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Cover Page Footnote

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Finite Tropical Semirings

By *Caden Zonnefeld*

Abstract. The focus of this paper lies at the intersection of the fields of tropical algebra and graph theory. In particular the interaction between tropical semirings and directed graphs is investigated. Originally studied in [7], the directed graph of a ring is useful in identifying properties within the algebraic structure of a ring. This work builds off the research done by [2, 5, 1] in constructing directed graphs from rings. However, we will investigate the relationship $(x, y) \to (\min(x, y), x+y)$ as defined by the operations of tropical algebra and applied to tropical semirings.

1 Introduction

The advent of graph theory has served to explore numerous mathematical phenomena. These include binary relations of objects, planarity, coloring, critical path analysis, and network flow as detailed in [6] and [11]. While there are numerous types of graphs, directed graphs are of particular interest to the contents of this paper. The value in directing the edges of a graph manifests itself in the study of Markov chains and, as discovered in [7], rings.

Maclagan and Sturmfels [8] detail the fundamentals of tropical algebra by redefining addition as the minimum of two numbers and multiplication as the sum of two numbers. The structure this describes is commonly known as the tropical semiring, though alternatively it is called the min-plus algebra. The term tropical geometry was coined in honor of Brazilian mathematician Imre Simon; however, the adjective tropical holds no deeper meaning to the field of study. Though initially limited to optimization and discrete mathematics, the applications of tropical geometry have expanded into fields including computational algebra and combinatorics [3]. Dynamic programming is another application of tropical geometry [8] that, when utilized in computational biology, creates algorithms capable of predicting genes. Though most applications of tropical geometry define the tropical semiring on $\mathbb{R} \cup \{\infty\}$ we will define an analagous structure on a finite set in Definition 2.17.

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This paper investigates finite tropical semirings as revealed by directed graphs. Section 2 details the foundational concepts for understanding the remainder of the paper. This includes graph theory and ring theory definitions as well as defining a finite tropical semiring. Section 3 explores findings from investigating the directed graph of finite tropical semirings. Section 4 shares ideas for further research in this line of inquiry.

2 Preliminaries

The findings of this paper rely upon concepts drawn from graph theory, ring theory, and tropical geometry. The following definitions are fundamental graph theory concepts that are based upon definitions found in [6] and [11]. We will begin by defining directed graphs; then we will examine characteristics of directed graphs.

Definition 2.1. A *directed graph* is a collection of vertices and arcs $G = (V, E, p)$ defined by the mapping $p : E \to V^2$ where V and E are sets. We call V or sometimes $V(G)$ the set of all vertices in G. Likewise, we call E or sometimes $E(G)$ the set of all arcs in G. Lastly, p is the incidence mapping which consists of mappings o and t where $o, t : E \to V$. The mappings o and t map arcs to vertices and $o(e), t(e)$ represent the *origin* and *tail* of an arc respectively.

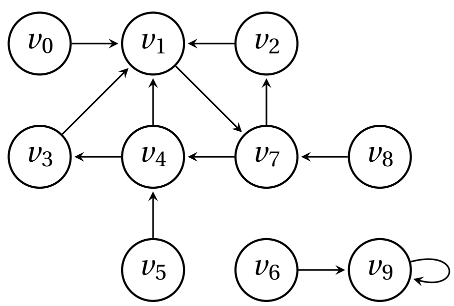

Figure 1: Directed Graph Example

Definition 2.2. A directed graph G is *one-sided connected* if for all vertices $a, b \in V(G)$ there exists a path from a to b or from b to a.

Definition 2.3. Consider the graph $G = (V, E, p)$. The graph $G' = (V', E', p')$ is a *subgraph* of G if $V(G') \subseteq V(G)$, $E(G') \subseteq E(G)$, and $p' = p\,|_{E'}$.

Definition 2.4. A *one-sided component* of directed graph G is a maximal one-sided connected subgraph.

All components examined in this paper will be one-sided connected so we will use the term component to mean one-sided component.

Figure 1 exemplifies a directed graph with more examples available in Figure 3 and 4. The following examples make use of Figure 1. Each circle represents a vertex and arrow an arc. The set $V = \{v_0, v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9\}$ consists of each of the 10 vertices in the directed graph. Likewise, the set

$$E = \{(v_0, v_1), (v_1, v_7), (v_2, v_1), (v_3, v_1), (v_4, v_1), (v_4, v_3), \\ (v_5, v_4), (v_6, v_9), (v_7, v_2), (v_7, v_4), (v_8, v_7), (v_9, v_9)\}$$

consists of each of the 12 arcs in the directed graph. Consider the edge (v_0, v_1) which we will call e_0. The origin of e_0 or $o(e_0)$ is v_0. The tail of e_0 or $t(e_0)$ is v_1. The incidence mapping of e_0 or $p(e_0)$ maps v_0 to v_1. We can see that no path exists from v_9 to v_0 or from v_0 to v_9; therefore, Figure 1 is not a one-sided connected graph. Lastly, the vertices v_6 and v_9 along with the edges (v_6, v_9) and (v_9, v_9) form a component since they are a maximal one-sided connected subgraph.

The ensuing definitions further describe features of a directed graph.

Definition 2.5. A *path* connects v_0 to v_n and consists a sequence of arcs $e_1, e_2, \dots e_n$. Furthermore, a *simple path* describes a path where no vertex appears more than once.

Definition 2.6. A *loop* is an edge e where $o(e) = t(e)$.

For an arbitrary vertex v and edge e where $o(e) = t(e) = v$ we say that v loops back to itself.

Definition 2.7. The *out-degree* of a vertex v is the number of arcs that point from v. Conversely, the *in-degree* of v is the number of arcs that point to v. A *source* is a vertex with an in-degree of 0.

Consider Figure 1 once more. Notice the edges (v_0, v_1) and (v_1, v_7), this pair of edges creates a simple path between the vertices v_0 and v_7. The origin and tail of the edge (v_9, v_9) are each v_9 such that it forms a loop. Consider v_5, this vertex has an in-degree of 0 and an out-degree of 1. Since v_5 has an in-degree of 0 it is a source.

The next several definitions are key to modular arithmetic and are adapted from [10].

Definition 2.8. Let $a, b, m \in \mathbb{Z}$ and $m \geq 2$. We say that a is *congruent* to b modulo m when $m|(a-b)$. Equivalently, a is *congruent* to b modulo m when a and b have the same remainder after division by m. We denote congruence as $a \equiv b \bmod m$.

Example 2.9. Given $a = 13$, $b = 3$, and $m = 5$ notice that $13 \equiv 3 \bmod 5$ since $5|(13-3)$.

Recall the division algorithm which, as written in [10], states that for any integers a and b where b is positive there exist unique integers q and r with the property that $a = mq + r$ with $0 \leq r < m$.

Definition 2.10. Let $a, m \in \mathbb{Z}$ where m is positive, if $a = mq + r$ and $0 \le r < m$, we call r the *least non-negative residue* of a mod m.

The least non-negative residue can also be understood as the remainder of the division algorithm.

Example 2.11. Consider the conditions from Example 2.9. We can see that $13 = 5 \cdot 2 + 3$ where 3 is the least non-negative residue of 13 mod 5.

Definition 2.12. The set of integers congruent mod m are an *equivalence relation*. The collection of all integers congruent to a given integer x is an *equivalence class*.

Example 2.13. The elements of an equivalence class for a mod m are denoted as $\overline{a} = \{a + km : k \in \mathbb{Z}\}$.

Though an equivalence class can be represented by any of its members, we will make use of the least non-negative residue of our equivalence classes. Thus, for an arbitrary equivalence class $\overline{b}$ usage of b will refer to the least non-negative residue of $\overline{b}$.

The definition we will employ for a semiring is adapted from [4]. A semiring is an algebraic structure that possesses properties similar to that of a ring.

Definition 2.14. A *semiring* is a nonempty set S together with two binary operations + and · satisfying the following properties:

1. **Associativity of Addition** Given any $a, b, c \in \mathrm{S}$, $(a + b) + c = a + (b + c)$.

2. **Commutativity of Addition** Given any $a, b \in \mathrm{S}$, $a + b = b + a$.

3. **Additive Identity** There exists a $0_\mathrm{S} \in \mathrm{S}$ such that $a + 0_\mathrm{S} = 0_\mathrm{S} + a = a$.

4. **Associativity of Multiplication** Given any $a, b, c \in \mathrm{S}$, $(ab)c = a(bc)$.

5. **Left Distributive over Addition** Given any $a, b, c \in \mathrm{S}$, $a(b + c) = ab + ac$.

6. **Right Distributive over Addition** Given any $a, b, c \in \mathrm{S}$, $(a + b)c = ac + bc$.

The following properties define special kinds of *semirings*.

A semiring S that possesses a multiplicative identity is called a semiring with identity.

7. **Multiplicative Identity** There exists a $1_\mathrm{S} \in \mathrm{S}$ such that for all $a \in \mathrm{S}$, $a1_\mathrm{S} = 1_\mathrm{S}a = a$.

A semiring S that satisfies multiplicative commutativity is called a commutative semiring.

8. **Multiplicative Commutativity** Given $a, b \in \mathrm{S}$, $ab = ba$.

Notice that the only property of rings lacking from semirings is that of an additive inverse. In this paper we will make use of commutative semirings with identity.

Example 2.15. Consider the set $\mathbb{N}$ with traditional addition and multiplication as operations. Notice that $5 \in \mathbb{N}$; however, -5, the additive inverse of 5, is not in the set of natural numbers. All other ring properties are fulfilled by the natural numbers, thus it is a semiring.

Example 2.16. Consider the set of all $n \times n$ matrices with nonnegative entries as seen in [9] with traditional addition and multiplication as operations. This set is a semiring.

We will now define the finite tropical semiring, the algebraic structure that will be the basis of our study for the remainder of the paper, as $\mathrm{S}_n := \mathbb{Z}_n \cup \{\infty\}$. Furthermore, we will demonstrate the finite tropical semiring's properties to show that it is a semiring. We establish the convention that if an element has an overline it is an equivalence class in $\mathbb{Z}_n$; conversely, if an element does not have an overline it is a general element of S_n.

Theorem 2.17. *Let* $\mathrm{S}_n := \mathbb{Z}_n \cup \{\infty\}$*, and define operations on* S_n *as follows.*

- *Given* $a_1, a_2 \in \mathbb{Z}_n, a_1 \oplus a_2 := \min(r_1, r_2)$*, where* r_i *is the least non-negative residue of* a_i*, and for all* $b \in \mathbb{Z}_n, b \oplus \infty = \infty \oplus b = \min(b, \infty) = b$.
- *Given* $a_1, a_2 \in \mathbb{Z}_n, a_1 \otimes a_2 := a_1 + a_2$*, and for all* $b \in \mathbb{Z}_n, b \otimes \infty = \infty \otimes b = \infty$.

Then:

(a) The operations $\oplus$ *and* $\otimes$ *are well-defined on* S_n.

(b) Under the operations $\oplus$ *and* $\otimes$*,* S_n *is a commutative semiring with identity given by* ∞.

Proof.
Addition is Well-Defined To demonstrate that addition is well-defined we will consider finite and infinite elements.

Case 1: Let $\overline{a}, \overline{b}, \overline{c}, \overline{d} \in \mathrm{S}_n$, $\overline{a} = \overline{b}$, and $\overline{c} = \overline{d}$. We can represent $\overline{a}, \overline{b}$ by $\overline{r_1}$ the least non-negative residue of $\overline{a}$ and $\overline{b}$. Likewise, we can represent $\overline{c}, \overline{d}$ by $\overline{r_2}$ the least non-negative residue of $\overline{c}$ and $\overline{d}$, then

$$\begin{aligned}\overline{a} \oplus \overline{c} &= \overline{r_1} \oplus \overline{r_2} \\ &= \min(\overline{r_1}, \overline{r_2}) \\ &= \overline{b} \oplus \overline{d}\end{aligned}$$

Case 2: Let $\overline{a}, \overline{b} \in \mathrm{S}_n$ and $\overline{a} = \overline{b}$, then

$$\begin{aligned}
\overline{a} \oplus \infty &= \min(\overline{a}, \infty) \\
&= \overline{a} \\
&= \overline{b} \\
&= \min(\overline{b}, \infty) \\
&= \overline{b} \oplus \infty
\end{aligned}$$

Since any representative of the equivalence class can be used addition is well-defined for S_n.

Multiplication is Well-Defined To demonstrate that multiplication is well-defined we will consider finite and infinite elements.

Case 1: Let $\overline{a}, \overline{b}, \overline{c}, \overline{d} \in \mathrm{S}_n$, $\overline{a} = \overline{b}$, and $\overline{c} = \overline{d}$. We can represent $\overline{a}, \overline{b}$ by $\overline{r_1}$ the least non-negative residue of $\overline{a}$ and $\overline{b}$. Likewise, we can represent $\overline{c}, \overline{d}$ by $\overline{r_2}$ the least non-negative residue of $\overline{c}$ and $\overline{d}$, then

$$\begin{aligned}
\overline{a} \otimes \overline{c} &= \overline{a} + \overline{c} \\
&= \overline{b} + \overline{d} \\
&= \overline{b} \otimes \overline{d}
\end{aligned}$$

Case 2: Let $\overline{a}, \overline{b} \in \mathrm{S}_n$ and $\overline{a} = \overline{b}$, then

$$\begin{aligned}
\overline{a} \otimes \infty &= \overline{a} + \infty \\
&= \infty \\
&= \overline{b} + \infty \\
&= \overline{b} \otimes \infty
\end{aligned}$$

Since any representative of the equivalence class can be used multiplication is well-defined for S_n.

Associativity of Addition Let $a, b, c \in \mathrm{S}_n$, then

$$\begin{aligned}
(a \oplus b) \oplus c &= \min(a, b) \oplus c \\
&= \min(\min(a, b), c) \\
&= \min(a, \min(b, c)) \\
&= a \oplus \min(b, c) \\
&= a \oplus (b \oplus c)
\end{aligned}$$

Commutativity of Addition Let $a, b \in \mathrm{S}_n$, then

$$\begin{aligned}
a \oplus b &= \min(a, b) \\
&= \min(b, a) \\
&= b \oplus a
\end{aligned}$$

Additive Identity We claim the additive identity in S_n is ∞. By definition, for all $a \in S_n$, $a \oplus \infty = \infty \oplus a = a$. Thus, ∞ is the additive identity.
Associativity of Multiplication Let $a, b, c \in S_n$, then

$$\begin{aligned} a \otimes (b \otimes c) &= a \otimes (b + c) \\ &= a + (b + c) \\ &= (a + b) + c \\ &= (a + b) \otimes c \\ &= (a \otimes b) \otimes c \end{aligned}$$

Left Distributive over Addition Let $a, b, c \in S_n$, then

$$\begin{aligned} a \otimes (b \oplus c) &= a \otimes \min(b, c) \\ &= a + \min(b, c) \\ &= \min(a + b, a + c) \\ &= \min(a \otimes b, a \otimes c) \\ &= a \otimes b \oplus a \otimes c \end{aligned}$$

Right Distributive over Addition Let $a, b, c \in S_n$, then

$$\begin{aligned} (a \oplus b) \otimes c &= \min(a, b) \otimes c \\ &= \min(a, b) + c \\ &= \min(a + c, b + c) \\ &= \min(a \otimes c, b \otimes c) \\ &= a \otimes c \oplus b \otimes c \end{aligned}$$

Commutativity of Multiplication Let $a, b \in S_n$, then

$$\begin{aligned} a \otimes b &= a + b \\ &= b + a \\ &= b \otimes a \end{aligned}$$

Multiplicative Identity We claim the multiplicative identity in S_n is 0. By definition, for all $a \in S_n$, $a \otimes 0 = 0 \otimes a = a$. Thus, 0 is the multiplicative identity.

Since S_n is well defined and fulfills the aforementioned properties, it is a commutative semiring with identity defined under the operations of $\oplus$ and $\otimes$. □

Now that we have established the finite tropical semiring, we will define the directed graph of a semiring in a similar fashion as [2]. The directed graph of finite tropical semirings will be the basis of the research completed in this paper.

Definition 2.18. The directed graph of a semiring S_n, denoted $\Psi(S_n)$, is the graph with $V(\Psi(S_n)) = S_n \times S_n$. For $(a,b),(c,d) \in S_n \times S_n$, $(a,b) \to (c,d)$ if and only if $a \oplus b = \min(a,b) = c$ and $a \otimes b = a+b = d$. Equivalently, $(a,b) \to (a \oplus b, a \otimes b) = (\min(a,b), a+b)$. Since S_n is commutative, $(b,a) \to (c,d)$ when $(a,b) \to (c,d)$.

We conclude this section with an observation about the structure of $\Psi(S_n)$.

Lemma 2.19. *Every vertex in $\Psi(S_n)$ can point to only one vertex.*

Proof. Let $(x,y) \in \Psi(S_n)$. If $\min(x,y) = x$, then $(x,y) \to (x, x+y)$. If $\min(x,y) = y$, then $(x,y) \to (y, x+y)$. Whether x or y is minimum (x,y) only points to one vertex. □

3 Results

The focus of our investigation is the finite tropical semiring; in particular, we will utilize the directed graph to explore finite tropical semirings. We will begin by examining the behavior of vertices in S_n that include the element infinity.

Proposition 3.1. *For all $n \in \mathbb{N}$, $(\infty,\infty) \in \Psi(S_n)$ is the only vertex in its component.*

Proof. Consider the vertex (∞,∞). We observe that $(\infty,\infty) \to (\infty,\infty)$ since $\infty \oplus \infty = \infty$ and $\infty \otimes \infty = \infty$. Likewise, any vertex that might point to (∞,∞) must consist of only infinite elements to satisfy tropical addition. Therefore, for all $n \in \mathbb{N}$ the vertex $(\infty,\infty) \in \Psi(S_n)$ is the only vertex in its component. □

Proposition 3.2. *For all $\overline{a} \in S_n$, there exists a component consisting of $(\infty,\overline{a}) \to (\overline{a},\infty) \to (\overline{a},\infty)$.*

Proof. Let $\overline{a} \in S_n$ and consider the vertex $(\infty,\overline{a})$. No vertex can point to $(\infty,\overline{a})$ by the definition of tropical addition in Theorem 2.17. We observe that $\infty \oplus \overline{a} = \overline{a}$ and $\infty \otimes \overline{a} = \infty$, such that $(\infty,\overline{a}) \to (\overline{a},\infty)$. Similarly, $\overline{a} \oplus \infty = \overline{a}$ and $\overline{a} \otimes \infty = \infty$ such that $(\overline{a},\infty) \to (\overline{a},\infty)$. Therefore, the component containing $(\infty,\overline{a})$ is $(\infty,\overline{a}) \to (\overline{a},\infty) \to (\overline{a},\infty)$. □

The outcome of Proposition 3.1 addresses the behavior of the vertex (∞,∞). The result of Proposition 3.2 indicates that for all $\overline{a} \in S_n$, the components that contain vertices with ∞ as an element are simple. The vertex $(\infty,\overline{a})$ is a source that points to $(\overline{a},\infty)$ where $(\overline{a},\infty)$ points to itself. Thus, from now on only vertices consisting of elements in $\mathbb{Z}_n$ from the finite tropical semiring will be considered.

The next several propositions begin to examine the larger structure of $\Psi(S_n)$. Our initial focus will be the behavior of the directed graph at vertices of the form $(\overline{0},\overline{a})$.

Definition 3.3. We define the *terminal vertex* of a component in $\Psi(S_n)$ as a vertex that points to itself.

Proposition 3.4. *All terminal vertices must be of the form* $(\overline{0}, \overline{a})$.

Proof. Let $(\overline{x}, \overline{y}) \in \Psi(S_n)$. For $(\overline{x}, \overline{y}) \to (\overline{x}, \overline{y})$ we must have $\min(\overline{x}, \overline{y}) = \overline{x}$ and $\overline{x} + \overline{y} = \overline{y}$. After subtracting $\overline{y}$ from the previous equation we can see that $\overline{x} = \overline{0}$. Similarly by substitution $\overline{0} + \overline{y} = \overline{y}$ such that $\overline{y}$ can be any equivalence class in $\mathbb{Z}_n$. Therefore, since $\overline{x} = \overline{0}$ and $\overline{y} \in \mathbb{Z}_n$ all terminal vertices must be of the form $(\overline{0}, \overline{a})$. □

Proposition 3.5. *All points of the form* $(\overline{0}, \overline{a})$ *are terminal vertices.*

Proof. Let $(\overline{0}, \overline{a}) \in \Psi(S_n)$. Then $(\overline{0}, \overline{a}) \to (\overline{0}, \overline{a})$ since $\overline{0} \oplus \overline{a} = \overline{0}$ and $\overline{0} \otimes \overline{a} = \overline{a}$. Therefore, all points of the form $(\overline{0}, \overline{a})$ are terminal vertices. □

The results of Proposition 3.4 and Proposition 3.5 indicate that the set of all terminal vertices in $\Psi(S_n)$ is precisely the set of vertices of the form $(\overline{0}, \overline{a})$.

Example 3.6. In $\Psi(S_5)$, the set of terminal vertices is $\{(\overline{0}, \overline{0}), (\overline{0}, \overline{1}), (\overline{0}, \overline{2}), (\overline{0}, \overline{3}), (\overline{0}, \overline{4})\}$.

Lemma 3.7. *For all* $\overline{a} \in S_n$, $(\overline{a}, \overline{0}) \to (\overline{0}, \overline{a}) \to (\overline{0}, \overline{a})$.

Proof. Let $\overline{a} \in S_n$ and consider the vertex $(\overline{a}, \overline{0})$. We can see that $(\overline{a}, \overline{0}) \to (\overline{0}, \overline{a})$ since $\overline{a} \oplus \overline{0} = \overline{0}$ and $\overline{a} \otimes \overline{0} = \overline{a}$. Furthermore, it is evident that $(\overline{0}, \overline{a}) \to (\overline{0}, \overline{a})$ as a result of Proposition 3.5. Therefore, for all $\overline{a} \in S_n$, $(\overline{a}, \overline{0}) \to (\overline{0}, \overline{a}) \to (\overline{0}, \overline{a})$. □

Proposition 3.8. *For all* $\overline{a} \in S_n$ *satisfying* $\frac{n}{2} < a \le n-1$, *there is a component that consists of 2 vertices such that* $(\overline{a}, \overline{0}) \to (\overline{0}, \overline{a}) \to (\overline{0}, \overline{a})$.

Proof. Let $\overline{a} \in S_n$ where $\frac{n}{2} < a \le n-1$. Lemma 3.7 can be referenced to show that $(\overline{a}, \overline{0}) \to (\overline{0}, \overline{a}) \to (\overline{0}, \overline{a})$. For an arbitrary vertex $(\overline{x}, \overline{y}) \in S_n$ to point to $(\overline{0}, \overline{a})$ we must have $\min(\overline{x}, \overline{y}) = \overline{0}$ and $\overline{x} + \overline{y} = \overline{a}$. Let $\min(\overline{x}, \overline{y}) = \overline{x} = \overline{0}$, then by substitution $\overline{0} + \overline{y} = \overline{a}$ such that $\overline{y} = \overline{a}$ and $(\overline{x}, \overline{y})$ is not unique. Thus, $(\overline{0}, \overline{a})$ is the only vertex that points to $(\overline{0}, \overline{a})$.

Consider an arbitrary vertex $(\overline{x}, \overline{y}) \in \Psi(S_n)$. Suppose $(\overline{x}, \overline{y}) \to (\overline{a}, \overline{0})$ such that $\min(\overline{x}, \overline{y}) = \overline{a}$ and $\overline{x} + \overline{y} = \overline{0}$. Let $\min(\overline{x}, \overline{y}) = \overline{x} = \overline{a}$. Assume without loss of generality that a, x, and y are the least non-negative residues of their respective equivalence classes. We know that $0 < y < n$ by Definition 2.10 such that $0 < x + y < 2n$. Thus, $x + y = n$ as n is the only element of $\overline{0}$ that fulfills $\overline{x} + \overline{y} = \overline{0}$. Since $x = a > \frac{n}{2}$ we can see that $y < \frac{n}{2}$ such that $y < x$; however, this causes a contradiction with the earlier statement that $\min(\overline{x}, \overline{y}) = \overline{x}$. Thus, no vertices can point to $(\overline{a}, \overline{0})$

Therefore, for all $\overline{a} \in S_n$ satisfying $\frac{n}{2} < a \le n-1$ there is a component that consists of 2 vertices such that $(\overline{a}, \overline{0}) \to (\overline{0}, \overline{a}) \to (\overline{0}, \overline{a})$. □

Proposition 3.9. *For all* $n \in 2\mathbb{Z}^+$, *there is a component of* $\Psi(S_n)$ *that consists of 3 vertices such that* $\left(\overline{\frac{n}{2}}, \overline{\frac{n}{2}}\right) \to \left(\overline{\frac{n}{2}}, \overline{0}\right) \to \left(\overline{0}, \overline{\frac{n}{2}}\right) \to \left(\overline{0}, \overline{\frac{n}{2}}\right)$.

Proof. Let $n \in 2\mathbb{Z}^+$ and consider the vertex $\left(\overline{\frac{n}{2}}, \overline{\frac{n}{2}}\right)$. We can see that $\left(\overline{\frac{n}{2}}, \overline{\frac{n}{2}}\right) \to \left(\overline{\frac{n}{2}}, \overline{0}\right)$ since $\overline{\frac{n}{2}} \oplus \overline{\frac{n}{2}} = \overline{\frac{n}{2}}$ and $\overline{\frac{n}{2}} \otimes \overline{\frac{n}{2}} = \overline{n} = \overline{0}$. Taken in conjunction with Lemma 3.7, we can see that $\left(\overline{\frac{n}{2}}, \overline{\frac{n}{2}}\right) \to \left(\overline{\frac{n}{2}}, \overline{0}\right) \to \left(\overline{0}, \overline{\frac{n}{2}}\right) \to \left(\overline{0}, \overline{\frac{n}{2}}\right)$. The argument from Proposition 3.8 can be referenced to show that no other vertices point to $(\overline{\frac{n}{2}}, \overline{0})$ or $(\overline{0}, \overline{\frac{n}{2}})$.

For an arbitrary vertex $(\overline{x}, \overline{y}) \in \Psi(\mathrm{S}_n)$ to point to $\left(\overline{\frac{n}{2}}, \overline{\frac{n}{2}}\right)$ we must have $\min(\overline{x}, \overline{y}) = \overline{\frac{n}{2}}$ and $\overline{x} + \overline{y} = \overline{\frac{n}{2}}$. We will use contradiction to show that no arbitrary vertex can point to $\left(\overline{\frac{n}{2}}, \overline{\frac{n}{2}}\right)$. Let $\min(\overline{x}, \overline{y}) = \overline{x} = \overline{\frac{n}{2}}$. Then by substitution $\overline{\frac{n}{2}} + \overline{y} = \overline{\frac{n}{2}}$ and $\overline{y} = \overline{0}$. Now we arrive at a contradiction since $\overline{x} > \overline{y}$ despite $\min(\overline{x}, \overline{y}) = \overline{x}$. Since no vertex can point to $\left(\overline{\frac{n}{2}}, \overline{\frac{n}{2}}\right)$ it is a source. Thus, there is a component of $\Psi(\mathrm{S}_n)$ that consists of $\left(\overline{\frac{n}{2}}, \overline{\frac{n}{2}}\right) \to \left(\overline{\frac{n}{2}}, \overline{0}\right) \to \left(\overline{0}, \overline{\frac{n}{2}}\right) \to \left(\overline{0}, \overline{\frac{n}{2}}\right)$. □

Though the results of Proposition 3.8 and 3.9 are important to our investigation, the components they yield are simple and easily described. Figure 2 depicts a pair of components from $\Psi(\mathrm{S}_{12})$. Proposition 3.8 describes the component shaded in grey and Proposition 3.9 describes the component shaded in white.

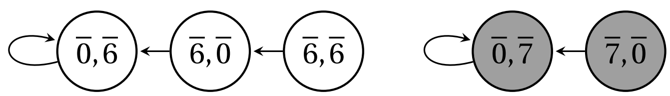

Figure 2: $\Psi(\mathrm{S}_n)$, $n = 12$

We will now begin to investigate components with terminal vertices where $a < \frac{n}{2}$. These components will be more complex and reveal information about the nature of the finite tropical semiring. Qualities of these components that we will examine include directed graph structure, source vertex form, and the maximum in-degree of a vertex.

Proposition 3.10. *For all $\overline{a} \in \mathrm{S}_n$ satisfying $\frac{n+1}{2} < a < n$ where $n \in 2\mathbb{Z}$ and $n \geq 2$, the vertex $(\overline{a}, \overline{1})$ is a source that points to $(\overline{1}, \overline{a+1})$.*

Proof. Let $\overline{a} \in \mathrm{S}_n$ satisfying $\frac{n+1}{2} < a < n$ where $n \in 2\mathbb{Z}$ and $n \geq 2$. We can see that $(\overline{a}, \overline{1}) \to (\overline{1}, \overline{a+1})$ since $\overline{a} \oplus \overline{1} = \overline{1}$ and $\overline{a} \otimes \overline{1} = \overline{a+1}$. Consider the arbitrary vertex $(\overline{x}, \overline{y}) \in \Psi(\mathrm{S}_n)$. We will now use contradiction to show that no vertex points to $(\overline{a}, \overline{1})$.

Suppose $(\overline{x}, \overline{y}) \to (\overline{a}, \overline{1})$ such that $\min(\overline{x}, \overline{y}) = \overline{a}$ and $\overline{x} + \overline{y} = \overline{1}$. Assume without loss of generality that $\min(\overline{x}, \overline{y}) = \overline{x} = \overline{a}$. Furthermore, we may assume without loss of generality that a, x, and y are the least non-negative representatives of their respective equivalence classes. We know that $0 < x, y < n$ by Definition 2.10 such that $0 < x + y < 2n$. The only way that $\overline{x} + \overline{y} = \overline{1}$ is if $x + y = n + 1$. Since $x = a > \frac{n+1}{2}$ we can see that $y < \frac{n+1}{2}$ such that

$x > y$; however, this causes a contradiction with the earlier statement that $\min(\overline{x}, \overline{y}) = \overline{x}$. Thus, no vertices can point to $(\overline{a}, \overline{1})$.

Therefore $(\overline{a}, \overline{1})$ is a source that points to $(\overline{1}, \overline{a+1})$. □

Proposition 3.11. *For all $\overline{a} \in S_n$ where $a > 0$ the vertex $(\overline{a}, \overline{a})$ is a source that points to a vertex with an in-degree of 1.*

Proof. Let $\overline{a} \in S_n$ and $a > 0$, then consider the vertex $(\overline{a}, \overline{a})$. For $(\overline{a}, \overline{a})$ to be a source it must have an in-degree of 0. We will use contradiction to show no vertex can point to $(\overline{a}, \overline{a})$. Consider the arbitrary vertex $(\overline{x}, \overline{y}) \in \Psi(S_n)$ and assume that $(\overline{x}, \overline{y}) \to (\overline{a}, \overline{a})$. As a result $\min(\overline{x}, \overline{y}) = \overline{a}$ and $\overline{x} + \overline{y} = \overline{a}$. The only values that $\overline{x}$ and $\overline{y}$ can assume are in $\mathbb{Z}_n = \{\overline{0}, \overline{1}, \overline{2}, \ldots, \overline{n-2}, \overline{n-1}\}$. Let $\min(\overline{x}, \overline{y}) = \overline{x} = \overline{a}$, then $\overline{a} + \overline{y} = \overline{a}$ such that $\overline{y} = \overline{0}$ and $\overline{x} > \overline{y}$. However, a contradiction arises with the earlier statement that $\min(\overline{x}, \overline{y}) = \overline{x}$. Therefore, there does not exist a vertex $(\overline{x}, \overline{y}) \in \Psi(S_n)$ that points to $(\overline{a}, \overline{a})$ implicating that $(\overline{a}, \overline{a})$ is a source.

We can see that $(\overline{a}, \overline{a}) \to (\overline{a}, \overline{2a})$ since $\overline{a} \oplus \overline{a} = \overline{a}$ and $\overline{a} \otimes \overline{a} = \overline{2a}$. We will now show that no other vertex $(\overline{w}, \overline{z}) \in \Psi(S_n)$ can point to $(\overline{a}, \overline{2a})$. Assume $(\overline{w}, \overline{z}) \to (\overline{a}, \overline{2a})$ such that $\min(\overline{w}, \overline{z}) = \overline{a}$ and $\overline{w} + \overline{z} = \overline{2a}$. Let $\min(\overline{w}, \overline{z}) = \overline{w} = \overline{a}$, then substitution shows that $\overline{a} + \overline{z} = \overline{2a}$. Now it is evident that $\overline{z} = \overline{a}$. Since $\overline{w} = \overline{z} = \overline{a}$ only $(\overline{a}, \overline{a})$ points to $(\overline{a}, \overline{2a})$.

Therefore, for all $\overline{a} \in S_n$ the vertex $(\overline{a}, \overline{a})$ is a source that points to a vertex with an in-degree of 1. □

Proposition 3.12. *The in-degree of every vertex $(\overline{x}, \overline{y}) \in \Psi(S_n)$ is at most 2.*

Proof. Consider the vertex $(\overline{x}, \overline{y}) \in \Psi(S_n)$ that is not a source. Definition 2.7 ensures that any vertex with an in-degree of 0 is a source. Let $(\overline{w}, \overline{z}) \in \Psi(S_n)$ such that $(\overline{w}, \overline{z}) \to (\overline{x}, \overline{y})$. Then $\overline{w} = \overline{x}$ and $\overline{z} = \overline{y} - \overline{x}$. Definition 2.18 implicates that $(\overline{z}, \overline{w}) \to (\overline{x}, \overline{y})$ where $\overline{z} = \overline{x}$ and $\overline{w} = \overline{y} - \overline{x}$. We will now show that no other unique vertex points to $(\overline{x}, \overline{y})$.

Consider $(\overline{u}, \overline{v}) \in \Psi(S_n)$ and assume that $(\overline{u}, \overline{v}) \to (\overline{x}, \overline{y})$. Then $\min(\overline{u}, \overline{v}) = \overline{x}$ and $\overline{u} + \overline{v} = \overline{y}$. Using substitution we can see that $\overline{x} + \overline{v} = \overline{y}$ such that $\overline{v} = \overline{y} - \overline{x}$. However, $\overline{u} = \overline{w}$ and $\overline{v} = \overline{z}$ such that $(\overline{u}, \overline{v}) \in \{(\overline{w}, \overline{z}), (\overline{z}, \overline{w})\}$ and thus is not unique. Likewise when $\min(\overline{u}, \overline{v}) = \overline{v}$ the vertex $(\overline{u}, \overline{v})$ is not unique.

Therefore, the in-degree of each vertex in $\Psi(S_n)$ is at most 2. □

The following material broadens the focus of study to the entire structure of the directed graph. The topics considered include the presence of cycles and connections in the directed graph of finite tropical semirings.

Lemma 3.13. *For all $(\overline{x}, \overline{y}) \in \Psi(S_n)$ where the least non-negative representatives of each equivalence class satisfy $xk = n - y$ and $x \leq y$, $(\overline{x}, \overline{y}) \to (\overline{x}, \overline{x+y}) \to (\overline{x}, \overline{2x+y}) \to \cdots \to (\overline{x}, \overline{(k-1)x+y}) \to (\overline{x}, \overline{0}) \to (\overline{0}, \overline{x})$.*

Proof. Let $(\overline{x},\overline{y}) \in \Psi(\mathrm{S}_n)$ and without loss of generality assume that the least non-negative representative of each equivalence class satisfies $xk = n-y$ and $x \le y$. Now we can see that $(\overline{x},\overline{y}) \to (\overline{x},\overline{x+y}) \to (\overline{x},\overline{2x+y}) \to \cdots \to (\overline{x},\overline{(k-1)x+y}) \to (\overline{x},\overline{0})$. Likewise we can see that $(\overline{x},\overline{0}) \to (\overline{0},\overline{x})$ as a result of Lemma 3.7. Thus when the least non-negative representatives of $(\overline{x},\overline{y})$ satisfy $xk = n-y$ and $x \le y$, $(\overline{x},\overline{y}) \to (\overline{x},\overline{x+y}) \to (\overline{x},\overline{2x+y}) \to \cdots \to (\overline{x},\overline{(k-1)x+y}) \to (\overline{x},\overline{0}) \to (\overline{0},\overline{x})$. □

Remark 3.14. If the least non-negative representatives of a vertex $(\overline{x},\overline{y}) \in \Psi(\mathrm{S}_n)$ satisfy $y|n-x$ and $y \le x$ the commutativity of tropical operations allows us to apply an argument symmetrical to Lemma 3.13. In that case, a path exists from $(\overline{x},\overline{y})$ to $(\overline{0},\overline{y})$.

Theorem 3.15. *The only cycles in* $\Psi(\mathrm{S}_n)$ *are of length 1 and consist of loops at terminal vertices.*

Proof. Let $(\overline{x_0},\overline{y_0}) \in \Psi(\mathrm{S}_n)$ be an arbitrary vertex that is not of the form of a terminal vertex. All vertices considered by Proposition 3.8 and 3.9 will not be addressed as their respective components only have cycles at terminal vertices. Assume that a path exists between $(\overline{x_0},\overline{y_0})$ and $(\overline{x_k},\overline{y_k})$ where $k \in \mathbb{N}$, $\overline{x_0} = \overline{x_k}$, and $\overline{y_0} = \overline{y_k}$. This path is a cycle. Furthermore, $\overline{x_0} \ne \overline{y_0}$ as a result of Proposition 3.11. Since $\overline{x_0} \ne \overline{y_0}$, we can assume without loss of generality that $\overline{0} < \overline{x_0} < \overline{y_0} < \overline{n}$. The nature of tropical addition implies that $\overline{x_0} \ge \overline{x_1} \ge \overline{x_2} \ge \cdots \ge \overline{x_{k-1}} \ge \overline{x_k}$ such that $\overline{x_0} = \overline{x_1} = \overline{x_2} = \cdots = \overline{x_{k-1}} = \overline{x_k}$. Tropical multiplication shows that $\overline{x_0} + \overline{y_0} = \overline{y_1}$, $\overline{x_1} + \overline{y_1} = \overline{y_2}, \ldots, \overline{x_{k-1}} + \overline{y_{k-1}} = \overline{y_k}$. We can use substitution to show that $\overline{x_0}+\overline{y_0} = \overline{y_1}$, $\overline{x_0}+\overline{y_1} = \overline{y_2}, \ldots, \overline{x_0}+\overline{y_{k-1}} = \overline{y_k}$. Furthermore, we can substitute $\overline{y_1}$ into the the second equation to see that $\overline{x_0}+\overline{x_0}+\overline{y_0} = \overline{2x_0}+\overline{y_0} = \overline{y_2}$. Likewise, we can substitute $\overline{y_2}$ into the third equation to see that $\overline{x_0} + \overline{2x_0} + \overline{y_0} = \overline{3x_0} + \overline{y_0} = \overline{y_3}$. Now it is clear that each copy of $\overline{x_0}$ added to $\overline{y_0}$ increases the index of the $\overline{y_k}$ term by 1, giving rise to the general form $j\overline{x_0} + \overline{y_0} = \overline{y_j}$ where $j \in \mathbb{N}$. We will now use contradiction to show that no $\overline{x_0}$ preserves the integrity of the cycle.

Case 1: Let $\overline{x_0}|(\overline{n} - \overline{y_0})$ as in Lemma 3.13. Then there exists a $c \in \mathbb{N}$ that satisfies $0 < c < k$ such that $\overline{cx_0} + \overline{y_0} = \overline{n}$ as a result of $\overline{y_0} > \overline{x_0}$. We can see from the general form that $\overline{y_c} = \overline{n} = \overline{0}$. As a result $\overline{x_{c+1}} = \overline{0}$ which is in contradiction with earlier statements that $\overline{0} < \overline{x_0}$ and $\overline{x_0} = \overline{x_1} = \cdots = \overline{x_{k-1}} = \overline{x_k}$. Therefore, $(\overline{x_0},\overline{y_0})$ cannot be in a cycle when $\overline{x_0}|(\overline{n} - \overline{y_0})$.

Case 2: Let $\overline{x_0} \nmid (\overline{n} - \overline{y_0})$, then, by the division algorithm, there exists a $c \in \mathbb{N}$ that satisfies $0 < c < k$ such that $\overline{cx_0} + \overline{y_0} = \overline{n} + \overline{r}$ where $\overline{0} < \overline{r} < \overline{x_0}$ and r, the least non-negative representative of $\overline{r}$, is an integer. We can substitute the general form to see that $\overline{y_c} = \overline{n} + \overline{r} = \overline{r}$. As a result $\overline{x_{c+1}} = \overline{r}$. However, this contradicts the earlier statement that $\overline{x_0} = \overline{x_1} = \cdots = \overline{x_c} = \overline{x_{c+1}}$ since $\overline{r} < \overline{x_0}$. Therefore, $(\overline{x_0},\overline{y_0})$ cannot be in a cycle when $\overline{x_0} \nmid (\overline{n} - \overline{y_0})$.

Thus, $(\overline{x_0},\overline{y_0})$ cannot be a part of a cycle. However for all $a \in \mathrm{S}_n$ the loop at the terminal vertex where $(\overline{0},\overline{a}) \to (\overline{0},\overline{a})$ constitutes a cycle of length 1. Therefore, the only cycles in $\Psi(\mathrm{S}_n)$ occur at terminal vertices and are of length 1. □

Lemma 3.16. *A path exists from every non-terminal vertex $(\overline{x},\overline{y}) \in \Psi(\mathrm{S}_n)$ to a terminal vertex.*

Proof. Consider an arbitrary vertex $(\overline{x},\overline{y}) \in \Psi(\mathrm{S}_n)$ and without loss of generality let x_0 and y_0 be the least non-negative representative of each equivalence class. Theorem 3.15 indicates that the only cycles in $\Psi(\mathrm{S}_n)$ occur at terminal vertices and are of length 1. Definition 2.18 ensures that every vertex must point to some vertex. Therefore, every non-terminal vertex $(x_0, y_0) \in \Psi(\mathrm{S}_n)$ is not a part of a cycle and must point to some vertex. Then $(x_0, y_0) \to (x_1, y_1)$. If (x_1, y_1) is a terminal vertex then we have accomplished our goal. If not, the same conditions will apply to (x_1, y_1) and the next vertex (x_2, y_2) all the way until (x_{k-1}, y_{k-1}) such that $(x_1, y_1) \to (x_2, y_2) \to \cdots (x_{k-1}, y_{k-1}) \to (x_k, y_k)$ where $0 < k < n^2$ and $x_k = 0$.

Therefore, a path exists from every non-terminal vertex $(\overline{x},\overline{y}) \in \Psi(\mathrm{S}_n)$ to a terminal vertex. □

Theorem 3.17. *The number of components consisting of finite elements in $\Psi(\mathrm{S}_n)$ is equal to n.*

Proof. A path exists from every non-terminal vertex $(\overline{x},\overline{y}) \in \Psi(\mathrm{S}_n)$ to a terminal vertex as a result of Lemma 3.16. Now we must determine the number of terminal vertices. As described in Definition 3.3 a terminal vertex must take on the form $(\overline{0},\overline{a})$ where $\overline{a} \in \mathrm{S}_n$. The set $\mathrm{S}_n = \{\overline{0},\overline{1},\overline{2},\ldots,\overline{n-2},\overline{n-1}\}$ such that there are n terminal vertices in $\Psi(\mathrm{S}_n)$. Therefore, there are n components consisting of finite elements in $\Psi(\mathrm{S}_n)$. □

The next definitions establish terminology that describes the behavior we will encounter and explore regarding different types of source vertices.

Definition 3.18. We define a *true source* as any source $(\overline{x},\overline{y}) \in \Psi(\mathrm{S}_n)$ where either $\overline{x} = \overline{y}$ or where $(\overline{x},\overline{y})$ and $(\overline{y},\overline{x})$ are both sources and the same distance from the terminal vertex of the component.

Definition 3.19. We define an *adjacent source* as any source in $\Psi(\mathrm{S}_n)$ that is not a true source as described by Definition 3.18. (*Note:* Sources of the form $(\overline{a},\overline{0})$ fall into this category; their components are described in Proposition 3.8.)

Figure 3 depicts true sources in dark grey and adjacent sources in light grey. For the sake of clarity, Figure 3 and 4 will follow the convention established after Proposition 3.2 by omitting vertices including ∞ as an element. The distinction between true and adjacent sources becomes important in describing the larger structure of each component in $\Psi(\mathrm{S}_n)$. We will begin to examine the difference between each type of source in the following propositions.

Proposition 3.20. *There exists a path $\mathrm{P} : (x_0, y_0) \to (x_1, y_1) \to \cdots \to (x_t, y_t)$ from every true source (x_0, y_0) to a terminal vertex (x_t, y_t). For all (x_k, y_k) in P where $0 < k < t$, $k \in \mathbb{Z}$, and (y_k, x_k) is a source, (y_k, x_k) is an adjacent source.*

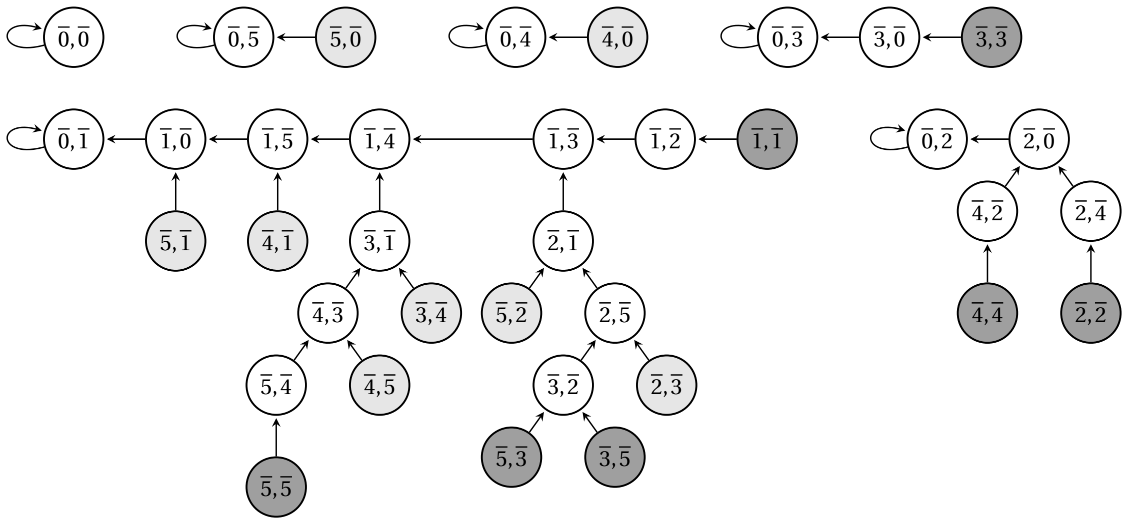

Figure 3: $\Psi(\mathrm{S}_n)$, $n=6$

Proof. Consider the true source $(x_0, y_0) \in \Psi(\mathrm{S}_n)$. The result of Lemma 3.16 informs us that a path P exists from (x_0, y_0) to a terminal vertex (x_t, y_t). Furthermore, we reference Definition 2.18 to show that for each $(x_k, y_k) \to (x_{k+1}, y_{k+1})$ in P where $k \in \mathbb{Z}$ and satisfies $0 \le k < t$, there exists a vertex $(y_k, x_k) \to (x_{k+1}, y_{k+1})$. Since the in-degree of (x_{k+1}, y_{k+1}) is 2, (y_k, x_k) cannot be a true source where $y_k = x_k$ as a result of Proposition 3.11. Furthermore, only when $k = 0$ is (x_k, y_k) a source; thus, when (y_k, x_k) is a source it is only a true source if $k = 0$. Thus, for all $0 < k < t$ along P where the vertex (y_k, x_k) is a source it is an adjacent source. □

4 Future Research

There are many directions for future research in this area of study. We will begin by sharing conjectures that we observed and could be used to guide future research into the directed graph of finite tropical semirings.

The first two conjectures examine the presence and proportion of sources in the directed graph of finite tropical semirings.

Conjecture 4.1. The number of sources in $\Psi(\mathrm{S}_n)$ for $n > 4, n \in \mathbb{Z}$ is equal to $\sum_{i=1}^{n-1} i$.

Conjecture 4.2. The proportion of sources in $\Psi(\mathrm{S}_n)$ approaches a limit of 0.5 as n approaches ∞.

The next two conjectures aim to address the structure of each component. The objective of the first conjecture is to demonstrate that true sources are fundamental to the construction of the directed graph.

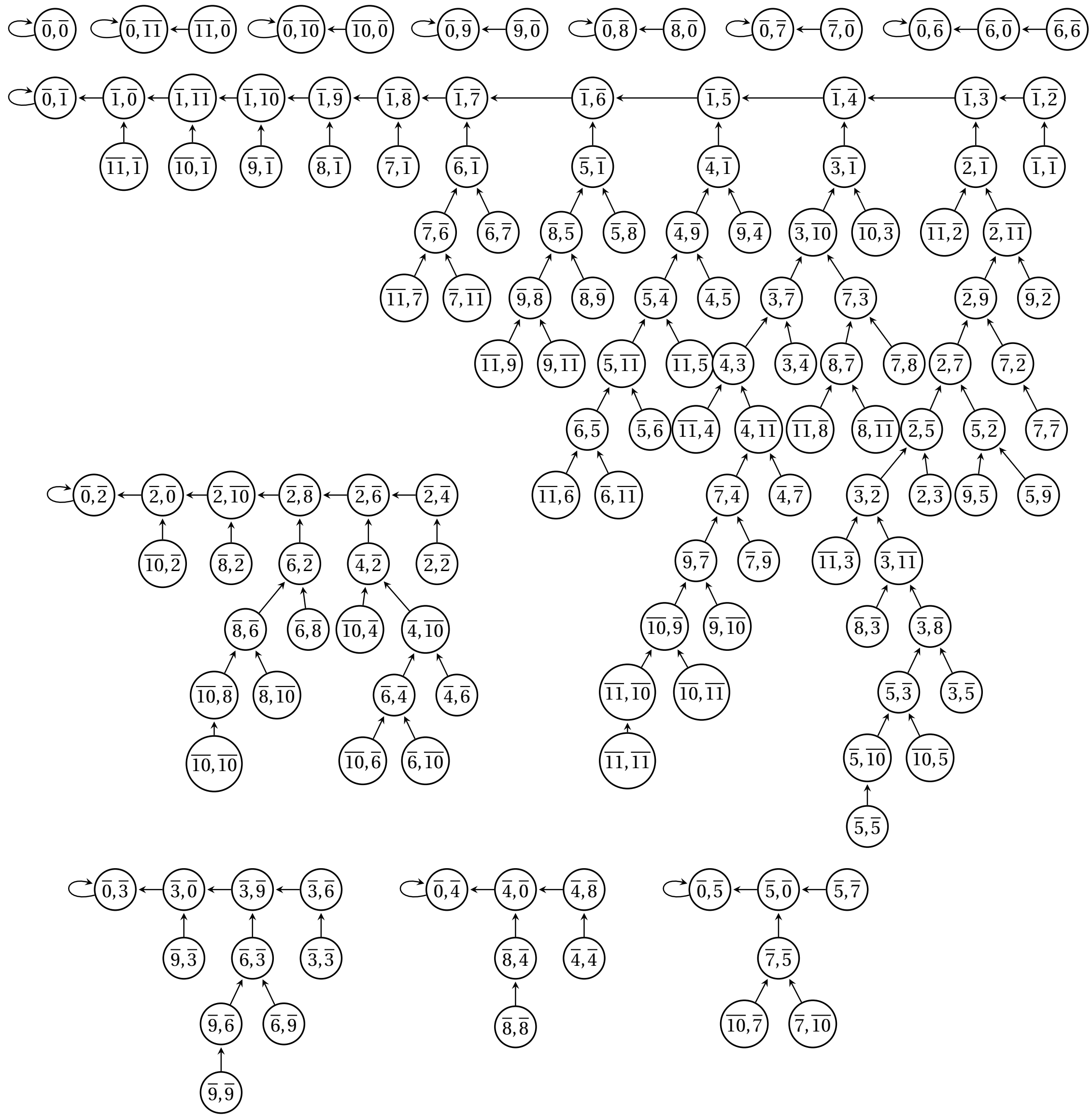

Figure 4: $\Psi(S_n)$, $n = 12$

Conversely, the second conjecture illustrates that adjacent sources are merely adjacent additions to the directed graph.

Conjecture 4.3. The exists a path from the set of true sources in $\Psi(S_n)$ to every vertex that is not an adjacent source.

Conjecture 4.4. There does not exist an adjacent source $(\overline{x}, \overline{y}) \in \Psi(S_n)$ that points to a vertex with in-degree 1.

The final conjecture from our research inspects the form of vertices in special components.

Conjecture 4.5. For $\overline{a} \in S_n$ where $a|n$, the component containing the vertex $(\overline{0}, \overline{a})$ consists only of elements from the generating set $\langle a \rangle$, the ideal generated by a.

This phenomena is best illustrated by the components in Figure 4 that include $(\overline{0}, \overline{2})$, $(\overline{0}, \overline{3})$, and $(\overline{0}, \overline{4})$.

Additional ideas to investigate include determining a general form for each type of vertex, identifying a model that predicts the maximal distance of a directed graph, and searching for a pattern in the number of vertices in each component. However, our ideas for research are from exhaustive. The study of tropical geometry in relation to graph theory boasts numerous opportunities for further study.

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