K. Begin with a Dirichlet form $\mathscr{E}_0 \colon \ell(V_0) \times \ell(V_0) \to \mathbb{R}$, as well as an N-tuple $(r_i)_{i\in S}$ of renormalization factors. The subsequent Dirichlet forms $\mathscr{E}_1, \mathscr{E}_2, \ldots$ are then determined by Equation (1), and we must also check that they satisfy Equation (2) (this check is usually how the renormalization factors are determined).

Since $\mathscr{DF}(V_m)$ and $\mathscr{LA}(V_m)$ are in bijective correspondence by **Lemma 3.4**, we may equivalently define a sequence $\{H_m\}_{m\geq 0}$ of Laplacians. Given a Laplacian $H_0 \colon \ell(V_0) \to \ell(V_0)$, the subsequent Laplacians are defined by

$$H_m = \sum_{w\in W_m} \frac{1}{r_w} R_w^\top H_0 R_w$$

where $r_w = r_{w_1} \ldots r_{w_m}$ with $w_i \in S$, and $R_w \colon \ell(V_m) \to \ell(V_0)$ is defined by $R_w u = u \circ F_w$.

We illustrate these constructions with a simple example.

Example 3.9 (Unit interval)**.** Let $I = [0,1]$. The reader may check that I is the unique self-similar set with respect to contractions $f_1(z) = \frac{z}{2}$ and $f_2(z) = \frac{z+1}{2}$. By **Proposition 2.15**, the boundary of I is $V_0 = \{0,1\}$.

(i) Define a Dirichlet form $\mathscr{E}_0 \colon \ell(V_0) \times \ell(V_0) \to \mathbb{R}$ by

$$\mathscr{E}_0(u,v) = (u(1) - u(0))(v(1) - v(0)).$$

for $u, v \in \ell(V_0)$. By Equation (1), we have

$$\mathscr{E}_1(u,v) = \frac{1}{r_1}\left(u\left(\frac{1}{2}\right) - u(0)\right)\left(v\left(\frac{1}{2}\right) - v(0)\right) + \frac{1}{r_2}\left(u(1) - u\left(\frac{1}{2}\right)\right)\left(v(1) - v\left(\frac{1}{2}\right)\right).$$

Given a function $u \in \ell(\mathrm{V}_0)$, we hope that, by choosing suitable renormalization factors, Equation (2) will be satisfied. Since $\mathscr{E}_0(u,u) = (u(1)-u(0))^2$, we have

$$\mathscr{E}_1(\tilde{u},\tilde{u}) = \frac{1}{r_1}\left(\tilde{u}\Big(\frac{1}{2}\Big) - \tilde{u}(0)\right)^2 + \frac{1}{r_2}\left(\tilde{u}(1) - \tilde{u}\Big(\frac{1}{2}\Big)\right)^2.$$

$\tilde{u}(1) = u(1)$ and $\tilde{u}(0) = u(0)$. If we let $\tilde{u}\left(\frac{1}{2}\right) = \frac{1}{2}(u(1)+u(0))$ and $r_1 = r_2 = \frac{1}{2}$, we have

$$\begin{aligned}\mathscr{E}_1(\tilde{u},\tilde{u}) &= 2\left(\frac{1}{2}(u(1)+u(0)) - u(0)\right)^2 + 2\left(u(1) - \frac{1}{2}(u(1)+u(0))\right)^2 \\ &= 4\left(\frac{1}{2}(u(1)-u(0))\right)^2 \\ &= (u(1)-u(0))^2\end{aligned}$$

which means $\widetilde{\mathscr{E}_1} = \mathscr{E}_0$, as desired.

(ii) Equivalently, we can check construct a sequence of Laplacians instead. The reader may check that the Laplacian corresponding to $\mathscr{E}_0$ defined above is

$$\mathrm{H}_0 = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$$

Since $r_1 = r_2 = \frac{1}{2}$ and $\mathrm{H}_1 = \sum_{w \in \mathrm{W}_1} \frac{1}{r_w} \mathrm{R}_w^\top \mathrm{H}_0 \mathrm{R}_w$,

$$\begin{aligned}\mathrm{H}_1 &= 2\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + 2\begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} -2 & 0 & 2 \\ 0 & -2 & 2 \\ 2 & 2 & -4 \end{pmatrix}\end{aligned}$$

Then, we compute the restriction $\widetilde{\mathrm{H}_1}$ to V_0, which is

$$\begin{aligned}\widetilde{\mathrm{H}_1} = \mathrm{T} - \mathrm{J}^{\mathrm{T}}\mathrm{X}^{-1}\mathrm{J} &= \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix} - \begin{pmatrix} 2 \\ 2 \end{pmatrix}\left(-\tfrac{1}{4}\right)\begin{pmatrix} 2 & 2 \end{pmatrix} \\ &= \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \\ &= \mathrm{H}_0\end{aligned}$$

Note also that $\tilde{u}\left(\frac{1}{2}\right) = \frac{1}{2}(u(1)+u(0)) = -\mathrm{X}^{-1}\mathrm{J}\begin{pmatrix} u(0) \\ u(1) \end{pmatrix}$, as shown in **Proposition 3.8**.

Remark 3.10. The Dirichlet form $\mathscr{E}_m : \ell(V_m) \times \ell(V_m) \to \mathbb{R}$ is sometimes referred to in the literature as **energy**.

Definition 3.11. Let $p, q \in V_m$. Define the equivalence relation $\underset{m}{\sim}$ by

$$p \underset{m}{\sim} q \iff \exists\, w \in W_m \text{ such that } p, q \in F_w(V_0).$$

If $p \underset{m}{\sim} q$, we say that p and q are **neighbors** in V_m.

Recall the case of SG, and refer to Figure 2. Notice that neighboring points in V_0, V_1, and V_2, which we can think of as vertices on a graph, are connected by an edge.

Definition 3.12. Given a discrete Laplacian $H_0 : \ell(V_0) \to \ell(V_0)$, we say that

$$(V_0, H_0) \le (V_1, H_1) \iff V_0 \subseteq V_1 \text{ and } \widetilde{H_1} = H_0.$$

For the sequence of Laplacians $\{H_m\}_{m\ge 0}$, it would be lovely if

$$(V_{m-1}, H_{m-1}) \le (V_m, H_m)$$

for *all* $m \ge 1$. Thankfully, the next proposition says that it is enough to check that $(V_0, H_0) \le (V_1, H_1)$.

Proposition 3.13. *If* $(V_0, H_0) \le (V_1, H_1)$*, then* $(V_{m-1}, H_{m-1}) \le (V_m, H_m)$ *for all* $m \ge 1$.

Proof. Proceeding by induction, assume $(V_{m-1}, H_{m-1}) \le (V_m, H_m)$ for some $m \ge 1$. For any $u \in \ell(V_m)$ and $i \in S$, $\mathscr{E}_{m-1}(u \circ F_i, u \circ F_i) = \mathscr{E}_m(\tilde{u} \circ F_i, \tilde{u} \circ F_i)$. By definition of $\mathscr{E}_m$ in Equation (1), this means that $\mathscr{E}_m(u, u) = \mathscr{E}_{m+1}(\tilde{u}, \tilde{u})$, and so $(V_m, H_m) \le (V_{m+1}, H_{m+1})$. □

Given a self-similar structure $\mathscr{L}$ and its boundary V_0, we say that $(H_0, \mathbf{r})$ is a *harmonic structure* if, for the corresponding sequence $\{(V_m, H_m)\}_{m\ge 0}$, we have $(V_{m-1}, H_{m-1}) \le (V_m, H_m)$ for all $m \ge 1$. While it is not known if a harmonic structure exists for every self-similar structure $\mathscr{L}$, there are several known examples of PCF self-similar structures with harmonic structures, as we will show in the next section.

4 Limits of Discrete Laplacians and Harmonic Functions

In the previous section, we constructed a sequence $\{H_m : \ell(V_m) \to \ell(V_m)\}_{m\ge 0}$ of discrete Laplacians, where V_m is the mth approximation of a self-similar set K. These Laplacians agree with the self-similarity of the set (Equation (1)) and is compatible with the harmonic extension of functions to greater approximations (Equation (2)).

In this section we will take the limit, in some sense, of the sequence $\{(V_m, H_m)\}_{m\ge 0}$. The result will be a continuous Laplacian Δ_μ on K. To begin, we will define a self-similar *measure* μ on K, on which Δ_μ depends. We will establish the notion of functions with finite energy, i.e. functions u for which the Dirichlet form $\mathcal{E}_m(u,u)$ remains finite as m approaches infinity. Then, we will show that all functions with finite energy defined on

$$V_* := \cup_{m\geq 0} V_m$$

are uniformly continuous, and that the set V_* is dense in K. The upshot of these two facts is that any function of finite energy defined on V_* will have a unique continuous extension to all of K. Finally, we will define the domain of the Laplacian for a self-similar set K, which consists of functions for which Δ_μ is defined. This domain will always include *harmonic functions*, for which the discrete Laplacian vanishes for all $m \geq 0$.

Definition 4.1. For $w \in W_m$, we call $F_w(K)$ a **cell** of level m. Informally, this is just a subset of K that is a "smaller version" of K (in the sense of self-similarity).

Definition 4.2. Let $C = F_w(K)$ for some $w \in W_m$. The **self-similar measure** μ is a measure on K satisfying the following conditions.

(i) (Positivity) $\mu(C) > 0$,

(ii) (Additivty) If $C = \cup_{j=1}^{N} C_j$, where the cells $\{C_j\}_{j\in S}$ only intersect at boundary points, then
$$\mu(C) = \sum_{j=1}^{N} \mu(C_j),$$

(iii) (Continuity) As $|C| \to 0$ (where $|C|$ is the cardinality of the cell C), $\mu(C) \to 0$,

(iv) (Probability) $\mu(K) = 1$.

The definition of μ largely depends on the distribution allocated to different cells, which changes the definition of the Laplacian. For example, the standard Laplacian on SG distributes weight to all cells equally, meaning that $\mu(SG) = 1$ and $\mu(F_i(SG)) = \frac{1}{3}$ for $i = 1,2,3$.

Definition 4.3. Recall that $V_* := \cup_{m\geq 0} V_m$. Define

$$\mathcal{F} := \left\{ u \in \ell(V_*) \mid \lim_{m\to\infty} \mathcal{E}_m\left(u|_{V_m}, u|_{V_m}\right) < +\infty \right\},$$
$$\mathcal{E}(u,v) := \lim_{m\to\infty} \mathcal{E}_m\left(u|_{V_m}, v|_{V_m}\right)$$

We are allowed to take the limit of the energy, since the sequence $\{\mathcal{E}_m(u,u)\}$ is nondecreasing in $\mathbb{R}$. Functions in $u \in \mathcal{F}$ have *finite energy*, i.e. $\mathcal{E}(u,u) < +\infty$.

We want to show that any $u \in \mathcal{F}$ is also uniformly continuous in V_*. We first define the following metric on V_*.

Definition 4.4. For any $p, q \in V_*$, define the **resistance metric**,

$$\mathcal{R}(p,q) := \left(\min\left\{\mathcal{E}(u,u) \;\middle|\; u \in \mathcal{F} \text{ and } u(p)=1, u(q)=0\right\}\right)^{-1}$$

The reader may check that $\mathcal{R}^{1/2} := \sqrt{\mathcal{R}(\ ,\ )}$ is a metric on V_*.

Proposition 4.5. $\mathcal{F} \subseteq C\left(V_*, \mathcal{R}^{1/2}\right)$*, the set of uniformly continuous functions on* V_*.

Proof. Let $u \in \mathcal{F}$. For any $p, q \in V_*$,

$$\begin{aligned}\mathcal{R}(p,q) &= \left(\min\left\{\mathcal{E}(u,u) \;\middle|\; u \in \mathcal{F} \text{ and } u(p)=1, u(q)=0\right\}\right)^{-1} \\ &= \max\left\{\frac{|u(p)-u(q)|^2}{\mathcal{E}(u,u)} \;\middle|\; u \in \mathcal{F}, \mathcal{E}(u,u) > 0\right\}.\end{aligned}$$

Thus, we have that $|u(p)-u(q)|^2 \le \mathcal{R}(p,q)\mathcal{E}(u,u)$, and so

$$|u(p)-u(q)| \le \mathcal{R}^{1/2}(p,q)\sqrt{\mathcal{E}(u,u)}.$$

Thus, u is Lipschitz continuous on $\left(V_*, \mathcal{R}^{1/2}\right)$, and therefore $u \in C\left(V_*, \mathcal{R}^{1/2}\right)$. □

Proposition 4.6. V_* *is dense in* K.

Proof. Recall the map $\pi: \Sigma \to K$ as defined in **Definition 2.13**. Since π is surjective, for any $x \in K$ there exists $w = w_1 w_2 w_3 \cdots \in \Sigma$ such that $\pi(w) = x$. Let $\tau \in \mathcal{P}$, the post-critical set of K, and define a sequence $\{x_m\}$, where $x_m = \pi(w_1 \dots w_m \tau)$. Since $V_m = \cup_{w \in W_m} f_w(V_0)$, and $\pi(\tau) \in V_0$, we have that $x_m \in V_m \subseteq V_*$. As $m \to \infty$, $x_m \to x$, so V_* is dense in K. □

Since V_* is dense in K, we have that any function u of finite energy (which is uniformly continuous on $(V_*, \mathcal{R}^{1/2})$) has a unique continuous extension to K.

Finally, we define the notion of harmonic functions on K. This definition is equivalent to the harmonic extension of a function $\rho \in \ell(V_0)$ to V_* (and hence all of K), as discussed in the previous section.

Definition 4.7. For any $\rho \in \ell(V_0)$, there exists a unique $u \in \mathcal{F}$ such that $u|_{V_0} = \rho$ and $\mathcal{E}(u,u) = \min\{\mathcal{E}(v,v) : v \in \mathcal{F}, u|_{V_0} = \rho\}$. Equivalently, u is the unique function that satisfies

$$\begin{cases}(H_m u)|_{(V_m \setminus V_0)} &= 0 \quad \text{for all} \quad m \ge 1, \\ u|_{V_0} &= \rho.\end{cases}$$

We call u a **harmonic function on** K. More generally, if the initial function ρ is in $\ell(V_m)$ for some $m \ge 1$ (and not just in $\ell(V_0)$), we call the corresponding (continuous) harmonic extension an m**-harmonic function**.

By the discussion in the first part of this section, a harmonic function u can be extended uniquely to a continuous function on all of K, and so we may identify all harmonic functions u with their continuous extensions. Also note that any harmonic function $K \to \mathbb{R}$ is completely determined by its values on the boundary V_0, which is analogous to the situation in many smooth settings with boundary!

Example 4.8 (Hata's tree-like set)**.** Let $M = \mathbb{C}$, and define contractions $f_1(z) = c\overline{z}$, $f_2(z) = (1-|c|^2)\overline{z} + |c|^2$ for $|c| \in (0,1)$. We call the PCF self-similar set corresponding to these contractions **Hata's tree-like set**. Its boundary is $V_0 = \{c, 0, 1\}$. Define

$$H_0 = \begin{pmatrix} -h & h & 0 \\ h & -(h+1) & 1 \\ 0 & 1 & -1 \end{pmatrix}, \quad \mathbf{r} = (a, 1-a^2),$$

where $a = \frac{1}{h}$. The reader may check via one of the techniques used in **Example 3.9** that $(H_0, \mathbf{r})$ is a harmonic structure. See Figure 4 and Figure 5 for illustrations.

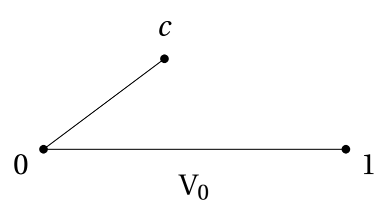

Figure 4: The boundary V_0 of Hata's tree-like set with $c = 0.4 + 0.3i$.

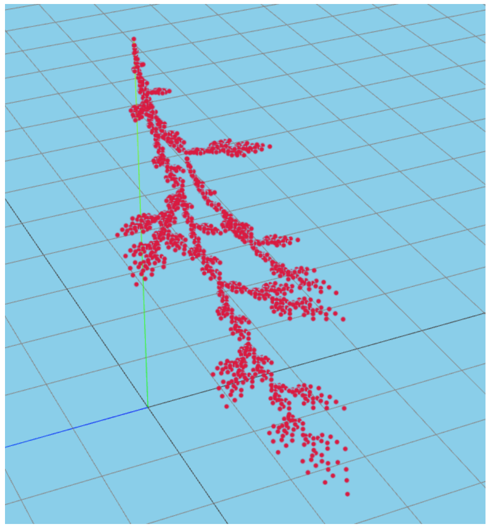

Figure 5: The unique harmonic function u with boundary values $u(c) = 0.5, u(0) = 1, u(1) = 0.5$. Note that here $a = 0.7$ and $c = 0.4 + 0.3i$. This image was created using the Javascript library *three.js* with points in the approximation V_{10}.

Definition 4.9. For $p \in V_0$, let ψ_p be the harmonic function satisfying $\psi_p|_{V_0} = \chi_p^{V_0}$. Note that this function takes on the value 1 at p, and 0 everywhere else in V_0. Define

$$D_\mu := \left\{ u \in C(K) \,\middle|\, \exists f \in C(K) \text{ such that } \lim_{m\to\infty} \max_{p\in V_m\setminus V_0} \left|\mu_{m,p}^{-1}(H_m u)(p) - f(p)\right| = 0 \right\}$$

where $\mu_{m,p} = \int_K \psi_p^m d\mu$ and $C(K)$ is the collection of continuous functions $f : K \to \mathbb{R}$. For $u \in D_\mu$, we write $f = \Delta_\mu u$, where f is the function in the definition above. Δ_μ is called the Laplacian associated with (H_0, r) and μ.

Note that if u is a harmonic function, then $(H_m u)|_{V_m\setminus V_0} = 0$ for all $m \geq 1$, so $u \in \mathscr{D}_\mu$ and $\Delta_\mu u$ vanishes on all of $K \setminus V_0$.

Example 4.10 (Unit interval)**.** Let $I = [0,1]$ as in **Example 3.9**. Define $H_0 \in \mathscr{LA}(V_0)$ by

$$H_0 = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$$

Then $(H_0, \mathbf{r})$ is a harmonic structure for any $\mathbf{r} = (r_1, r_2)$ such that $r_1 + r_2 = 1$ and $0 < r_i < 1$. Ideally, we want the Laplacian to agree with its classical counterpart, i.e. $\Delta_\mu u = \sum_i \frac{\partial^2 u}{\partial x_i^2}$ for $u \in D_\mu$. Set $\mathbf{r} = (\frac{1}{2}, \frac{1}{2})$. Then for $p = \frac{i}{2^m} \in V_m$,

$$(H_m u)(p) = \frac{1}{2^{-m}} \begin{cases} u(p+2^{-m}) + u(p-2^{-m}) - 2u(p) & \text{if } p \neq 0,1; \\ u(2^{-m}) - u(0) & \text{if } p = 0; \\ u(1-2^{-m}) - u(1) & \text{if } p = 1. \end{cases}$$

Then if we let μ be the self-similar measure with weight $(\frac{1}{2},\frac{1}{2})$, we have that $\mu_{m,p} = 2^{-m}$, and so

$$\mu_{m,p}^{-1}(H_m u)(p) = \frac{1}{4^{-m}}(u(p+2^{-m}) + u(p-2^{-m}) - 2u(p)) = \frac{u(p+\frac{1}{2^m}) + u(p-\frac{1}{2^m}) - 2u(p)}{(\frac{1}{2^m})^2}$$

which is precisely the second difference quotient.

Example 4.11 (Sierpiński gasket)**.** Define

$$H_0 = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}, \mathbf{r} = \left(\frac{3}{5}, \frac{3}{5}, \frac{3}{5}\right)$$

Then $(H_0, \mathbf{r})$ is a harmonic structure, with

$$(H_m u)(p) = \left(\frac{5}{3}\right)^m \sum_{p \underset{m}{\sim} q} \big(u(q) - u(p)\big)$$

If we let μ be the self-similar measure with weight $\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$, then

$$\int_K \psi_p^m \, d\mu = \begin{cases} \frac{2}{3^{m+1}} & \text{if } p \in V_m \setminus V_0; \\ \frac{1}{3^{m+1}} & \text{if } p \in V_0. \end{cases}$$

Thus we have that

$$\Delta_\mu u = \lim_{m\to\infty} \frac{3}{2} 5^m \sum_{p \underset{m}{\sim} q} (f(q) - f(p)).$$

This is known as the standard Laplacian on the Sierpiński gasket.

5 Green's function

This section is concerned with the Dirichlet problem for Poisson's equation where the domain is a PCF self-similar fractal K along with Laplacian Δ_μ as defined in the previous section. That is, given a function $f \in C(K)$ and $\rho \in \ell(V_0)$, we seek $u \in D_\mu$ such that

$$\begin{cases} \Delta_\mu u & = f, \\ u|_{V_0} & = \rho. \end{cases}$$

To this end, we will construct a Green's function $g\colon K \times K \to \mathbb{R}$. By the end of the section, we will show that the equation above has a unique solution u, which satisfies

$$u(x) = \sum_{p \in V_0} u(p)\psi_p(x) - \int_K g(x,y) f(y) \mu(dy),$$

where ψ_p is once again the harmonic function with value 1 at p and value 0 elsewhere on the boundary V_0.

As in our construction of Δ_μ, we begin with the discrete case. Recall that for $H_m\colon \ell(V_m) \to \ell(V_m)$ on V_m, we have

$$H_m = \begin{pmatrix} T_m & J_m^\top \\ J_m & X_m \end{pmatrix}$$

Definition 5.1. Define $\Psi\colon K \to \mathbb{R}$ by

$$\Psi(x,y) = \sum_{p,q \in V_1 \setminus V_0} \left(-X_1^{-1}\right)_{pq} \psi_p(x)\psi_q(y)$$

for $x, y \in K$. For $w \in W_*$, further define

$$\Psi_w(x,y) = \begin{cases} \Psi\left(F_w^{-1}(x), F_w^{-1}(y)\right), & \text{if } x, y \in K_w; \\ 0, & \text{otherwise.} \end{cases}$$

We write $\Psi^x(y) := \Psi(x,y)$ and $\Psi_w^x(y) := \Psi_w(x,y)$. Note that Ψ_w is a nonnegative continuous function on $K \times K$ and Ψ_w^x is an $(m+1)$-harmonic function if $w \in W_m$.

Definition 5.2. For $u \in \mathscr{F}$, define

$$u_m := \sum_{p \in V_m} u(p)\psi_p^m,$$

where ψ_p^m is the m-harmonic function with boundary value $\chi_p^{V_m}$ (this function has value 1 at $p \in V_m \setminus V_{m-1}$, and value 0 everywhere else in V_m).

Lemma 5.3. *Let u be an m-harmonic function and let $f \in \mathscr{F}$. If $f|_{V_m} = 0$, then $\mathscr{E}(u,f) = 0$.*

Proof. For $n > m$, we have $(H_n u)(p) = 0$ if $p \in V_n \setminus V_m$ and $f(p) = 0$ if $p \in V_m$. Hence,

$$\mathscr{E}_n(u,f) = -\sum_{p \in V_n} f(p)(H_n u)(p) = 0.$$

□

Lemma 5.4. *For any $u \in \mathscr{F}$,*

$$\mathscr{E}(\Psi_w^x, u) = \begin{cases} r_w^{-1}(u_{m+1}(x) - u_m(x)), & \textit{if } x \in K_w; \\ 0, & \textit{otherwise.} \end{cases}$$

Proof.

$$\begin{aligned}\mathscr{E}\left(\Psi^{x}, u\right) &= \mathscr{E}\left(\Psi^{x}, u-u_0\right) && (\textbf{Lemma 5.3})\\ &= -\sum_{p,q\in V_1\setminus V_0} X_{pq}\Psi^{x}(p)\cdot\left(u(q)-u_0(q)\right)\\ &= \sum_{q\in V_1\setminus V_0}\left(u(q)-u_0(q)\right)\cdot\psi_q(x)\\ &= u_1(x)-u_0(x)\end{aligned}$$

Thus if $x \in K_w$, $w \in W_m$, and $z := F_w^{-1}(x)$, then

$$\begin{aligned}\mathscr{E}\left(\Psi_w^{x}, u\right) &= \sum_{v\in W_m} r_v^{-1}\mathscr{E}\left(\Psi_w^{x}\circ F_v, u\circ F_v\right)\\ &= r_w^{-1}\mathscr{E}\left(\Psi^{z}, u\circ F_w\right)\\ &= r_w^{-1}\left((u\circ F_w)_1(z)-(u\circ F_w)_0(z)\right)\\ &= r_w^{-1}\left(u_{m+1}(x)-u_m(x)\right).\end{aligned}$$

□

Definition 5.5. Define $g_m\colon K \to \mathbb{R}$ by

$$g_m(x,y) = \sum_{k=0}^{m-1}\sum_{w\in W_k} r_w\Psi_w(x,y)$$

and set $g_m^x(y) := g_m(x,y)$.

Note that g_m^x is an m-harmonic function and $g_m^x(y) = 0$ if $y \in V_0$. By **Lemma 5.4**,

$$\mathscr{E}(g_m^x, u) = u_m(x) - u_0(x)$$

for any $u \in \mathscr{F}$. So we get that

$$g_m(x,y) = \sum_{p,q\in V_m\setminus V_0}(-X_m^{-1})_{pq}\psi_p^m(x)\psi_q^m(y).$$

Definition 5.6. Define

$$g(x,y) = \lim_{m\to\infty} g_m(x,y) = \sum_{w\in W_*} r_w\Psi(x,y).$$

Theorem 5.7. *Define* $R_m^t(\mu) := \max_{w\in W_m} r_w\mu(K_w)^{1/t}$ *for* $1 \le t \le \infty$*. Assume* $\sum_{m\ge0} R_m^t(\mu) < \infty$*. If* s *is the constant satisfying* $\frac{1}{t}+\frac{1}{s} = 1$*, then the following conditions hold.*

(i) Let $L^s(K,\mu)$ *be the space of functions* f *such that*

$$\|f\|_s = \left(\int_S |f|^s\,d\mu\right)^{1/s} < \infty.$$

For $f \in \mathrm{L}^s(\mathrm{K},\mu)$,

$$\left(\mathrm{G}_\mu f\right)(x) = \int_{\mathrm{K}} g(x,y) f(y)\mu(dy)$$

is well-defined for all $x \in \mathrm{K}$ *and* $\mathrm{G}_\mu f \in \mathrm{C}(\mathrm{K}) \cap \mathscr{F}_0$, *where* $\mathscr{F}_0 := \left\{u \in \mathscr{F} : u|_{\mathrm{V}_0} = 0\right\}$.

(ii) $\mathrm{G}_\mu : \mathrm{L}^s(\mathrm{K},\mu) \to \mathrm{C}(\mathrm{K})$ *is a compact operator.*

(iii) $\mathscr{E}\left(u, \mathrm{G}_\mu f\right) = \int_{\mathrm{K}} (u - u_0) f \, d\mu$ *for any* $u \in \mathscr{F}$.

We call G_μ *the (extended) Green's operator.*

We need several lemmas to prove the above theorem.

Lemma 5.8. *If* $\sum_{m\geq 0} \mathrm{R}^t_m(\mu) < \infty$, *then* $g^x \in \mathrm{L}^t(\mathrm{K},\mu)$ *for any* $x \in \mathrm{K}$. *Moreover,* $x \mapsto g^x$ *is a continuous map from* K *to* $\mathrm{L}^t(\mathrm{K},\mu)$.

Proof. For $u \in \mathrm{L}^t\left(\mathrm{K},\mu\right)$, set $\|u\|_{\mu,t} = \left(\int_{\mathrm{K}} |u|^t \, d\mu\right)^{1/t}$. Let $x = \pi(w_1 w_2 \ldots)$. Then,

$$g^x(y) = \sum_{m\geq 0} r_{w_1\ldots w_m} \Psi_{w_1\ldots w_m}(x,y).$$

As such,

$$\begin{aligned}
\left\|g^x\right\|_{\mu,t} &\leq \sum_{m\geq 0} r_{w_1\ldots w_m} \left\|\Psi_{w_1\ldots w_m}\right\|_{\mu,t} \\
&\leq \sum_{m\geq 0} r_{w_1\ldots w_m} \mu\left(\mathrm{K}_{w_1\ldots w_m}\right)^{1/t} \quad \left(\text{see the definition of } \Psi^x(y)\right) \\
&\leq \sum_{m\geq 0} \mathrm{R}^t_m(\mu) \\
&< \infty.
\end{aligned}$$

So $g^x \in \mathrm{L}^t(\mathrm{K},\mu)$. Then,

$$\left\|g^x - g^y\right\|_{\mu,t} \leq \left\|g^x_m - g^y_m\right\|_{\mu,t} + \left\|g^x_m - g^x\right\|_{\mu,t} + \left\|g^y_m - g^y\right\|_{\mu,t}.$$

Since $\left\|g^x_m - g^x\right\|_{\mu,t} \leq \sum_{k\geq m+1} \mathrm{R}^t_k(\mu) < \infty$ and g_m is continuous on $\mathrm{K} \times \mathrm{K}$ (by continuity of harmonic functions), we have that $\left\|g^x - g^y\right\|_{\mu,t} \to 0$ as $d(x,y) \to 0$. □

Lemma 5.9.

$$\sum_{q\in \mathrm{V}_m} (\mathrm{H}_m)_{pq} \, g^q(y) = \begin{cases} -\psi^m_p(y), & \text{if } p \in \mathrm{V}_m \setminus \mathrm{V}_0; \\ -\psi^m_p(y) + \psi_p(y), & \text{if } p \in \mathrm{V}_0. \end{cases}$$

Proof. For any $u \in \mathscr{F}$, we have that $\mathscr{E}\left(\psi_p^m, u\right) = \mathscr{E}_{H_m}\left(\psi_p^m, u\right) = -\left(H_m u\right)(p)$. Since $\mathscr{E}\left(g^p, u\right) = u(p) - u_0(p)$, it follows that

$$\mathscr{E}\left(\sum_{p\in V_m} (H_m)_{pq}\, g^q, u\right) = (H_m u)(p) - (H_m u_0)(p) = \mathscr{E}\left(-\psi_p^m, u\right) + \mathscr{E}\left(\psi_p^m, u_0\right).$$

Since u_0 is harmonic, **Lemma 5.3** implies that $\mathscr{E}\left(\psi_p^m - \psi_p, u_0\right) = 0$ for $p \in V_0$. Furthermore, $(H_m u_0)(p) = 0$ for $p \in V_m \setminus V_0$. Hence,

$$\mathscr{E}\left(\sum_{p\in V_m} (H_m)_{pq}\, g^q, u\right) = \begin{cases} \mathscr{E}\left(-\psi_p^m, u\right) & \text{if } p \in V_m \setminus V_0; \\ \mathscr{E}\left(-\psi_p^m + \psi_p, u\right) & \text{if } p \in V_0. \end{cases}$$

for any $u \in \mathscr{F}$, which implies the result. □

Lemma 5.10.

$$\left(H_m(G_\mu f)\right)(p) = \begin{cases} -\int_K \psi_p^m f\, d\mu, & \textit{if } p \in V_m \setminus V_0; \\ -\int_K \left(\psi_p^m - \psi_p\right) f\, d\mu, & \textit{if } p \in V_0. \end{cases}$$

Proof. Note that

$$\left(H_m\left(G_\mu f\right)\right)(p) = \sum_{q\in V_m} (H_m)_{pq} \int_K g^q f\, d\mu = \int_K \left(\sum_{q\in V_m} (H_m)_{pq} g^q\right) f\, d\mu$$

So **Lemma 5.9** implies the result. □

Now we are ready for the proof of **Theorem 5.7**.

Proof of ***Theorem*** *5.7.* Recall that we set $\|f\|_{\mu,s} := \left(\int_K |f|^s\, d\mu\right)^{1/s}$.

(i) By Hölder's inequality,

$$\left|\left(G_\mu f\right)(x) - \left(G_\mu f\right)(y)\right| = \left|\int_K (g(x,p) - g(y,p)) f(p) \mu(dp)\right| = \|(g^x - g^y) f\|_{\mu,1} \le \|g^x - g^y\|_{\mu,t} \|f\|_{\mu,s}.$$

We have showed in **Lemma 5.8** that $\|g^x - g^y\|_{\mu,t} \to 0$ as $d(x,y) \to 0$, so this implies that $G_\mu f \in C(K)$. By **Lemma 5.10**,

$$\begin{aligned} \mathscr{E}_{H_m}\left(u, G_\mu f\right) &= -\sum_{p\in V_m} u(p)\left(H_m(G_\mu f)\right)(p) \\ &= -\sum_{p\in V_m} u(p)\left(-\int_K \left(\psi_p^m - \psi_p\right) f\, d\mu\right) \\ &= \int_K (u_m - u_0) f\, d\mu \end{aligned}$$

(Recall that $u_m = \sum_{p\in V_m} u(p)\psi_p^m$.) If we set $u = G_\mu f$, then

$$\begin{aligned}\mathcal{E}_{H_m}\left(G_\mu f, G_\mu f\right) &= \int_K \left((G_\mu f)_m - (G_\mu f)_0\right) f\, d\mu \\ &= \int_K \left(G_\mu f\right)_m f\, d\mu \\ &\le \int_K \sup_{x\in K} \left|(G_\mu f)(x)\right| \cdot |f|\, d\mu \\ &\le \|f\|_{\mu,1} \cdot \sup_{x\in K} \left|(G_\mu f)(x)\right|.\end{aligned}$$

Note that $(G_\mu f)_0 = 0$ since $g = 0$ if $x \in V_0$ or $y \in V_0$. Since we already showed that $G_\mu f$ is continuous and K is compact, $G_\mu f$ is bounded and so $\mathcal{E}_{H_m}\left(G_\mu f, G_\mu f\right) < \infty$, meaning that $G_\mu f \in \mathcal{F}_0$.

(ii) Let $\{f_n\}_{n\ge1}$ be a bounded sequence in $L^s(K,\mu)$ such that $\|f_n\|_{\mu,s} \le M$ for all n. Now we want to show that $\{G_\mu f_n\}_{n\ge1} = \{\int_K g(x,p) f(p)\mu(dp)\}_{n\ge1}$ is equicontinuous.

$$\left|(G_\mu f_n)(x) - (G_\mu f_n)(y)\right| = \left|\int_K (g(x,p) - g(y,p)) f_n(p)\mu(dp)\right| = \left\|(g^x - g^y) f_n\right\|_{\mu,1} \le M\left\|g^x - g^y\right\|_{\mu,t}$$

By **Lemma 5.8**, this shows $\{G_\mu f_n\}_{n\ge1}$ is equicontinuous.

Now we want to show $\{G_\mu f_n\}_{n\ge1}$ is uniformly bounded.

$$\left|(G_\mu f_n)(x)\right| = \left|\int_K g(x,p) f_n(p)\mu(dp)\right| = \left\|g^x f_n\right\|_{\mu,1} \le M \sup_{x\in K} \left\|g^x\right\|_{\mu,t}$$

Note that $\sup_{x\in K} \|g^x\|_{\mu,t} < \infty$ since g^x is continuous on K. Again by **Lemma 5.8**, this shows that $\{G_\mu f_n\}_{n\ge1}$ is uniformly bounded.

By Ascoli-Arzelà's theorem, $\{G_\mu f_n\}_{n\ge1}$ contains a subsequence that is convergent in C(K). Hence G_μ is a compact operator from $L^s(K,\mu)$ to C(K).

(iii) For any $u \in \mathcal{F}$, if we let $m \to \infty$ for $\mathcal{E}_{H_m}(u, G_\mu f) = \int_K (u_m - u_0) f\, d\mu$, we get

$$\mathcal{E}(u, G_\mu f) = \lim_{m\to\infty} \mathcal{E}_{H_m}\left(u, G_\mu f\right) = \lim_{m\to\infty} \int_K (u_m - u_0) f\, d\mu$$

We already know that $(u_m - u_0) f$ converges to $(u - u_0) f$. Since K is compact, it has finite measure, and so there exists an (integrable) constant function g on K such that $|(u_m - u_0) f| \le g$ for all m. By the Lebesgue Dominated Convergence Theorem, we have that

$$\lim_{m\to\infty} \int_K (u_m - u_0) f\, d\mu = \int_K (u - u_0) f\, d\mu.$$

as desired.

□

Lemma 5.11. *Let* $\mathscr{D}_{\mathrm{D}} := \{u \in \mathscr{D}_\mu : u|_{V_0} = 0\}$. *For any* $f \in C(K)$, *we have*

$$G_\mu f \in \mathscr{D}_{\mathrm{D}} \text{ and } \Delta_\mu(G_\mu f) = -f.$$

Proof. Recall that

$$\mathscr{D}_\mu := \left\{ u \in C(K) \,\middle|\, \exists f \in C(K) \text{ such that } \lim_{m\to\infty} \max_{p\in V_m \setminus V_0} |\mu_{m,p}^{-1}(H_m u)(p) - f(p)| = 0 \right\}.$$

where $\mu_{m,p} = \int_K \psi_p^m \, d\mu$.
Let $f \in C(K)$. Note that since K is compact, $f \in L^s(K,\mu)$. Using **Lemma 5.10**, for $p \in V_m \setminus V_0$,

$$\begin{aligned}
\left|\mu_{m,p}^{-1} H_m(G_\mu f)(p) + f(p)\right| &= \mu_{m,p}^{-1}\left| -\int_K \psi_p^m(y) f(y)\mu(dy) + f(p)\right| && (\textbf{Lemma 5.10})\\
&= \mu_{m,p}^{-1}\int_{\cup_{w\in W_m} K_w} \psi_p^m(y)|f(p)-f(y)|\mu(dy)\\
&= \mu_{m,p}^{-1}\int_{K_{m,p}} \psi_p^m(y)\left|f(p)-f(y)\right|\mu(dy) && (\text{since } \psi_p^m(y)=0 \text{ if } y\notin K_{m,p})\\
&\le \max_{p\in V_m}\left\{\sup_{y\in K_{m,p}} \left|f(p)-f(y)\right|\right\}.
\end{aligned}$$

where $K_{m,p} := \cup_{w\in W_m, p\in K_w} K_w$ for $p \in K$. Note that another characterization of an m-harmonic function ρ is that $\rho \circ F_w$ is a harmonic function for any $w \in W_m$. Thus if $y \in K_w$ but $y \notin K_{m,p}$, we have that $\psi_p^m(y) = 0$. Then since f is uniformly continuous on K,

$$\lim_{m\to\infty} \max_{p\in V_m}\left\{\sup_{y\in K_{m,p}} \left|f(y)-f(p)\right|\right\} = 0.$$

By definition of D_μ, this implies that $G_\mu f \in D_\mu$ with $\Delta_\mu(G_\mu f) = -f$. Since $G_\mu f|_{V_0} = 0$, we have $G_\mu f \in \mathscr{D}_{\mathrm{D}}$. □

Theorem 5.12. *Let* $f \in C(K)$ *and* $\rho \in \ell(V_0)$. *There exists a unique function* $u \in D_\mu$ *such that*

$$\begin{cases} \Delta_\mu u &= f,\\ u|_{V_0} &= \rho. \end{cases}$$

Moreover,

$$u(x) = \sum_{p\in V_0} \rho(p)\psi_p(x) - \int_K g(x,y) f(y)\mu(dy)$$

Note that if $u|_{V_0} = 0$ then we have $u(x) = -\int_K g(x,y)f(y)\mu(dy)$.

Proof. Let $u = (\sum_{p\in V_0} \rho(p)\psi_p) - G_\mu f$. By **Lemma 5.11**, u satisfies the above.
To show uniqueness, assume u_a and u_b satisfy the above. Then $v := u_a - u_b \in \mathscr{D}_{\mathrm{D}}$ and $\Delta_\mu v = 0$. Since $-G_\mu = (\Delta_\mu|_{\mathscr{D}_{\mathrm{D}}})^{-1}$, it follows that $v = 0$, i.e. $u_a = u_b$. □

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