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Cover Page Footnote
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The Degeneration of the Hilbert Metric on Ideal Pants and its Application to Entropy

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Abstract. Entropy is a single value that captures the complexity of a group action on a metric space. We are interested in the entropies of a family of ideal pants groups Γ_T, represented by projective reflection matrices depending on a real parameter $T > 0$. These groups act on convex sets Ω_{Γ_T} which form a metric space with the Hilbert metric. It is known that entropy of Γ_T takes values in the interval $\left(\frac{1}{2}, 1\right]$; however, it has not been proven whether $\frac{1}{2}$ is the sharp lower bound. Using Python programming, we generate approximations of tilings of the convex set in the projective plane and estimate the entropies of these groups with respect to the Hilbert metric. We prove a theorem that, along with the images and data produced by our code, suggests that the lower bound is indeed sharp. This theorem regards the degeneration of the Hilbert metric on the convex set Ω_{Γ_T}.

1 Introduction

Entropy is a measure of chaos or disorder, both inside and outside the field of mathematics. In mathematics, we define entropy more precisely as a number that captures the complexity of certain group actions. Despite its usefulness in topology and geometry and the variety of formulas that describe it, entropy is notoriously difficult to compute. In order to better understand entropy, it would be beneficial to have a tool to approximate it. As coding languages become more accessible, there is reason to believe that a software program could approximate entropy.

By using this tool in the context of projective geometry, we bring a rich theory into a more flexible setting where there is plenty of active research and much unknown, as referenced in SECTION 1.1. In particular, Xin Nie has shown that the entropy of projective triangle groups takes all values in the interval $(0, 1]$ [11]. Analogously, it is natural to ask if such a statement might hold for ideal projective pants groups.

Motivated by the difficulty of calculating the entropy of ideal projective pants groups and Nie's statement on the entropy of projective triangle groups, we produced Python

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code which approximates the entropy corresponding to the action of a given ideal projective pants group, as discussed in SECTION 6. The results from our code mirror those of Nie's, suggesting the degeneration of entropy in this new setting. Visual tilings generated by the code provide an avenue for proving our first main result of this paper, given by the following theorem, discussed and proven in SECTION 4:

Theorem 1.1 (Degeneration of the Hilbert Metric)**.** *As* T *goes to* ∞*, the convex set* $\Omega_{\Gamma_{\mathrm{T}}}$ *which is tiled by the ideal pants group* Γ_{T} *converges to a triangle.*

Though this theorem only explicitly states the convergence of a tiling, it implies that the Hilbert metric, which we use in our definition for entropy (Definition 2.2), converges to the Hilbert metric on a triangle as T approaches infinity. Indeed, when $\mathrm{T} = 1$, Ω_{Γ_1} is elliptic and the geometry is hyperbolic, and the group action of Γ_{T} has an entropy of 1. As T approaches infinity, the set $\Omega_{\Gamma_{\mathrm{T}}}$ which the metric is defined on changes from an ellipse to a triangle. This triangle has minimal entropy, since polygons with the Hilbert metric are quasi-isometric to $\mathbb{R}^2$ [9]. The metric is now defined on a polygon and has lost its sense of complexity, becoming something simple; this is what we mean when we say the Hilbert metric degenerates.

Estimations of entropy generated from our code also suggested a symmetric nature of entropy around the value of $\mathrm{T} = 1$. These observations gave rise to the second main result of this paper, discussed and proven in SECTION 5:

Theorem 1.2 (Duality Invariance of Entropy)**.** *For the ideal pants groups* Γ_{T} *and* $\Gamma_{\frac{1}{\mathrm{T}}}$ *with* $\mathrm{T} \in \mathbb{R}_{>0}$*, their entropies are equal; that is,* $h_{\Gamma_{\mathrm{T}}} = h_{\Gamma_{\frac{1}{\mathrm{T}}}}$.

This hinges on results from Benoist, Crampon-Marquis, Danciger-Guéritaud-Kassel, Kassel-Potrie, and Cooper-Long-Tillman, among others [3, 7, 8, 10, 5]. See SECTION 5 for more precise statements and references.

Finally, our coded approximations presented in SECTION 6 serve as strong evidence that the entropy of the family of ideal projective pants groups, $h_{\Gamma_{\mathrm{T}}}$ (from [12]), takes all values in the interval $\left(\frac{1}{2}, 1\right]$, which motivates Conjecture 3.2.

1.1 Historical Context

Within the past decade, there have been many papers published on entropies in projective space, especially in relation to Hilbert geometry. Beginning in 2009, Crampon showed that any divisible strictly convex set in $\mathbb{RP}^n$ has entropy less than or equal to $n-1$, with equality if and only if the Hilbert geometry is hyperbolic (ie. Ω is an ellipsoid and supports the projective model of hyperbolic geometry) [7]. And as recently as 2017, Barthelme, Marquis, and Zimmer showed that if a properly convex set admits a proper, free, finite co-volume group action by automorphisms, the entropy is equal to $n-1$ if and only if the Hilbert metric is Riemannian [2]. These two papers both look at the

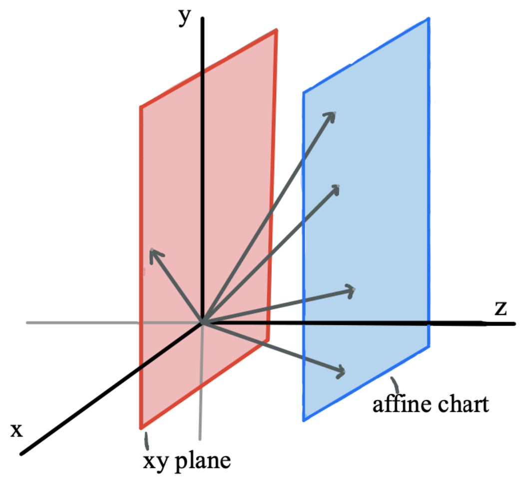


Figure 1: A visual representation of $\mathbb{RP}^2$

entropy of properly convex sets in real projective space, similar to what we set out to explore. Finally, another recent piece of research that directly relates to our work is Nie's paper, "On the Hilbert geometry of simplicial Tits sets" [11]. Nie's work with the entropy of projective triangle groups features an entropy degeneration result. This inspired our original conjectures about ideal pants and will be further discussed in this paper.

2 Preliminaries

2.1 Real Projective Plane

We work in the real projective plane, $\mathbb{RP}^2$, which can be thought of as the set of lines through the origin in $\mathbb{R}^3$. A quirk of projective geometry is that there are no parallel lines. Two seemingly parallel lines are said to meet at infinity, just as the rails of a railroad track appear to meet at some point on the horizon [1]. One way to conceptualize $\mathbb{RP}^2$ is to pick any affine plane in $\mathbb{R}^3$. For example, consider the affine plane $\{(x, y, z) \in \mathbb{R}^3 \mid z = 1\}$. All the lines in $\mathbb{R}^3$ through the origin intersect this plane exactly once, except for the lines in the xy-plane themselves. This we designate as the 'plane at infinity'. Thus, we can view $\mathbb{RP}^2$ as equivalent to $\mathbb{R}^2 \cup$ (the 'plane at infinity') where the affine plane (also called *affine chart*) chosen earlier represents $\mathbb{R}^2$ [1].

2.2 Hilbert Metric

The metric we are using for computing and approximating entropy is the Hilbert metric which is defined for a properly convex, open set. A *properly convex,* open set is one that is open, bounded, and convex. The Hilbert distance between two points x, y in such a

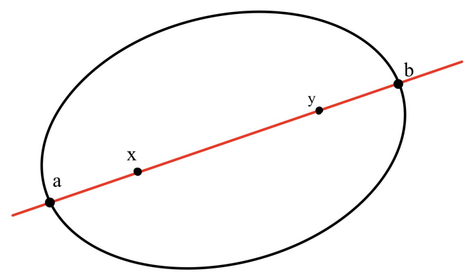


Figure 2: Hilbert metric visual: a and b are the points where the line intersects the boundary, $\partial\Omega$.

set, $\Omega \subset \mathbb{R}^3$, is given by

$$d_\Omega(x, y) := \frac{1}{2} \log \frac{|x-b||y-a|}{|x-a||y-b|},$$

where a and b are the points where the line through x and y meets the boundary of $\overline{\Omega}$ (the closure of Ω). Using the notation above, a is chosen so that it is closer to x and b is closer to y in the Euclidean metric (See Fig. 2). Although we will not verify that this indeed defines a metric, we note that the cross-ratio

$$[a, x, y, b] := \frac{|x-b||y-a|}{|x-a||y-b|} \geq 1,$$

guaranteeing that $d_\Omega(x, y)$ is non-negative. Finally, we also observe that as $x \to a$ or $y \to b$, the cross-ratio increases rapidly along with the Hilbert distance between x and y. Thus, in the Hilbert distance, the boundary of a properly convex set is infinitely far away.

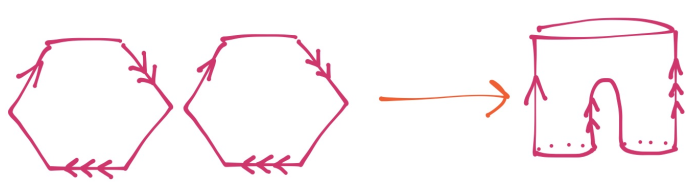


Figure 3: Forming a pair of pants.

2.3 Ideal Pants Group

If we identify every other side of two hexagons to each other, we get a topological object called a *pair of pants* (see Fig. 3).

By pinching and stretching the openings to 'infinity,' we create an *ideal pair of pants*. Cutting along the seams of these ideal pants yields 'ideal triangles' with vertices at 'infinity' (see Fig. 4).

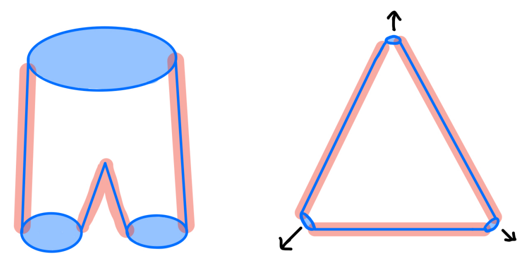


Figure 4: A topological pair of pants becoming an ideal pair of pants

The inspiration for this research came from Xin Nie's work on projective triangle groups which are generated by projective reflections over the sides of a compact triangle in $\mathbb{RP}^2$. Similarly, the group, Γ, associated to an pair of ideal (projective) pants is generated by projective reflections over the sides of the ideal triangles. Geometrically, we can think of the group as unfolding the pants onto the projective plane. In terms of computation, we can represent these projective reflections as projective transformations in $\mathbb{R}^3$. Then, the matrix representation of Γ is a subgroup of $\mathrm{PSL}(3,\mathbb{R})$. By exploring ideal pants, the project looks at a noncompact extension of the results of Nie which will be precisely stated in SECTION 3.

Note 2.1. Since we will be only considering ideal pants and projective reflections in this paper, we will often write *pants group* in place of ideal projective pants group.

2.4 Entropy

In a mathematical setting, entropy measures how 'chaotic' a group action is by examining how far away each group element carries an object in a metric space. There are multiple existing definitions for entropy, but the one primarily used in this project is given below.

Definition 2.2. Let (X, d) be a metric space and Γ be a group acting properly discontinuously on X by isometries. The *entropy* of group Γ, h_Γ, is given by

$$h_\Gamma = \limsup_{n\to\infty} \frac{\log \#\{\gamma \in \Gamma \mid d(o, \gamma \cdot o) \le n\}}{n},$$

for an arbitrary $o \in X$, and where n is a positive real number.

Roughly speaking, the entropy of Γ gives the *exponential growth rate* of 'the number of elements in Γ that move o within an increasing distance.' In this way, we can think of entropy as a value describing the complexity of a group as it acts on a particular metric space. In the context of our research, the metric space is the properly convex set Ω_{Γ_T} endowed with the Hilbert metric. The projective pants group, Γ_T acts properly discontinuously on Ω_{Γ_T} by isometries, so it is natural to explore the entropy of Γ_T.

In general, given a one-parameter family of groups Γ_T, $T \in \mathbb{R}_{>0}$ that acts by isometries on a metric space (X_T, d_{X_T}), it is known that the function which maps $T \mapsto h_{\Gamma_T}$ is continuous [4]. In his previous work on projective triangle groups, Nie uses this fact to give a non-constructive proof of the degeneration of entropy in that setting. Taking this as inspiration, we hope that Nie's result can be extended to the setting of ideal projective pants in a similar way.

2.5 Tiling

Given a group, Γ, that acts by isometries on a metric space, (X, d), there is a notion of tiling induced by that group. More specifically, we give the following definition.

Definition 2.3. Γ tiles X with generating polygon P provided that:

1. $X = \bigcup_{\gamma \in \Gamma} \gamma \cdot P$
2. $\operatorname{int}(\gamma \cdot P) \cap \operatorname{int} P \neq \emptyset \iff \gamma = 1.$

In our research, a projective pants group, Γ_T, acts on a properly convex set, Ω_{Γ_T}, equipped with the Hilbert metric. While the group can be represented by matrices in $\mathrm{PSL}(3,\mathbb{R})$, it is unclear what the convex set looks like. To resolve this, we examine approximations of tilings of the convex set. Given a triangle placed inside an affine chart and satisfying the above properties, we can use the group action on the triangle to get a picture of Ω_{Γ_T} inside an affine chart. Furthermore, if we consider the quotient $\Omega_{\Gamma_T}/\Gamma_T$, we recover the pair of projective pants we started with. In this way, a pair of projective pants is associated with both a group and a tiling.

The generating polygons in our tilings, the ideal triangles, touch the boundaries of the convex set at their vertices. This is consistent with the notion of ideal pants and the Hilbert metric: by placing the vertices of the triangles on the boundary of the convex set, we pinch and stretch the openings of our pants to 'infinity.'

Using computer programming, we can generate approximations of Γ_T and also approximations of Ω_{Γ_T}. The visuals generated by these tilings show a deformation of the properly convex set, Ω_{Γ_T} as $T \to \infty$. This observation of deformation in the tiling originally pointed us to our initial conjecture: the degeneration of entropy in the setting of ideal projective pants (Conjecture 3.2). While this does not offer a complete proof, it allows us to show the degeneration of the Hilbert metric on Ω_{Γ_T} which suggests that Conjecture 3.2 is true.

3 Background

Previous work by Xin Nie [11] explored projective triangle groups which have a similar construction to pants groups. When viewed in an affine chart of $\mathbb{RP}^2$, both groups generate tilings of a properly convex open set (endowed with the Hilbert metric) by acting on triangles in the chart. For projective triangle groups, the triangles in the tiling are strictly contained inside the properly convex, open set. This tiling phenomenon was ultimately used to prove the degeneration of entropy in that setting, the main theorem in Nie's paper.

Theorem 3.1 (Nie, [11, Theorem 1])**.** *Given a one parameter family of projective triangle groups, Γ_T, the function sending the group to its entropy, $\mathrm{T} \mapsto h_{\Gamma_\mathrm{T}}$, has image $(0,1]$.*

In the case of pants groups, Γ_T, however; the action takes place on triangles that touch the boundary of the closure of the properly convex set $\Omega_{\Gamma_\mathrm{T}}$. We hope to extend Nie's result to this noncompact setting, albeit with some slight modification.

Conjecture 3.2. *Given a one parameter family of pants groups, Γ_T, the function sending the group to its entropy, $\mathrm{T} \mapsto h_{\Gamma_\mathrm{T}}$, has image $\left(\frac{1}{2},1\right]$.*

A few results from hyperbolic and projective geometry give us some insight into this problem.

3.1 Representation of the Pants Group

First, let us look deeper into Nie's aforementioned paper [11] for a representation of pants groups. Using the notation from this paper, we consider a special case of the Theorem on page 6, referring to results of Tits and Vinberg [12], taking dimension $m = 2$, with the order of generators $m_{i,i} = 2$, and $m_{i,j} = \infty$ for all $i \neq j$. This Theorem gives us a recipe for computing the generators of the pants groups in terms of matrices in $\mathrm{PSL}(3,\mathbb{R})$.

$$\Gamma_\mathrm{T} := \left\langle \begin{pmatrix} -1 & 0 & 0 \\ 2\mathrm{T} & 1 & 0 \\ \frac{2}{\mathrm{T}} & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & \frac{2}{\mathrm{T}} & 0 \\ 0 & -1 & 0 \\ 0 & 2\mathrm{T} & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 2\mathrm{T} \\ 0 & 1 & \frac{2}{\mathrm{T}} \\ 0 & 0 & -1 \end{pmatrix} \right\rangle \qquad \mathrm{T} > 0.$$

Throughout this paper, the matrix generators of the group Γ_T are referred to as $\mathrm{R}_{\mathrm{T}1}$, $\mathrm{R}_{\mathrm{T}2}$, and $\mathrm{R}_{\mathrm{T}3}$, respective to the matrices above, left to right. Indeed, the version of Vinberg's theorem stated by Nie gives us that any one-parameter family of pants groups is conjugate to Γ_T.

By construction, the fundamental domain for this family of representations Γ_T acting on $\Omega_{\Gamma_\mathrm{T}}$ is "the fundamental simplex" P, defined on page 5 in the preliminaries of Nie's paper. In dimension 2, P is the first octant of $\mathbb{R}^3$, or in other words, the positive cone over the interior of the triangle with vertices e_1, e_2, e_3.

It is worth observing that $R_{T,1}$ fixes span(e_2, e_3), $R_{T,2}$ fixes span(e_1, e_3), and $R_{T,3}$ fixes span(e_1, e_2). So it is believable that the fundamental simplex P is indeed a fundamental domain for the action of Γ_T on Ω_T; we refer the reader to Nie's paper for a proof of this fact.

Now, it is a straightforward calculation to show that $R_{T,2}R_{T,3}$ is a unipotent matrix and hence is a parabolic isometry. It fixes only one point, e_1, which we can see quickly because e_1 is the unique intersection point of the fixed point sets span(e_1, e_3) and span(e_1, e_2) of $R_{T,2}$ and $R_{T,3}$, respectively. Then e_1 must be in the boundary of Ω because $R_{T,2}R_{T,3}$, being parabolic, has infinite order and moves points towards e_1.

3.2 Entropy of the Pants Group

Next, the hyperbolic pants group, Γ_1, has been computed to have an entropy of 1, and all other groups in the family, Γ_T for $T \neq 1$, are known to have a smaller entropy [2]. It also is known that the entropy of a given pants groups is strictly greater than $\frac{1}{2}$ [7]. However, it is unknown exactly what im$(T \mapsto h_{\Gamma_T})$ is; i.e. it has not been proven that $\frac{1}{2}$ is a sharp lower bound of this mapping. Our project involves making observations about pants groups and creating approximations of their entropies to gain intuition and build support for the conjecture.

While making these approximations of entropy, we also present approximations of the convex set Ω_{Γ_T} generated by approximating the Γ_T-tilings. From these pictures, we observed a deformation of the tiling of Ω_{Γ_T} to its "base triangle," described further in Theorem 4.1. While this does not prove Conjecture 3.2, it provides strong evidence that it is true and can be interpreted as a degeneration of the Hilbert metric on Ω_{Γ_T} as $T \to \infty$. In terms of Conjecture 3.2, this interpretation shows that the space Ω_{Γ_T} equipped with the Hilbert metric looks 'more like' $\mathbb{R}^2$ with the Euclidean metric as $T \to \infty$. As a Euclidean space, actions on $\mathbb{R}^2$ have minimum entropy. And in this way, Theorem 1.1 suggests Conjecture 3.2.

Note 3.3. Before we look at any visualizations, a note for clarity: the visuals in this paper are all two-dimensional. A natural question that the reader may have is how a group generated by three-dimensional matrices has become a two-dimensional tiling. Even though the generators are 3×3 matrices, the reason we have two-dimensional figures for different groups is because we are working in the projective space $\mathbb{RP}^2$, which is a projection of $\mathbb{R}^3$ onto a two-dimensional affine chart in union with the plane at infinity. Thus, we 'lose' a dimension moving from the group to a visualization on a two dimensional plane.

Now having defined all necessary terms and given thought to the practicality of Theorem 1.1, we move on to one outcome of our project.

4 Proof of Degeneration of Hilbert Metric

As we worked on our code and building empirical evidence for Conjecture 3.2, we made interesting observations which lead us to Theorem 1.1 and several other results to prove it. This section details these results and their proofs.

4.1 Convergence to Base Triangle

With our code's visual outputs (detailed in SECTION 6), we observe that for the family of ideal projective pants groups Γ_T varying under a real value T, the properly open convex set Ω_{Γ_T} converges to a triangle as T increases. This convergence becomes evident very quickly, even for T = 16, and can be seen in Figure 6. We turn our focus to proving this observation, stated in Theorem 1.1:

Theorem (Degeneration of Hilbert Metric)**.** As T goes to ∞, Ω_{Γ_T} converges to the base triangle.

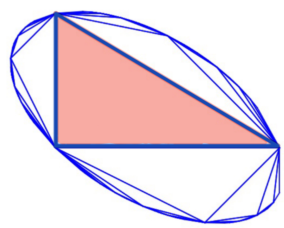


Figure 5: A pants tiling of Ω_{Γ_1} with the base triangle highlighted in red

By "base triangle," we refer to the triangle which the group elements of Γ_T act upon. This triangle is chosen based on the attracting eigenvectors of each matrix generator. For the relevant images in this report, we take the base triangle to be simply the triangle with vertices at e_1, e_2, e_3 (the standard basis vectors) projected onto a fixed affine chart. More on this specific affine chart and how it is chosen can be found in SECTION 6. By the way our matrix generators are chosen, no matter the value of T, Ω_{Γ_T} must contain this triangle.

4.2 Results Leading to Theorem 1.1

To prove Theorem 1.1, we proceed to define a triangle in the affine chart which bounds Ω_{Γ_T} and converges to the base triangle. It then follows by the squeeze theorem that Ω_{Γ_T} converges to the base triangle, as desired. The following five lemmas and their proofs define such a bounding triangle, prove its containment of Ω_{Γ_T} and prove its convergence to the base triangle.

First, we identify the vertices of the desired bounding triangle, $f_{\mathrm{T}1}, f_{\mathrm{T}2}, f_{\mathrm{T}3}$ to be the eigenvectors of the matrix generators $\mathrm{R}_{\mathrm{T}1}, \mathrm{R}_{\mathrm{T}2}, \mathrm{R}_{\mathrm{T}3}$ respectively, with eigenvalue of -1. Lemmas 4.1 and 4.2 are necessary for use in Lemma 4.3, which proves that the bounding triangle contains $\Omega_{\Gamma_{\mathrm{T}}}$.

Note 4.1. Recall from SECTION 3, that the matrices $\mathrm{R}_{\mathrm{T}1}, \mathrm{R}_{\mathrm{T}2}$, and $\mathrm{R}_{\mathrm{T}3}$ depend on T since they are associated to the group Γ_{T}. We have moved the parameter T to the subscript to clean up the notation (writing $\mathrm{R}_{\mathrm{T}1}$ instead of $\mathrm{R}_1(\mathrm{T})$. We introduce a similar format for the eigenvectors of each matrix: $f_{\mathrm{T}1}, f_{\mathrm{T}2}$, and $f_{\mathrm{T}3}$.

Lemma 4.1. *Let i, j, k be three distinct numbers in $\{1,2,3\}$ and $\mathrm{R}_{\mathrm{T}i}, \mathrm{R}_{\mathrm{T}j}$ be two generators of the group Γ_{T}. Then for $f_{\mathrm{T}i}, f_{\mathrm{T}j}$, the eigenvectors of $\mathrm{R}_{\mathrm{T}i}, \mathrm{R}_{\mathrm{T}j}$ respectively with eigenvalue -1, we have $f_{\mathrm{T}i} + f_{\mathrm{T}j} = e_k$.*

Proof. By symmetry, it suffices to check this for the case of $f_{\mathrm{T}1}$ and $f_{\mathrm{T}2}$. The statement of the lemma for other values of i, j can be proven by a similar method. Let $f_{\mathrm{T}1}$ and $f_{\mathrm{T}2}$ be the eigenvectors of $\mathrm{R}_{\mathrm{T}1}$ and $\mathrm{R}_{\mathrm{T}2}$ both corresponding to the eigenvalue $\lambda = -1$. We can compute $f_{\mathrm{T}2}$ and $f_{\mathrm{T}2}$ by looking at the system of equations generated by $\mathrm{R}_{\mathrm{T}1} + \mathrm{I}$ and $\mathrm{R}_{\mathrm{T}2} + \mathrm{I}$ respectively. Solving the system of equations produces the following eigenvectors:

$$f_{\mathrm{T}1} = \begin{pmatrix} -1 \\ \mathrm{T} \\ \frac{1}{\mathrm{T}} \end{pmatrix}, \quad f_{\mathrm{T}2} = \begin{pmatrix} -\frac{1}{\mathrm{T}} \\ 1 \\ -\mathrm{T} \end{pmatrix}.$$

Before adding the eigenvectors together, we multiply by scalars to simplify the summation. Since we are ultimately working in $\mathbb{RP}^2$, this scaling is permitted. Multiply $f_{\mathrm{T}1}$ by T and $f_{\mathrm{T}2}$ by T^2, to get

$$f_{\mathrm{T}1} = \begin{pmatrix} -\mathrm{T} \\ \mathrm{T}^2 \\ 1 \end{pmatrix}, \quad f_{\mathrm{T}2} = \begin{pmatrix} \mathrm{T} \\ -\mathrm{T}^2 \\ \mathrm{T}^3 \end{pmatrix}$$

and we can observe that

$$f_{\mathrm{T}2} + f_{\mathrm{T}1} = \begin{pmatrix} 0 \\ 0 \\ 1 + \mathrm{T}^3 \end{pmatrix} = (1 + \mathrm{T}^3) e_3.$$

Thus, barring scaling, we see that $f_{\mathrm{T}1} + f_{\mathrm{T}2} = e_3$. □

Thus, we have that the vertices of the base triangle are linear combinations of the vertices of the bounding triangle.

Lemma 4.2. *Let $f_{\mathrm{T}n}$ be the eigenvector for the generator $\mathrm{R}_{\mathrm{T}n}$ which corresponds to the eigenvalue -1. The projective line through $f_{\mathrm{T}i}$ and $f_{\mathrm{T}j}$, for $i, j \in \{1,2,3\}$ $i \neq j$, is preserved by the generators $\mathrm{R}_{\mathrm{T}i}$ and $\mathrm{R}_{\mathrm{T}j}$.*

Proof. Let $i = 1$, $j = 2$. We can see the projective line through f_{T1} and f_{T2} is a subspace containing both f_{T1} and f_{T2}. In projective space, f_{T1} being the eigenvector of R_{T1} for the eigenvalue of -1 means that f_{T1} is preserved by R_{T1}. If we can show that R_{T1} sends f_{T2} to a point on the projective line through f_{T1} and f_{T2} then R_{T1} will preserve the entire projective line. We have found the following matrices for the general form of R_{T1}, f_{T2} with parameter T.

$$R_{T1} = \begin{pmatrix} -1 & 0 & 0 \\ 2T & 1 & 0 \\ \frac{2}{T} & 0 & 1 \end{pmatrix}, f_{T2} = \begin{pmatrix} -\frac{1}{T} \\ 1 \\ -T \end{pmatrix}$$

We find the following vector for $R_{T1} f_{T2}$ and can break it into a linear combination of f_{T1} and f_{T2}.

$$R_{T1} f_{T2} = \begin{pmatrix} \frac{1}{T} \\ -1 \\ -T - \frac{2}{T^2} \end{pmatrix} = -\frac{2}{T} \begin{pmatrix} -1 \\ T \\ \frac{1}{T} \end{pmatrix} + \begin{pmatrix} \frac{-1}{T} \\ 1 \\ -T \end{pmatrix} = -\frac{2}{T} f_{T1} + f_{T2}$$

We have shown that $R_{T1} f_{T2}$ is on the projective line between f_{T1} and f_{T2}, which means that R_{T1} preserves the entire projective line.

To show R_{T2} preserves the projective line, we similarly show that $R_{T2} f_{T1}$ is on the line. Computing, we have

$$R_{T2} f_{T1} = \begin{pmatrix} 1 \\ -T \\ 2T^2 + \frac{1}{T} \end{pmatrix} = \begin{pmatrix} -1 \\ T \\ \frac{1}{T} \end{pmatrix} + -2T \begin{pmatrix} \frac{-1}{T} \\ 1 \\ -T \end{pmatrix} = f_{T1} + -2T f_{T2}.$$

We have shown that $R_{T2} f_{T1}$ can be written as a linear combination of f_{T1} and f_{T2}, meaning that $R_{T2} f_{T1}$ is on the projective line. We can conclude that R_{T2} also preserves this projective line through f_{T1} and f_{T2}.

The other cases follow by symmetry. □

Equipped with these two lemmas, we are now ready to prove that the bounding triangle contains Ω_{Γ_T}.

Lemma 4.3. *For $n = 1, 2, 3$, let f_{Tn} be the eigenvector for the generator R_{Tn} which corresponds to the eigenvalue -1, and let Ω be the properly convex set generated by $\langle R_{Tn} \rangle$. Then the triangle formed by f_{T1}, f_{T2}, and f_{T3} contains Ω.*

Proof. Let A be an affine chart containing Ω. Suppose the triangle $\triangle'$ formed by f_{T1}, f_{T2}, and f_{T3} on A does not contain Ω. Note that $f_{T1} + f_{T2} = e_3 \in \partial\Omega$, $f_{T1} + f_{T3} = e_2 \in \partial\Omega$, and $f_{T2} + f_{T3} = e_1 \in \partial\Omega$, as shown in Lemma 4.1. Fix distinct $i, j, k \in \{1, 2, 3\}$ and let L denote the line containing f_{Ti} and f_{Tj}. Then we know $L \cap \partial\Omega = e_k$. Since $\triangle'$ does not contain Ω, L must intersect $\partial\Omega$ at another point p, other than e_k. Because $\partial\Omega$ is preserved under

Γ and, as proven earlier in Lemma 4.2, L is preserved under $R_{Ti}R_{Tj}$, either $R_{Ti}R_{Tj}p = p$ or $R_{Ti}R_{Tj}p = e_k$. But we already know that $R_{Ti}R_{Tj}e_k = e_k$, so we must have $R_{Ti}R_{Tj}p = p$, so e_k and p are both eigenvectors of $R_{Ti}R_{Tj}$. $R_{Ti}R_{Tj}$ is parabolic, meaning it only has one eigenvector.[1] Thus, $p = e_k$, which contradicts our definition of p. Thus, L is tangent to $\partial\Omega$ and $\triangle'$ contains Ω. $\square$

So, we have that the bounding triangle defined by f_{T1}, f_{T2}, f_{T3} contains Ω_{Γ_T}. The next step toward proving Theorem 1.1 is to prove that the bounding triangle converges to the base triangle using the notion of Gromov-Hausdorff convergence. That is, we say that a sequence of sets converges to another set if the Gromov-Hausdorff distance between the sets goes to 0. To prove that the bounding triangle converges to the base triangle, then, we need to show that the Gromov-Hausdorff distance between between the bounding triangles and the base triangles goes to 0. We do this in two lemmas, first proving that the vertices of the bounding triangle converge to the vertices of the base triangle, and second proving that the sides of the bounding triangle converge to the sides of the base triangle.

Lemma 4.4. *Inside an affine chart containing e_1, e_2, and e_3, the vertices of the bounding triangle converge to the vertices of the base triangle in the following way:*

$$f_{T1} \to e_2 \qquad f_{T2} \to e_3 \qquad f_{T3} \to e_1 \qquad \textit{as } T \to \infty.$$

Proof. Recall that

$$f_{T1} = \begin{pmatrix} 1 \\ -T \\ \frac{-1}{T} \end{pmatrix}, \; f_{T2} = \begin{pmatrix} \frac{-1}{T} \\ 1 \\ -T \end{pmatrix}, \; \text{and} \; f_{T3} = \begin{pmatrix} -T \\ \frac{-1}{T} \\ 1 \end{pmatrix}.$$

Consider now an affine chart containing e_1, e_2, and e_3; that is, one determined by a plane $\{ax + by + cz = 1\}$ for $a, b \neq 0$. Let's first place the base triangle in the chart by projecting it onto the plane $\{ax + by + cz = 1\}$ and then projecting it down onto the xy-plane.

$$e_1 \mapsto \frac{1}{a}e_1 \mapsto \begin{pmatrix} \frac{1}{a} \\ 0 \end{pmatrix} \qquad e_2 \mapsto \frac{1}{b}e_2 \mapsto \begin{pmatrix} 0 \\ \frac{1}{b} \end{pmatrix} \qquad e_3 \mapsto \frac{1}{c}e_3 \mapsto \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Similarly, we can also place the vertices of the bounding triangle, f_{T1}, f_{T2}, and f_{T3}, inside the chart and see where they move as T tends to infinity.

[1] A concrete proof of $R_{T1}R_{T2}$ having only one eigenvector can be found in Appendix A.

$$f_{\mathrm{T}1} \longmapsto \frac{1}{a-b\mathrm{T}-\frac{c}{\mathrm{T}}}\begin{pmatrix}1\\-\mathrm{T}\\\frac{-1}{\mathrm{T}}\end{pmatrix} \longmapsto \begin{pmatrix}\frac{1}{a-b\mathrm{T}-\frac{c}{\mathrm{T}}}\\\frac{-\mathrm{T}}{a-b\mathrm{T}-\frac{c}{\mathrm{T}}}\end{pmatrix} \xrightarrow[\mathrm{T}\to\infty]{} \begin{pmatrix}0\\\frac{1}{b}\end{pmatrix} = e_2$$

$$f_{\mathrm{T}2} \longmapsto \frac{1}{\frac{-a}{\mathrm{T}}+b-c\mathrm{T}}\begin{pmatrix}\frac{-1}{\mathrm{T}}\\1\\-\mathrm{T}\end{pmatrix} \longmapsto \begin{pmatrix}\frac{-1}{-a-b\mathrm{T}-c\mathrm{T}^2}\\\frac{1}{\frac{-a}{\mathrm{T}}+b-c\mathrm{T}}\end{pmatrix} \xrightarrow[\mathrm{T}\to\infty]{} \begin{pmatrix}0\\0\end{pmatrix} = e_3$$

$$f_{\mathrm{T}3} \longmapsto \frac{1}{-a\mathrm{T}-\frac{b}{\mathrm{T}}+c}\begin{pmatrix}-\mathrm{T}\\\frac{-1}{\mathrm{T}}\\1\end{pmatrix} \longmapsto \begin{pmatrix}\frac{-\mathrm{T}}{-a\mathrm{T}-\frac{b}{\mathrm{T}}+c}\\\frac{1}{a\mathrm{T}^2+b-c\mathrm{T}}\end{pmatrix} \xrightarrow[\mathrm{T}\to\infty]{} \begin{pmatrix}\frac{1}{a}\\0\end{pmatrix} = e_1$$

Thus, $f_{\mathrm{T}1} \to e_2$, $f_{\mathrm{T}2} \to e_3$, and $f_{\mathrm{T}3} \to e_1$, so that the vertices of the bounding triangle converge to those of the base triangle inside a suitable chart. □

Finally, we must show that the edges of the bounding triangle converge to the edges of the base triangle by Gromov-Hausdorff convergence.

Lemma 4.5. *Let ℓ be a line segment in $\mathbb{R}^2$, and let ℓ_{T} be a sequence of line segments whose endpoints converge to those of ℓ. More specifically, suppose ℓ has endpoints x, y and ℓ_{T} has endpoints $x_{\mathrm{T}}, y_{\mathrm{T}}$ where $x_{\mathrm{T}} \to x$ and $y_{\mathrm{T}} \to y$ as T tends to ∞. Assume furthermore that for all T, the intersection of the segments ℓ and ℓ_{T} is x_{T}. Then, ℓ_{T} converges to ℓ as T tends to ∞ with respect to the Gromov-Hausdorff distance.*

This can be proven using parameterization and convexity of the metric on $\mathbb{R}^2$. However, we provide an alternate proof in APPENDIX A labeled as Proof (A). Using these results, we are now able to prove Theorem 1.1.

Proof of Theorem 1.1. By Lemma 4.5 we know that the sides of the bounding triangles converge to the sides of the base triangle. We also know from lemma 4.4 that the vertices of the bounding triangle converge to the vertices of the base triangle. Knowing this is true, assume the two triangles do not converge to each other. Then there is at least one point in one of the triangles that is not within the set of the other as T goes to infinity. However, this cannot be in a triangle side or a vertex. Thus, this point cannot exist and we have a contradiction. We now know our assumption is false and the two triangles must converge. □

As mentioned in SECTION 1, Theorem 1.1 can be interpreted as a degeneration of the Hilbert metric. The main idea in this interpretation is the fact that a triangle endowed with the Hilbert metric is quasi-isometric to $\mathbb{R}^2$ with the Euclidean metric [9]. As $\Omega_{\Gamma_{\mathrm{T}}}$ converges, its geometry degenerates, becoming 'flatter.' In terms of the entropy of Γ_{T}, this flattening of the geometry is evidence for a decrease in h_{Γ_T}. The extent of this decrease and the bounds on h_{Γ_T} can be explored using a computer program, detailed in SECTION 6. In total, the results and approximations strongly suggest the sharp bounds mentioned in Conjecture 3.2.

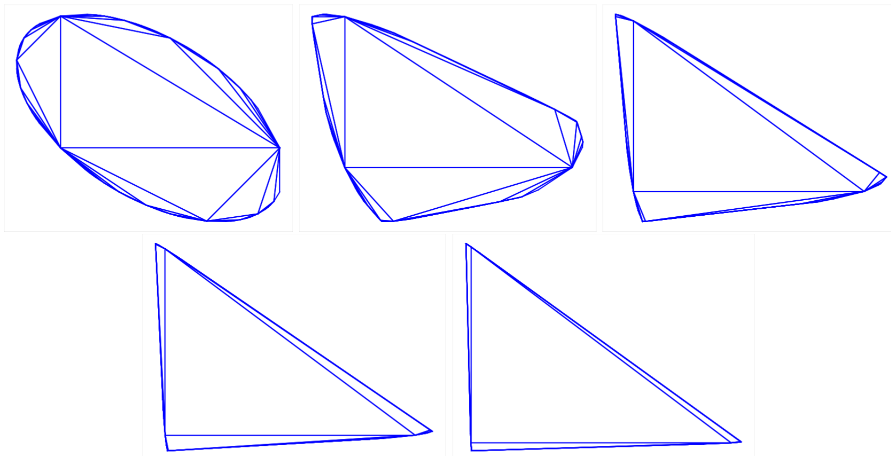


Figure 6: Projection of Ω_{Γ_T} onto the xy-plane with T values 1, 2, 4 on top , and 8, 16 on bottom.

5 Proof of Duality Invariance of Entropy

While developing our code, we also found empirical evidence to support the duality invariance of entropy. In this section, we build support and work to prove our other main result, Theorem 1.2.

5.1 Projective Dual

Our results from SECTION 6 display a symmetric trend around $T = 1$ (see Figure 9). Looking deeper, we found that this was caused by the projective duality of the pants group, leading us to Theorem 1.2.

Theorem (Duality Invariance of Entropy)**.** For the ideal pants groups Γ_T and $\Gamma_{\frac{1}{T}}$ with $T \in \mathbb{R}_{>0}$, their entropies are equal; that is, $h_{\Gamma_T} = h_{\Gamma_{\frac{1}{T}}}$.

If we consider the projective dual of Ω_{Γ_T}, we can uncover a relationship between the entropy of Γ_T and the entropy of the group that acts on the dual convex set, $(\Omega_{\Gamma_T})^*$. In theory, the dual exchanges points and lines in $\mathbb{RP}^2$. For example, the vertices of the base triangle become the edges of the bounding triangle of $(\Omega_{\Gamma_T})^*$ and vice versa. In this way, the base triangle and the bounding triangle are swapped in the dual picture (see Fig. 7).

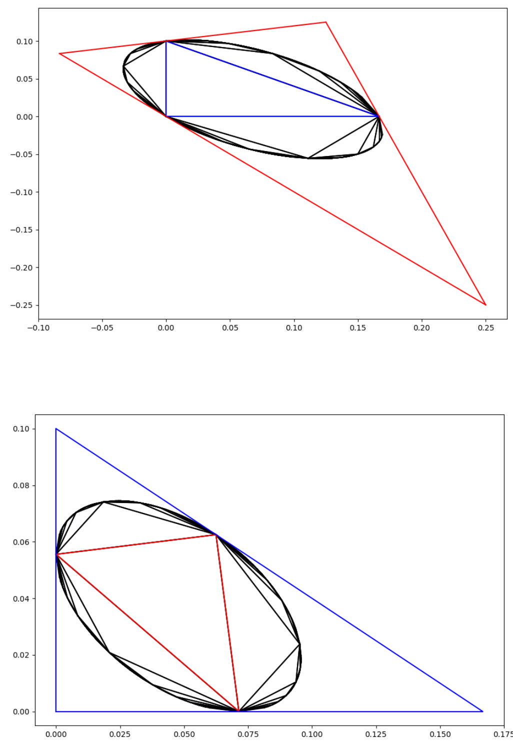


Figure 7: The figure on the top shows the original convex set Ω_{Γ_1} and the one on the bottom is its dual, $(\Omega_{\Gamma_1})^*$.

To determine exactly where the vertices of the new base triangle, $\hat{e_1}, \hat{e_2}, \hat{e_3}$ in $(\Omega_{\Gamma_T})^*$ are, we can examine the intersections of the kernels (e.g. $\operatorname{span}\hat{e_1} = \ker f_{T2}^{\top} \cap \ker f_{T3}^{\top}$):

$$\hat{e_1} = \begin{pmatrix} 0 \\ T \\ 1 \end{pmatrix},\ \hat{e_2} = \begin{pmatrix} 1 \\ 0 \\ T \end{pmatrix},\ \hat{e_3} = \begin{pmatrix} T \\ 1 \\ 0 \end{pmatrix}.$$

In essence, The base triangle in Ω_{Γ_T} and the bounding triangle in $(\Omega_{\Gamma_T})^*$ are dual to each other.

To understand the dual convex set $(\Omega_{\Gamma_T})^*$, we consider the dual map

$$\phi : \{\text{lines in } \mathbb{RP}^2\} \to \left(\mathbb{RP}^2\right)^*$$
$$\text{given by } \phi : L \mapsto \text{the } 1 \times 3 \text{ matrix with kernel} = L.$$

$$(\Omega_{\Gamma_T})^* = \phi(\{\text{lines in } \mathbb{RP}^2 \text{ that do not intersect } \Omega_{\Gamma_T}\})$$

These definitions are designed so that if Γ_T preserves Ω_{Γ_T}, then $(\Gamma_T^{-1})^{\top}$ preserves $(\Omega_{\Gamma_T})^*$. Let $\ell_{i,T}$ denote the line between $f_{j,T}$ and $f_{k,T}$ in $\mathbb{RP}^2$, where $\{i, j, k\} = \{1, 2, 3\}$. It is straightforward to check that if Γ_T preserves Ω_{Γ_T}, then $(\Gamma_T^{-1})^{\top}$ preserves $(\Omega_{\Gamma_T})^*$. That is,

$$\bigcup_{\gamma \in \Gamma_T} \gamma \cdot \triangle(e_1, e_2, e_3) = \Omega_{\Gamma_T} \quad \Longrightarrow \bigcup_{\gamma \in (\Gamma_T^{-1})^*} \gamma \cdot \triangle(\phi(\ell_{1,T}), \phi(\ell_{2,T}), \phi(\ell_{3,T})) = \Omega^*_{\Gamma_T},$$

where $\triangle(a, b, c)$ denotes the triangle with vertices a, b, c. This implies that if we can show that $(\Gamma_T^{-1})^{\top}$ is a group of reflections, then it must tile $(\Omega_{\Gamma_T})^*$. We proceed by showing that the generators $R_{1,T}^{-1}$, $R_{2,T}^{-1}$, and $R_{1,T}^{-1}$ of $(\Gamma_T^{-1})^{\top}$ are reflections; that is, $(R_{i,T}^{-1})^{\top}(\phi(\ell_{j,T})) = \phi(\ell_{j,T})$ for $1 \le i \ne j \le 3$. By symmetry, we can look at just $(R_{1,T}^{-1})^{\top}$. We have,

$$R_{T1} = R_{T1}^{-1} = \begin{pmatrix} -1 & 0 & 0 \\ 2T & 1 & 0 \\ \frac{2}{T} & 0 & 1 \end{pmatrix} \text{ so } (R_{1,T}^{-1})^{\top} = \begin{pmatrix} -1 & 2T & \frac{2}{T} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Since e_2 lies on $\ell_{2,T}$, we have $\phi(\ell_{2,T}) = \begin{pmatrix} 1 \\ 0 \\ T \end{pmatrix} = \hat{e}_2$. Then,

$$(R_{1,T}^{-1})^{\top}\phi(\ell_{2,T}) = \begin{pmatrix} -1 & 2T & \frac{2}{T} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ T \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ T \end{pmatrix}.$$

Thus, $\hat{e}_2$ is a fixed point of $(R_{1,T}^{-1})^{\top}$. Similarly, for $j = 3$, $\hat{e}_3$ is a fixed point of $(R_{1,T}^{-1})^{\top}$, and therefore $(R_{1,T}^{-1})^{\top}$ must be a reflection matrix. As stated before, $(R_{2,T}^{-1})^{\top}$ and $(R_{3,T}^{-1})^{\top}$ must also be reflection matrices by a symmetrical argument. Hence, we have shown that $(\Gamma_T^{-1})^{\top}$ tiles $(\Omega_{\Gamma_T})^*$, and described the vertices $\hat{e}_1$, $\hat{e}_2$, and $\hat{e}_3$ of its base triangle.

The duality gives an algebraic relationship between the groups $(\Gamma_{\mathrm{T}}^{-1})^\top$ and $\Gamma_{\frac{1}{\mathrm{T}}}$ that we now discuss. Consider the matrix:

$$g_{\mathrm{T}} := \begin{pmatrix} | & | & | \\ \hat{e_1} & \hat{e_2} & \hat{e_3} \\ | & | & | \end{pmatrix} = \begin{pmatrix} 0 & 1 & \mathrm{T} \\ \mathrm{T} & 0 & 1 \\ 1 & \mathrm{T} & 0 \end{pmatrix}$$

Then, for $i = 1,2,3$,

$$\begin{aligned} g_{\mathrm{T}}^{-1}(\mathrm{R}_{i,\mathrm{T}}^{-1})^\top g_{\mathrm{T}} &= \frac{1}{\mathrm{T}^3+1}\begin{pmatrix} -\mathrm{T} & \mathrm{T}^2 & 1 \\ 1 & -\mathrm{T} & \mathrm{T}^2 \\ \mathrm{T}^2 & 1 & -\mathrm{T} \end{pmatrix}\begin{pmatrix} -1 & 2\mathrm{T} & \frac{2}{\mathrm{T}} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 & \mathrm{T} \\ \mathrm{T} & 0 & 1 \\ 1 & \mathrm{T} & 0 \end{pmatrix} \\ &= \begin{pmatrix} -1 & 0 & 0 \\ \frac{2}{\mathrm{T}} & 1 & 0 \\ 2\mathrm{T} & 0 & 0 \end{pmatrix} = \mathrm{R}_{i,\frac{1}{\mathrm{T}}}. \end{aligned}$$

We have shown that the generators of $(\Gamma_{\mathrm{T}}^{-1})^\top$ are similar to the generators of $\Gamma_{\frac{1}{\mathrm{T}}}$, and we can conclude $(\Gamma_{\mathrm{T}}^{-1})^\top$ and $\Gamma_{\frac{1}{\mathrm{T}}}$ have the same entropy. This indicates a relationship between the entropies h_{T} and $h_{\frac{1}{\mathrm{T}}}$ for all $\mathrm{T} > 0$; we are able to prove this relationship by applying a proposition from Danciger-Guéritaud-Kassel and a proposition from Kassel-Potrie, both found below. For $\gamma \in \Gamma$, we let $\sigma_i(\gamma)$ denote the i^{th} singular value of γ and λ_i denote the modulus of the i^{th} eigenvalue of γ (arranged in descending order). We also define

$$\begin{aligned} \mu(\gamma) &:= (\log\sigma_1(\gamma), \log\sigma_2(\gamma), \log\sigma_3(\gamma)) \\ \text{and } \lambda(\gamma) &:= (\log\lambda_1(\gamma), \log\lambda_2(\gamma), \log\lambda_3(\gamma)), \end{aligned}$$

where $\sigma_1 \geq \sigma_2 \geq \sigma_3 > 0$ and $\lambda_1 \geq \lambda_2 \geq \lambda_3 > 0$.

Proposition 5.1 (Danciger-Guéritaud-Kassel, [8, Prop. 10.1])**.** *Let Ω be a properly convex set of $\mathbb{P}(\mathrm{V})$. For any $z \in \Omega$, there exists $\kappa > 0$ such that for any automorphism g on Ω,*

$$\log\frac{\sigma_1(\gamma)}{\sigma_3(\gamma)} \geq 2d_\Omega(z, \gamma\cdot z) - \kappa.$$

Proposition 5.2 (Kassel-Potrie, [10, Fact 2.10])**.** *Let Γ be a group of 3×3 matrices whose Zariski closure is reductive. Then there exists a finite subset S of Γ and constant $\mathrm{M}' > 0$ such that for any $\gamma \in \Gamma$,*

$$\min_{s\in\mathrm{S}} \|\mu(\gamma) - \lambda(\gamma\cdot s)\| \leq \mathrm{M}',$$

(See also, Benoist [3]).

5.2 Results Leading to Theorem 1.2

To begin proving the duality invariance of entropy, we first begin by defining bounds in terms of the singular values of the group for the Hilbert distance.

Lemma 5.1. *For all $z \in \Omega_{\Gamma_T}$, there exists a uniform constant $M' > 0$ such that for all $\gamma \in \Gamma_T$,*

$$\frac{1}{2}\log\frac{\sigma_1(\gamma)}{\sigma_3(\gamma)} + M' \geq d_{\Omega_{\Gamma_T}}(z, \gamma \cdot z) \geq \frac{1}{2}\log\frac{\sigma_1(\gamma)}{\sigma_3(\gamma)} - M'.$$

After completing this proof, we learned that a more general version of this lemma is proven in work in progress by Konstantinos Tsouvalas.

Proof. The left-hand inequality,

$$\frac{1}{2}\log\frac{\sigma_1}{\sigma_3}(\gamma) + M' \geq d_{\Omega_{\Gamma_T}}(z, \gamma \cdot z),$$

follows from the Danciger-Guéritaud-Kassel result (see 5.1). We will prove the right hand equality:

$$d_{\Omega_{\Gamma_T}}(z, \gamma \cdot z) \geq \frac{1}{2}\log\frac{\sigma_1}{\sigma_3}(\gamma) - M'.$$

By Crampon-Marquis [6, Thm. 3.3] and Cooper-Long-Tillman [5, Prop. 2.1], we have the following result

$$d_{\Omega_{\Gamma_T}}(z, \gamma \cdot z) \geq \frac{1}{2}\log\frac{\lambda_1}{\lambda_3}(\gamma).$$

In order to connect this result to the singular values of g, we will need some more machinery. We would like to use Proposition 5.2, but first, we must show the following claim:

Claim 5.3. Γ_T *has a reductive Zariski closure.*

Proof sketch of the claim in our setting. This is a special case of a much more general statement. To see that all the necessary hypotheses hold in our setting, we observe that $R_{1,T}, R_{2,T}, R_{3,T}$ do not preserve a proper vector subspace of $\mathbb{R}^3$, therefore, the set of generators $\{R_{i,T}\}$ is irreducible. Thus, since we have an irreducible set of generators, we can apply Crampon Marquis [6, Thm. 7.28] to obtain that the Zariski closure of Γ is reductive. ∎

Now we can apply Proposition 5.2, which in our setting gives us a finite subset $S \subset \Gamma_T$ and $M > 0$ such that for any $\gamma \in \Gamma_T$,

$$\min_{s \in S} \|\mu(\gamma) - \lambda(\gamma \cdot s)\| \leq M$$

Consider such an $S \subset \Gamma_T$ such that the above statement holds. Let $s \in S$ such that

$$\|\mu(\gamma) - \lambda(\gamma \cdot s)\| \le M.$$

Then,

$$\begin{aligned}
&|\log \sigma_1(\gamma) - \log \lambda_1(\gamma \cdot s)| \le M \\
&\text{and } |\log \sigma_3(\gamma) - \log \lambda_3(\gamma \cdot s)| \le M, && \text{since } \|\mu(\gamma) - \lambda(\gamma \cdot s)\| \le M \\
&\implies \frac{1}{2}\left|\log \frac{\sigma_1(\gamma)}{\sigma_3(\gamma)} - \log \frac{\lambda_1(\gamma \cdot s)}{\lambda_3(\gamma \cdot s)}\right| \le M && \text{reverse triangle inequality.}
\end{aligned}$$

We thus have

$$\begin{aligned}
&\frac{1}{2}\log \frac{\lambda_1(\gamma \cdot s)}{\lambda_3(\gamma \cdot s)} \ge \frac{1}{2}\log \frac{\sigma_1(\gamma)}{\sigma_3(\gamma)} - M \\
\implies &d_{\Omega_{\Gamma_T}}(z, \gamma \cdot (s \cdot z)) \ge \frac{1}{2}\log \frac{\sigma_1(\gamma)}{\sigma_3(\gamma)} - M && \text{triangle inequality.}
\end{aligned}$$

To complete the proof, we make use of the following claim.

Claim 5.4. *There exists* $C \in \mathbb{R}$ *such that*

$$C + d_\Omega(z, \gamma \cdot z) \ge d_\Omega(z, \gamma \cdot s \cdot z)$$

for all $s \in S$.

Proof of claim. By the triangle inequality,

$$\begin{aligned}
d_\Omega(z, \gamma \cdot s \cdot z) &\le d_\Omega(z, \gamma \cdot z) + d_\Omega(\gamma \cdot z, \gamma \cdot s \cdot z) \\
&= d_\Omega(z, \gamma \cdot z) + d_\Omega(z, s \cdot z)
\end{aligned}$$

since g preserves distances in d_Ω. Noting that S is finite, we then set

$$C = \max_{s \in S} d_\Omega(z, sz). \qquad \blacksquare$$

Putting together multiple steps, we have,

$$\begin{aligned}
&d_{\Omega_{\Gamma_T}}(z, \gamma \cdot z) \ge d_{\Omega_{\Gamma_T}}(z, \gamma \cdot (s \cdot z)) - C && \text{from Claim 5.4} \\
&d_{\Omega_{\Gamma_T}}(z, \gamma \cdot (s \cdot z)) \ge \frac{1}{2}\log \frac{\sigma_1(\gamma)}{\sigma_3(\gamma)} - M && \text{as shown earlier} \\
\implies &d_{\Omega_{\Gamma_T}}(z, \gamma \cdot z) \ge \frac{1}{2}\log \frac{\sigma_1(\gamma)}{\sigma_3(\gamma)} - M - C && \text{substituting for } d_{\Omega_{\Gamma_T}}(z, \gamma \cdot (s \cdot z)) \\
&d_{\Omega_{\Gamma_T}}(z, \gamma \cdot z) \ge \frac{1}{2}\log \frac{\sigma_1(\gamma)}{\sigma_3(\gamma)} - M' && \text{where } M' = M + C.
\end{aligned}$$

This is the inequality we were looking for and thus completes our proof.

$\square$

Moving toward completing the proof for Theorem 1.2, we now prove and then make use of the following lemma. Starting from Definition 2.2, the proven result formalizes entropy in terms of singular values of the group.

Lemma 5.2. *Given a group of* 3×3 *real matrices,* Γ*, the entropy of* Γ *can be characterized by the singular values of the group elements* $\gamma\in\Gamma$ *in the following way*

$$h_\Gamma = \limsup_{n\to\infty} \frac{\log\left(\#\left\{\gamma\in\Gamma \;\middle|\; \frac{1}{2}\log\frac{\sigma_1(\gamma)}{\sigma_3(\gamma)} < n\right\}\right)}{n}.$$

Proof. We will prove equality by showing both inequalities hold.

$$\begin{aligned}
h_\Gamma &= \limsup_{n\to\infty} \frac{\log\left(\#\{\gamma\in\Gamma \mid d_\Omega(x,\gamma\cdot x) < n\}\right)}{n} && \text{definition of entropy}\\
&= \limsup_{n\to\infty} \frac{\log\left(\#\{\gamma\in\Gamma \mid d_\Omega(x,\gamma\cdot x) < n-\mathrm{M}'\}\right)}{n-\mathrm{M}'}\\
&\le \limsup_{n\to\infty} \frac{\log\left(\#\left\{\gamma\in\Gamma \;\middle|\; \frac{1}{2}\log\frac{\sigma_1(\gamma)}{\sigma_3(\gamma)} < n\right\}\right)}{n-\mathrm{M}'}\cdot\frac{n-\mathrm{M}'}{n} && \text{Lemma 5.1}\\
&\le \limsup_{n\to\infty} \frac{\log\left(\#\left\{\gamma\in\Gamma \;\middle|\; \frac{1}{2}\log\frac{\sigma_1(\gamma)}{\sigma_3(\gamma)} < n\right\}\right)}{n}.
\end{aligned}$$

Now, we show the other inequality holds.

$$\begin{aligned}
h_\Gamma &= \limsup_{n\to\infty} \frac{\log\left(\#\{\gamma\in\Gamma \mid d_\Omega(x,\gamma\cdot x) < n'\}\right)}{n} && \text{definition of entropy}\\
&= \limsup_{n\to\infty} \frac{\log\left(\#\{\gamma\in\Gamma \mid d_\Omega(x,\gamma\cdot x) < n-\mathrm{M}'\}\right)}{n-\mathrm{M}'}\\
&\ge \limsup_{n\to\infty} \frac{\log\left(\#\{\gamma\in\Gamma \mid \frac{1}{2}\log\ \frac{\sigma_1(\gamma)}{\sigma_3(\gamma)} < n-\mathrm{M}'\}\right)}{n}\cdot\frac{n}{n-\mathrm{M}'} && \text{Lemma 5.1}\\
&\ge \limsup_{n\to\infty} \frac{\log\left(\#\left\{\gamma\in\Gamma \;\middle|\; \frac{1}{2}\log\frac{\sigma_1(\gamma)}{\sigma_3(\gamma)} < n-\mathrm{M}'\right\}\right)}{n-\mathrm{M}'}\\
&\ge \limsup_{n\to\infty} \frac{\log\left(\#\left\{\gamma\in\Gamma \;\middle|\; \frac{1}{2}\log\ \frac{\sigma_1(\gamma)}{\sigma_3(\gamma)} < n\right\}\right)}{n}. \square
\end{aligned}$$

Now that we have defined entropy in terms of singular values, the proof for the theorem on duality invariance follows from our work above.

Proof of Theorem 1.2. To prove $h_{\Gamma_\mathrm{T}} = h_{\Gamma_{\frac{1}{\mathrm{T}}}}$, we show $h_{\Gamma_\mathrm{T}} = h_{(\Gamma_\mathrm{T}^{-1})^\top}$, as we already know $h_{(\Gamma_\mathrm{T}^{-1})^\top} = h_{\Gamma_{\frac{1}{\mathrm{T}}}}$ from the beginning of Section 5. We first note that it is simple to show, using linear algebra, that for $\gamma \in \Gamma_T$, $(\gamma^{-1})^\top$ and γ have the same singular values. Lemma 5.2 shows that entropy depends on (and can be computed) with only knowledge of the singular values. Hence, since the singular values of $(\gamma^{-1})^\top$ and γ are the same, the entropies of the groups Γ_T and $\Gamma_{\frac{1}{T}}$ are the same as well. $\square$

Returning to the symmetry observed in Figure 9, we see that our code provides numerical evidence that agrees with this theorem.

6 Numerical Approximations of Entropy

6.1 Estimating Entropy via Programming

Theorem 1.1 states that Ω_{Γ_T} converges to the base triangle in the affine chart and implies the degeneration of the Hilbert Metric. The proof would not be possible without us first seeing its likelihood when working on the software described in this section.

Our team made a program in order to approximate entropy and generate images of tilings of Ω_{Γ_T} for various values of T. We worked in Python to approximate and generate these properly convex sets (see Fig. 6 for pictures of ideal pants group tilings). For those who are interested, the code is accessible and open to the public at `https://gitlab.eecs.umich.edu/logm/wi20/entropy-project-outputs`. In the following section, we describe our method to estimate entropy, along with a flowchart of the method in Figure 8.

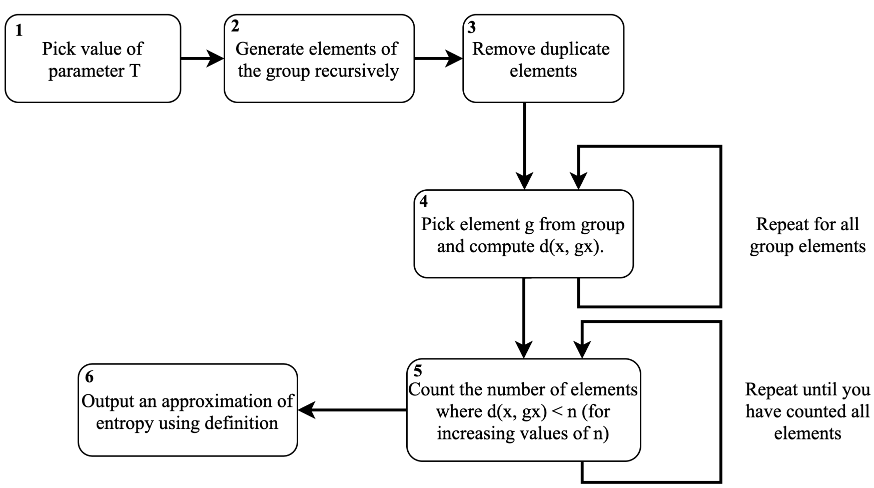


Figure 8: A chart describing how we approximated entropy.

6.2 Method of Approximating Entropy h_{Γ_T}

1. Choose the value of parameter T where T is the parameter used in the matrices that generate a finite area ideal pants groups, as mentioned in SECTION 3. T can be any positive value in $\mathbb{R}$.

2. Generate the elements of the group recursively by forming words in the generators. There is a parameter *Threshold* which controls how many group elements are generated (by controlling the possible word length). Our results use a *Threshold* value of 12. The group elements are generated by multiplying the initial generators together many times.

3. Multiplying the generators in such a way creates many duplicates of the same elements in our list. We then remove any duplicates to make sure we have a list of unique group elements.

4. In order to use the Hilbert metric, we must first approximate $\partial\Omega_{\Gamma_T}$. We can do this in two ways; the first is to use all the vertices of the triangles, and the second is to use the orbit of a point already on the boundary. In our work, we combined these two lists.

5. For simplicity, the Hilbert distance is measured inside the affine chart as opposed to inside $\mathbb{R}^3$ (A.1 proves why this simplification is valid). To find an approximation of entropy, we need to know the Hilbert distance between x and $\gamma \cdot x$ or how far a group element moves a point in the set Ω_{Γ_T}. Each group element γ is a 3×3 matrix. x can be any fixed value in Ω_{Γ_T}. This step needs to be repeated for every group element, attaching a distance to each element to be used in the next step.

6. To estimate entropy, we find a sequence of the values of

$$z_n = \frac{\log \#\{\gamma \in \Gamma \mid d_\Omega(x, \gamma \cdot x) < n\}}{n},$$

increasing n each time. As a reminder, our definition of entropy is

$$h_{\Gamma_T} = \limsup_{n\to\infty} \frac{\log\left(\#\{\gamma \in \Gamma \mid d_\Omega(x, \gamma \cdot x) < n\}\right)}{n}.$$

In terms of z_n, entropy is equal to the $\limsup$ of z_n as n goes to infinity. One way to approximate this limit is to increase n until we see a convergence in z_n. We calculate z_n for increasing values of n and stop once we reach $\max(d_{\Omega_{\Gamma_T}}(x, \gamma \cdot x)) < n$ where $\max(d_{\Omega_{\Gamma_T}}(x, \gamma \cdot x))$ is the largest distance measured for all $\gamma \in \Gamma_T$ or the maximum distance a group element moves the point x.

7. Once we have stopped increasing n, we output an estimation of h_{Γ_T} as the final value in the sequence.

6.3 Results of Entropy Estimates

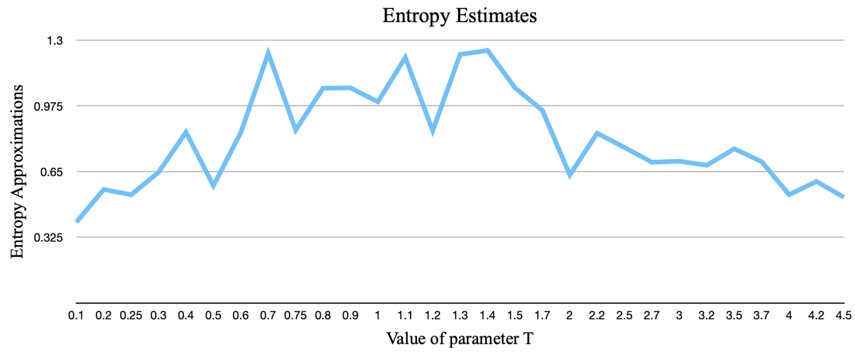


Figure 9: The Entropy estimates for Ideal Projective Pants Groups

Looking at our results, we see that many entropy estimates that are within $\left(\frac{1}{2},1\right]$, and it is clear that entropy varied for the different values of T we tested. This is a possible sign that the entropy of these groups surjects onto the interval $\left(\frac{1}{2},1\right]$, aligning with Conjecture 3.2.

6.3.1 Inaccuracies.

We also observe that our estimation achieves values outside of the range $\left(\frac{1}{2},1\right]$. We suspect that this is because our approximation for the Hilbert distance becomes less accurate as Ω_{Γ_T} converges to the base triangle. More specifically, as $T \to \infty$, the approximation for the intersection of the line through x and $\gamma \cdot x$ and $\partial\Omega_{\Gamma_T}$ is greatly compromised due to an inadequate representation of $\partial\Omega_{\Gamma_T}$.

6.4 Interpretation of Results

Our program's results and generated tilings of various Ω_{Γ_T} was one of the main motivations for Theorem 1.1. Initially, we saw Ω_{Γ_T} converging to a triangle as a problem and a source of inaccuracy for the Hilbert Metric, but the phenomenon quickly became interesting to explore. In terms of entropy, another observation that can be made is that the entropy estimates for a group Γ_T plotted against the value of T seem to be symmetric around $T = 1$. This is part of the empirical evidence that inspired our later work on the projective dual.

A Proofs and Computations

Lemma A.1. *Given* R_{T1}, R_{T2}, R_{T3} *as generators of our group. For any* i, j *where* $i \neq j$ *and* $i, j \in \{1,2,3\}$, $R_{Ti}R_{Tj}$ *have one eigenvector, specifically* e_k *where* $k \neq i, k \neq j, k \in \{1,2,3\}$.

Proof. We prove this lemma for case of generators R_{T1} and R_{T2}. The same method will prove the lemma for all other combinations of generators. We have that

$$R_{T1} = \begin{pmatrix} -1 & 0 & 0 \\ 2T & 1 & 0 \\ \frac{2}{T} & 0 & 1 \end{pmatrix}, R_{T2} = \begin{pmatrix} 1 & \frac{2}{T} & 0 \\ 0 & -1 & 0 \\ 0 & 2T & 1 \end{pmatrix}, \text{ and } R_{T1}R_{T2} = \begin{pmatrix} -1 & \frac{-2}{T} & 0 \\ 2T & 3 & 0 \\ \frac{2}{T} & \frac{4}{T^2}+2T & 1 \end{pmatrix}.$$

To find the eigenvalues of this parabolic, $R_{T1}R_{T2}$, we take the determinant of $R_{T1}R_{T2} - \lambda I$.

$$\begin{aligned} \det(R_{T1}R_{T2} - \lambda I) &= \det \begin{pmatrix} -1-\lambda & \frac{-2}{T} & 0 \\ 2T & 3-\lambda & 0 \\ \frac{2}{T} & \frac{4}{T^2}+2T & 1-\lambda \end{pmatrix} \\ &= (1-\lambda)\left[(\lambda+1)(\lambda-3) + \frac{4T}{T}\right]. \end{aligned}$$

Simplifying the characteristic polynomial, we see $\det(R_{T1}R_{T2} - \lambda I) = (\lambda - 1)^3$. And we observe $R_{T1}R_{T2}$ has only one eigenvalue $\lambda_1 = 1$. In order to find the of eigenvectors of $\lambda_1 = 1$ we compute $[R_{T1}R_{T2} - I]$.

$$[R_{T1}R_{T2} - I] = \begin{pmatrix} -2 & \frac{-2}{T} & 0 \\ 2T & 2 & 0 \\ \frac{2}{T} & \frac{4}{T^2}+2T & 0 \end{pmatrix}.$$

Looking at $[R_{T1}R_{T2} - I]\,v = 0$, we get the following system of equations that we solve to find the eigenvector(s):

$$\begin{cases} -2v_1 - \frac{2v_2}{T} = 0 \\ 2Tv_1 + 2v_2 = 0 \\ \frac{2v_1}{T} + (\frac{4}{T^2} + 2T)\,v_2 = 0. \end{cases}$$

The only values that fulfill this system of equations are $v_1 = v_2 = 0$, and $v_3 = c$ where c is any real number. We conclude $R_{T1}R_{T2}$ has one eigenvalue that corresponds to a one dimensional eigenspace. Thus, $R_{T1}R_{T2}$ fixes a projective point, namely e_3. □

Proposition A.1. *Let Ω be a properly convex set in an affine chart A in $\mathbb{RP}^2$. Let x and y be in Ω. Then, the Hilbert distance between x and y, d_Ω, is preserved by the projection onto the xy-plane.*

Proof. Let a and b be the points where the line through x and y intersects with the boundary of Ω. It suffices to show that the absolute value of the cross product is preserved by the projection onto the xy-plane.

Let P be the matrix which projects any point in $\mathbb{R}^3$ onto the xy-plane by left matrix multiplication, given by

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

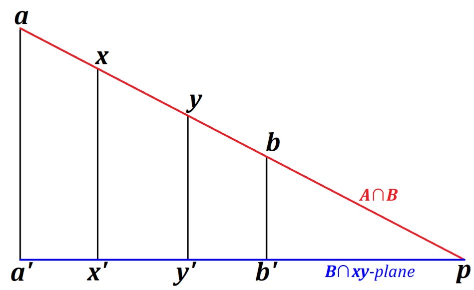


Figure 10: Similar triangles in B created by a, x, y, b and their projections a', x', y', b'.

Let $a' = \mathrm{P}a$, $b' = \mathrm{P}b$, $x' = \mathrm{P}x$, $y' = \mathrm{P}y$. Then the line through a, x, y, b is co-planar with the line through a', x', y', b', call this plane B. Note that because Ω is properly convex in A, A cannot be parallel to the xy-plane. Thus, A must intersect the xy-plane at some line $\bar{q}$. But this implies that B intersects $\bar{q}$ at one point, since it contains both the line through a, x, y, b and the line through a', x', y', b'. Denote this point p.

Now, we have the similar triangles $\triangle(a, p, a')$, $\triangle(x, p, x')$, $\triangle(y, p, y')$, and $\triangle(b, p, b')$, all contained in B (Figure 10). We can use properties of similar triangles to show $[a, x, y, b] = [a', x', y', b']$. Notice that since ratios of the lengths of segments are the same in similar triangles, we have

$$\frac{|x-b|}{|b-y|} = \frac{|x'-b'|}{|b'-y'|} \quad \text{and} \quad \frac{|y-a|}{|x-a|} = \frac{|y'-a'|}{|x'-a'|}.$$

Multiply the two inequalities to get

$$\frac{|x-b||y-a|}{|b-y||x-a|} = \frac{|x'-b'||y'-a'|}{|b'-y'||x'-a'|},$$

which gives us equality of the cross ratios, $[a, x, y, b] = [a', x', y', b']$. Thus, the cross ratio is preserved by P, and therefore $d_\Omega(x, y) = d_\Omega(\mathrm{P}x, \mathrm{P}y)$. □

Lemma (4.5)**.** Let ℓ be a line segment in $\mathbb{R}^2$, and let ℓ_T be a sequence of line segments whose endpoints converge to those of ℓ. More specifically, suppose ℓ has endpoints x, y and ℓ_T has endpoints $x_\mathrm{T}, y_\mathrm{T}$ where $x_\mathrm{T} \to x$ and $y_\mathrm{T} \to y$ as T tends to ∞. Assume furthermore that for all T, the intersection of the segments ℓ and ℓ_T is x_T. Then, ℓ_T converges to ℓ as T tends to ∞ with respect to the Gromov-Hausdorff distance.

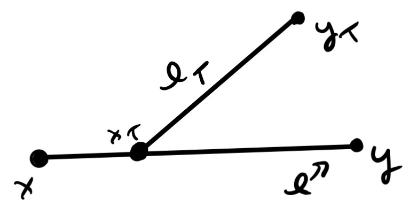
Proof (A). The Hausdorff distance, d_H between the two lines ℓ_T and ℓ is given by

$$d_\mathrm{H}(\ell_\mathrm{T}, \ell) := \max\left\{\sup_{a\in\ell_\mathrm{T}} \inf_{b\in\ell} d(a,b),\ \sup_{b\in\ell} \inf_{a\in\ell_\mathrm{T}} d(a,b)\right\}$$

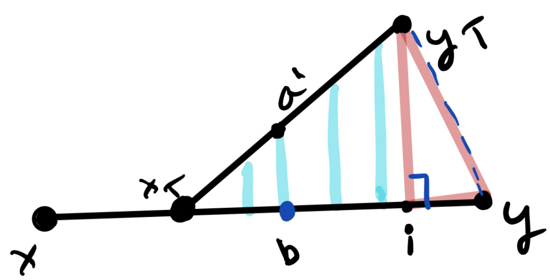
where d represents the Euclidean distance in $\mathbb{R}^2$. Define

$$m_T := \sup_{a \in \ell_T} \inf_{b \in \ell} d(a,b) \text{ and } n_T := \sup_{b \in \ell} \inf_{a \in \ell_T} d(a,b) \text{ so that}$$
$$d_H(\ell_T, \ell) = \max\{m_T, n_T\}.$$

The Gromov-Hausdorff distance, d_{GH}, is at most the Hausdorff distance, so we aim to show that $d_H(\ell, \ell_T)$ vanishes. Note that m_T and n_T both exist since ℓ_T and ℓ are compact sets (so in fact, sup and inf may be replaced with max and min). In order to show that ℓ_T converges to ℓ in this setting, we aim to prove that both m_T and n_T go to 0 as T tends to ∞. Let's first prove that m_T vanishes.



Without loss of generality, we may assume that x is the origin, ℓ lies on the x-axis, and ℓ_T has positive slope. Since ℓ_T and ℓ are compact sets, we also know that there exists a point a' on ℓ_T such that $\inf_{b \in \ell} d(a', b) = m_T$.



Then,

$\inf_{b \in \ell} d(a', b) \le \inf_{b \in \ell} d(y_T, b)$	ℓ_T has positive slope
$\sup_{a' \in \ell_T} \inf_{b \in \ell} d(a', b) \le \sup_{a' \in \ell_T} \inf_{b \in \ell} d(y_T, b) = \inf_{b \in \ell} d(y_T, b) \le m_T$	taking suprema
$m_T \le \inf_{b \in \ell} d(y_T, b) \le m_T \implies \inf_{b \in \ell} d(y_T, b) = m_T$	definition of m_T

Since we are working in Euclidean geometry, we know that

$$m_T = \inf_{b \in \ell} d(y_T, b) = d(y_T, i)$$

where i is the point on ℓ such that the line through i and y_T is perpendicular to ℓ. By the Pythagorean theorem,

$$m_T = d(y_T, i) < d(y_T, y),$$

and since y_T converges to y as T tends to ∞, we have that

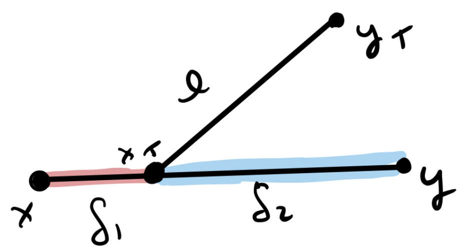
$$0 \leq \lim_{T\to\infty} m_T = \lim_{T\in\infty} d(y_T, i) \leq \lim_{T\to\infty} d(y_T, y) = 0.$$

Using the squeeze theorem, we conclude that m_T vanishes. Now we show that n_T vanishes.

Again, without loss of generality, we make the same assumptions that x is the origin, ℓ lies on the x-axis, and ℓ_T has positive slope. Separate ℓ into two δ_1 and δ_2 by 'cutting' at x_T. Then,

$$n_T := \sup_{b\in\ell} \inf_{a\in\ell_T} d(a,b) = \max\left\{ \sup_{b\in\delta_1} \inf_{a\in\ell_T} d(a,b),\ \sup_{b\in\delta_2} \inf_{a\in\ell_T} d(a,b) \right\}.$$

We will show that both quantities in the max go to 0 as T tends to ∞.



Let's show that $\sup_{b\in\delta_1} \inf_{a\in\ell_T} d(a,b)$ vanishes. Again, using the fact that ℓ_T and ℓ are compact sets, we also know that there exists a point b' on δ_1 such that

$$\sup_{b\in\delta_1} \inf_{a\in\ell_T} d(a,b) = \inf_{a\in\ell_T} d(a,b').$$

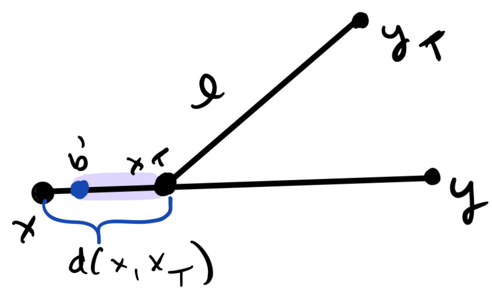
It is clear from Euclidean geometry that $\inf_{a\in\ell_T} d(a,b') = d(x_T, b')$. Since $b' \in \delta_1$, b' is between x_T and x, and we know $d(x_T, b') \leq d(x_T, x)$ which implies

$$0 \leq \sup_{b\in\delta_1} \inf_{a\in\ell_T} d(a,b) = d(x_T, b') \leq d(x_T, x).$$

Taking limits as T tends to ∞, we get

$$0 \leq \lim_{T\to\infty} \sup_{b\in\delta_1} \inf_{a\in\ell_T} d(a,b) \leq \lim_{T\to\infty} d(x_T, x) = 0.$$

And applying the squeeze theorem, we see that $\sup_{b\in\delta_1} \inf_{a\in\ell_T} d(a,b)$ vanishes.



Finally, we show that $\sup_{b\in\delta_2}\inf_{a\in\ell_T} d(a,b)$ vanishes. Using compactness, let b' be a point on δ_2 such that

$$\inf_{a\in\ell_T} d(a,b') = \sup_{b\in\delta_2}\inf_{a\in\ell_T} d(a,b),$$

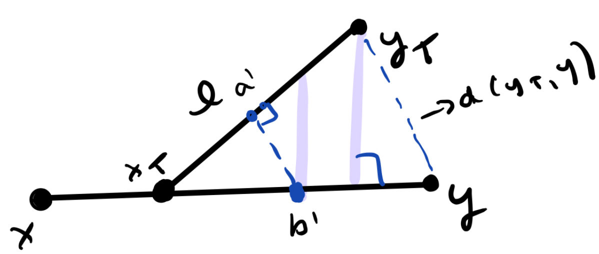
and let a' be a point on ℓ_T such that

$$\sup_{b\in\delta_2}\inf_{a\in\ell_T} d(a,b) = d(b',a').$$

Since we are now working with the Euclidean distance in $\mathbb{R}^2$, we know that the line through a', b' will be perpendicular to ℓ_T. Define $i \in \ell_T$ such that the line through b' and i is perpendicular to ℓ. Similarly, let's define $j \in \ell$ such that line through j and y_T is perpendicular to ℓ. Then,

$$\begin{aligned}
0 \le \sup_{b\in\delta_2}\inf_{a\in\ell_T} d(a,b) = d(b',a') &\qquad \text{by definition of } a', b' \\
\le d(b', i) &\qquad \text{by Pythagorean theorem} \\
\le d(j, y_T) &\qquad \text{since positive slope} \\
\le d(y_T, y) &\qquad \text{by Pythagorean theorem}
\end{aligned}$$

Taking limits and applying the squeeze theorem once more, we have that $\sup_{b\in\delta_2}\inf_{a\in\ell_T} d(a,b)$ vanishes.



Thus, we have shown that $\sup_{b\in\delta_2}\inf_{a\in\ell_T} d(a,b)$ and $\sup_{b\in\delta_1}\inf_{a\in\ell_T} d(a,b)$ go to 0 as T tends to ∞ which implies that n_T vanishes.

We conclude that $d_H(\ell_T,\ell) = \max\{m_T, n_T\}$ goes to 0 as T tends to ∞. Since the Gromov-Hausdorff distance is at most the Hausdorff distance we have that ℓ_T converges to ℓ in the Gromov-Hausdorff distance as T goes to infinity, proving the lemma. □

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