Lebesgue Measure Preserving Thompson Monoid and Its Properties of Decomposition and Generators

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Cover Page Footnote
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Lebesgue Measure Preserving Thompson Monoid and Its Properties of Decomposition and Generators

By William Li

Abstract. This paper defines the Lebesgue measure preserving Thompson monoid, denoted by $G$, which is modeled on the Thompson group $F$ except that the elements of $G$ preserve the Lebesgue measure and can be non-invertible. The paper shows that any element of the monoid $G$ is the composition of a finite number of basic elements of the monoid $G$ and the generators of the Thompson group $F$. However, unlike the Thompson group $F$, the monoid $G$ is not finitely generated. The paper then defines equivalence classes of the monoid $G$, use them to construct a monoid $H$ that is finitely generated, and shows that the union of the elements of the monoid $H$ is a set of equivalence classes, the union of which is $G$.

1 Introduction

In this paper we define the Lebesgue measure preserving Thompson monoid. This study is at an intersection of two subjects of research. The first subject concerns the Lebesgue measure preserving interval maps of $[0, 1]$ onto itself, which studies dynamical properties such as transitivity, mixing, periodic points, and metric entropy, and finds important applications in the abstract formulation of dynamical systems, chaos theory, and ergodic theory [1]. The author in [2] motivates the study of interval maps by stating that the “most interesting” part of some higher dimensional systems can be of lower dimensions, which in some cases allows them to boil down to systems in dimension one. In particular, a recent paper [3] uses a special form of interval maps, namely, piecewise affine maps, as a tool to prove results of generic maps. The second subject concerns the Thompson group $F$ [4, 5], which is the group of piecewise affine homeomorphisms from $[0, 1]$ onto itself whose derivatives are the integer powers of 2 and the points at which the derivatives are discontinuous are dyadic numbers. For simplicity, a homeomorphism

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in the Thompson group $\mathbb{F}$ is referred to as a Thompson group $\mathbb{F}$ map or simply a map in $\mathbb{F}$ in this paper. As the derivatives are always positive, the orientation is preserved. The Thompson group $\mathbb{F}$ has a collection of unusual algebraic properties that make it appealing in diverse areas of mathematics such as group theory, combinatorics [6], and cryptography [7].

Except for the identity map, any Thompson group $\mathbb{F}$ map does not preserve the Lebesgue measure, and any Lebesgue measure preserving interval map does not preserve the orientation and thus not belong to the Thompson group $\mathbb{F}$. These two subjects do not naturally intersect. We intend to build on the Thompson group $\mathbb{F}$ by making important changes to preserve the Lebesgue measure. More precisely, we define the Lebesgue measure preserving Thompson monoid, denoted by $\mathbb{G}$. The monoid $\mathbb{G}$ is similar to the group $\mathbb{F}$ except that the derivatives of piecewise affine maps can be negative. As a result, the maps in the monoid $\mathbb{G}$ are non-invertible, except for the identity maps, and exhibit very different properties from those in the group $\mathbb{F}$.

The main results of this paper are summarized as follows.

- We show that any element of the monoid $\mathbb{G}$ is the composition of a finite number of basic elements of the monoid $\mathbb{G}$ and the generators of the Thompson group $\mathbb{F}$.

- We show that unlike the Thompson group $\mathbb{F}$, the monoid $\mathbb{G}$ is not finitely generated.

- We define equivalence classes of the monoid $\mathbb{G}$ and use them to construct a new monoid $\mathbb{H}$ that has two properties. First, the monoid $\mathbb{H}$ is finitely generated. Second, the union of the elements of the monoid $\mathbb{H}$ is a set of equivalence classes, the union of which is the monoid $\mathbb{G}$.

The remainder of the paper is organized as follows. Section 2 reviews the basic properties of the measure preserving interval maps and Thompson group $\mathbb{F}$, and defines the measure preserving Thompson monoid $\mathbb{G}$. Section 3 shows that any map in the monoid $\mathbb{G}$ can be expressed as the composition of a finite number of basic maps in the monoid $\mathbb{G}$ and the generators in the group $\mathbb{F}$. Section 4 shows that unlike the group $\mathbb{F}$, the monoid $\mathbb{G}$ is not finitely generated. Section 5 defines equivalence relationship, equivalence classes and sets of equivalence classes, constructs the monoid $\mathbb{H}$ with the sets of equivalence classes, and shows that the monoid $\mathbb{H}$ has a finite number of generators and that any map in the monoid $\mathbb{G}$ is an element of an equivalence class in the monoid $\mathbb{H}$. Finally, Section 6 concludes the paper.

## 2 Basic Definitions and Properties

Consider continuous interval maps from $[0,1]$ to $[0,1]$. Let $h_1$ and $h_2$ be two maps. Denote by $h_1 \circ h_2$ the composition of $h_1$ and $h_2$ where $h_1 \circ h_2(x) = h_1(h_2(x))$. The composition of more than two maps can be recursively defined with this definition. For any $y \in [0,1]$, define $h^{-1}(y) = \{ x \in [0,1] : h(x) = y \}$.
Define positive identity map \( g_{0,+}(x) = x \) and negative identity map \( g_{0,-}(x) = 1 - x \) for \( x \in [0,1] \). Refer to \( g_{0,+}(x) \) and \( g_{0,-}(x) \) as the identity maps.

Let \( A \) be a point in the plane of \([0,1] \times [0,1]\). Denote by \( A_x \) and \( A_y \) the \( x \)- and \( y \)-coordinates of the point \( A \), respectively. If \( A \) is on the graph of map \( h, A_y = h(A_x) \).

Let \( \mathcal{I} \) be an interval in \([0,1]\). Let \( \mathcal{I}^o \) represent the interior of \( \mathcal{I} \). The left and the right endpoints of \( \mathcal{I} \) are denoted by \( \mathcal{I}^0, \mathcal{I}^1 \), respectively. If \( \mathcal{I} \) is closed, then \( \mathcal{I} = [\mathcal{I}^0, \mathcal{I}^1] \).

Let \( |\mathcal{I}| \) represent the measure of the interval: \( |\mathcal{I}| = \mathcal{I}^1 - \mathcal{I}^0 \). For two distinct intervals \( \mathcal{I}_1 \) and \( \mathcal{I}_2 \), \( \mathcal{I}_1 < \mathcal{I}_2 \) if \( x_1 \leq x_2, \forall x_1 \in \mathcal{I}_1, x_2 \in \mathcal{I}_2 \).

Let \( \mathcal{I}_1, \mathcal{I}_2 \) be two closed intervals of \([0,1]\) and \( f_1, f_2 \) be two maps. Let \( f_1(\mathcal{I}) \cong f_2(\mathcal{I}) \) if \( f_2 \) can be linearly transformed from \( f_1 \), meaning that if \( x_1 = \mathcal{I}^0 + \alpha(\mathcal{I}^1 - \mathcal{I}^0) \) and \( x_2 = \mathcal{I}^0 + \alpha(\mathcal{I}^1 - \mathcal{I}^0) \) for some \( \alpha \in [0,1] \), then \( f_1(x_1) = f_2(x_2) \). When \( f_2 \) is an identity map, \( f_1(\mathcal{I}) \cong \mathcal{I} \) if \( f_1 \) is an affine map. A horizontally flipped version of the relationship is defined as follow. \( f_1(\mathcal{I}) \cong f_2(\mathcal{I}) \) if \( x_1 = \mathcal{I}^0 + \alpha(\mathcal{I}^1 - \mathcal{I}^0) \) and \( x_2 = \mathcal{I}^1 - \alpha(\mathcal{I}^1 - \mathcal{I}^0) \) for some \( \alpha \in [0,1] \), then \( f_1(x_1) = f_2(x_2) \).

A set of distinct closed intervals \( \{\mathcal{I}_1, \ldots, \mathcal{I}_n\} \) is a partition of \([0,1]\) if \( \mathcal{I}_i^o \cap \mathcal{I}_j^o = \emptyset \) for any \( i \neq j \) and \( \bigcup_{i=1}^n \mathcal{I}_i = [0,1] \). A subset of \( \{\mathcal{I}_i\} \) may be a single point, i.e., \( \mathcal{I}_j^0 = \mathcal{I}_j^1 \) where some \( j \).

Denote by \( \lambda \) the Lebesgue measure on \([0,1]\) and \( \mathcal{B} \) all Borel sets on \([0,1]\).

**Definition 2.1 (Measure Preserving Interval Maps, [3]).** A continuous interval map \( h \) is measure preserving or \( \lambda \)-preserving for simplicity if \( \forall A \in \mathcal{B}, \lambda(A) = \lambda(h^{-1}(A)) \).

**Remark 2.2.** Definition 2.1 does not imply \( \lambda(A) = \lambda(h(A)) \) for \( \lambda \)-preserving \( h \). In fact, one can easily show that if \( h \) is \( \lambda \)-preserving, \( \lambda(A) \leq \lambda(h(A)) \) for any \( A \in \mathcal{B} \). Except for the identity maps \( g_{0,+} \) and \( g_{0,-} \), \( h \) is not invertible and \( \exists A \in \mathcal{B} \) such that \( \lambda(A) < \lambda(h(A)) \).

The Thompson group \( \mathbb{F} \) has a few representations, such as group presentations, rectangle diagrams, and piecewise linear homeomorphisms. The following focuses on the representation of piecewise linear homeomorphisms because it is closely related to the \( \lambda \)-preserving Thompson monoid to be introduced in this section.

**Definition 2.3 (Thompson Group \( \mathbb{F}, [4] \)).** A homeomorphism \( f \) from \([0,1]\) onto \([0,1]\) is an element of the Thompson group \( \mathbb{F} \) if \( f \) is piecewise affine and differentiable except at finitely many points, the \( x \)-coordinate of any point of non-differentiability is a dyadic number, i.e., a rational number whose denominator is an integer power of 2, and on the intervals where \( f \) is differentiable, the derivatives are the integer powers of 2.

In the remainder of this paper, \( f \) is referred to as an element in the Thompson group \( \mathbb{F} \).

**Remark 2.4.** It is easily to see that \( f(0) = 0, f(1) = 1 \), and \( f \) is strictly increasing on \([0,1]\) and is thus invertible. Except for the identity map \( f = g_{0,+} \), \( f \) is not \( \lambda \)-preserving.
Example 2.5. Define the following two maps in the group $F$.

$$f_A(x) = \begin{cases} 
\frac{x}{2}, & 0 \leq x \leq \frac{1}{2}, \\
\frac{x}{2} + \frac{1}{4}, & \frac{1}{2} \leq x \leq \frac{3}{4}, \\
2x - 1, & \frac{3}{4} \leq x \leq 1,
\end{cases} \quad f_B(x) = \begin{cases} 
\frac{x}{2}, & 0 \leq x \leq \frac{1}{2}, \\
\frac{x}{2} + \frac{1}{4}, & \frac{1}{2} \leq x \leq \frac{3}{4}, \\
2x - 1, & \frac{3}{4} \leq x \leq 1.
\end{cases} \quad (1)$$

The significance of $f_A$ and $f_B$ is that the Thompson group $F$ is generated by the two maps. That is, any $f \in F$ can be represented by the composition of a finite number of $f_A$ and $f_B$ in certain order [4].

Definition 2.6 ($\lambda$-Preserving Thompson Monoid $G$). A continuous interval map $g$ from $[0, 1]$ onto $[0, 1]$ is an element of the $\lambda$-preserving Thompson monoid $G$ if $g$ is $\lambda$-preserving, piecewise affine, and differentiable except at finitely many points, the $x$-coordinate of each of these points of non-differentiability is a dyadic number, and on an interval where $g$ is differentiable, the derivative is positive or negative and the absolute value of the derivative is an integer power of 2.

Remark 2.7. The difference between $G$ and $F$ is that the derivatives can be negative in the maps of $G$, which makes it possible for them to be $\lambda$-preserving.

Remark 2.8. As will be explained in Lemma 2.11, the absolute value of any derivative cannot be a negative power of 2 in the maps of $G$ due to the measure preserving property.

Remark 2.9. It is easy to see that if $g_1, g_2, g_3 \in G$, then $g_1 \circ g_2 \in G$ and $(g_1 \circ g_2) \circ g_3 = g_1 \circ (g_2 \circ g_3)$. The identity map $g_{0,+}$ is the identity element of $G$. However, an inverse may not always exist for any given $g \in G$. This is the reason that $G$ satisfying these conditions is a monoid.

In the remainder of this paper, $g$ is referred to as an element in the $\lambda$-preserving Thompson monoid $G$. When $g$ is an affine segment on an interval, for simplicity, refer to the derivative of $g$ on the interval as the slope of the affine segment. We say $(x, y)$ is a point of $g$ if and only if $y = g(x)$.

Definition 2.10 (Breakpoints). Let $g \in G$. A breakpoint of $g$ is either an endpoint at $x = 0$ or $x = 1$ or a point at which the derivative of $g$ is discontinuous. A breakpoint that is not an endpoint is referred to as an interior breakpoint. An interior breakpoint is further categorized into type I and type II. At a type I breakpoint, the left and the right derivatives are of the same sign. At a type II breakpoint, the left and the right derivatives are of the opposite signs.

A point $(x, y)$ is said to be dyadic if both $x$ and $y$ are dyadic. It can be shown that for any point $(x, y)$ of $g \in G$, $y$ is dyadic if and only if $x$ is dyadic.
Lemma 2.11. For $y \in [0,1]$, suppose that $g^{-1}(y) = \{x_1, \ldots, x_n\}$ and none of $x_1, \ldots, x_n$ are breakpoints. The map $g$ is $\lambda$-preserving if and only if
\[
\sum_{i=1}^{n} \frac{1}{|g'(x_i)|} = \sum_{i=1}^{n} 2^{-k_i} = 1,
\]
where $k_i$ is an integer and $|g'(x_i)| = 2^{k_i}$ is the absolute value of the slope of the affine segment on which $x_i$ resides.

Proof. Let $\mathcal{Y} = [y - \delta, y + \delta]$ for $\delta > 0$. For a sufficiently small $\delta$, $g^{-1}(\mathcal{Y}) = \bigcup_{i=1}^{n} I_i$, where the intervals $I_i$ are disjoint, $x_i \in I_i$, and $g(I_i) = \mathcal{Y}$ for $i = 1, \ldots, n$. $\lambda(\mathcal{Y}) = \lambda(g(I_i)) = |g'(x_i)|\lambda(I_i)$ as $\delta \to 0$. By the $\lambda$-preservation and because $I_i$ are disjoint,
\[
\lambda(\mathcal{Y}) = \lambda(g^{-1}(\mathcal{Y})) = \sum_{i=1}^{n} \lambda(I_i).
\]
Hence, (2) follows immediately. \hfill \Box

To satisfy (2), $k_i$ must be non-negative for any $i$. In contrast, for any $f \in \mathcal{F}$, a derivative can be a negative integer power of 2. Moreover, if $n > 1$, $g'(x_i)$ has alternating signs: $g'(x_i)g'(x_{i+1}) < 0$ for $i = 1, \ldots, n - 1$. Unlike $f$, $g$ is not orientation-preserving except for the identity maps.

Definition 2.12 (Legs, Affine Legs, $m$-fold Window Affine on an Interval, and Window Affine Maps). Let an interval $\mathcal{Y} \subset [0,1]$. If except for a finite number of points in the interval $\mathcal{Y}$, $\forall y \in \mathcal{Y}$, the set $g^{-1}(y)$ has $m$ elements, then the map $g$ is said to have $m$ legs on the set $g^{-1}(\mathcal{Y})$. It can be shown that $m$ intervals $I_1, \ldots, I_m$ with mutually disjoint interiors exist such that $g^{-1}(\mathcal{Y}) = \bigcup_{i=1}^{m} I_i$, and the map $g$ is monotone on every $I_i$ and $\mathcal{Y} = g(I_i)$ for any $i$. In this case, the graph of $g$ on $I_i$ is referred to as the $i$-th leg. Moreover, if $g$ is affine on every $I_i$, then the map $g$ is said to have $m$ affine legs on the set $g^{-1}(\mathcal{Y})$. Furthermore, if $\bigcup_{i=1}^{m} I_i$ is an interval $I$, the map $g$ is said to be an $m$-fold window affine on an interval $I$. In addition, if $g(x) = x$ or $g(x) = 1 - x$ for $x \in [0,1] \setminus I$, then $g$ is referred to as an $m$-fold window affine map. Whether $g(x) = x$ or $g(x) = 1 - x$ depends on the continuity of $g$. An $m$-fold window affine map is an $m$-fold window affine on a specific interval.

We will define the basic maps of $\mathcal{G}$ with the $m$-fold window affine maps in Definition 3.17 and introduce their notation later in Section 3.

Figure 1 illustrates the definitions of legs, affine legs, and an $m$-fold window affine on an interval.


3 Decomposition

Recall that the group $\mathcal{F}$ is generated by two generator maps. Theorem 3.18 will show that any map in the monoid $\mathcal{G}$ can be expressed as the composition of a finite number of basic maps in the monoid $\mathcal{G}$ and the generators in the group $\mathcal{F}$. To this end, first Theorem 3.11 shows that any map in $\mathcal{G}$ is the composition of the maps in $\mathcal{F}$ and the window affine maps, and then any window affine map is shown to be the composition of a few basic maps in $\mathcal{G}$ and the maps in $\mathcal{F}$. On the other hand, Theorem 4.2 shows that unlike $\mathcal{F}$, $\mathcal{G}$ is not finitely generated.

First, consider type I breakpoints.

**Lemma 3.1.** Let $\{\mathcal{I}_1 < \cdots < \mathcal{I}_{2n+1}\}$ be a partition of $[0,1]$. Let $g \in \mathcal{G}$. Suppose that for $i = 1, \ldots, n$, $g$ is an affine segment on $\mathcal{I}_{2i}$ with slope $(-1)^{p_i}2^{k_i}$, and $Y = g(\mathcal{I}_{2i})$ is the same for all $i$. If integers $\{l_i\}$ exist such that

$$\sum_{i=1}^n 2^{-k_i} = \sum_{i=1}^n 2^{-l_i}, \quad (3)$$

then $f_1 \in \mathcal{F}$, $g_1 \in \mathcal{G}$, and another partition of $[0,1] \{\mathcal{I}_1 < \cdots < \mathcal{I}_{2n+1}\}$ exist such that composition $g_1(f_1(x)) = g(x)$ for any $x \in [0,1]$, $g_1(\mathcal{I}_j) \simeq g(\mathcal{I}_j)$ for any $j$, and

- for odd $j$, $|\mathcal{I}_j| = |\mathcal{I}_j|$;
- for even $j = 2i$, $g_1$ on $\mathcal{I}_{2i}$ is an affine segment with slope $(-1)^{p_i}2^{l_i}$.

**Proof.** The set of intervals $\{\mathcal{I}_j\}$ is completely defined once their lengths are defined. Specifically, let

$$|\mathcal{I}_j| = \begin{cases} |\mathcal{I}_j|, & \text{for odd } j; \\ |\mathcal{Y}| \cdot 2^{-l_i}, & \text{for even } j = 2i. \end{cases} \quad (4)$$

Figure 1: Illustration of the definitions of legs (a), affine legs (b), and window affine on an interval (c). $m = 3$ in the figure.
The endpoints of any interval \( I \) are dyadic by construction. To show the above partition is a valid one, note that for \( i = 1, 2, \ldots, n \),

\[
|I_{2i}| = |\mathcal{Y}| \cdot 2^{-ki}, |I_{2i+1}| = |\mathcal{Y}| \cdot 2^{-li}
\]

By (3)

\[
\sum_{i=1}^{n} |I_{2i}| = \sum_{i=1}^{n} |I_{2i+1}| \sum_{j=1}^{2n+1} |I_j| = \sum_{j=1}^{n} |I_j| = 1.
\]

Construct \( g_1 \) as follows. With odd \( j \), for \( x \in I_j \), let \( g_1(x) = g(x - d_j) \), where \( d_j = I_{2i} - I_{2i-1} \). For even \( j = 2i \), the graph of \( g_1(x) \) on \( I_j \) is the affine segment that connects the right endpoint of \( g_1 \) on \( I_{j-1} \) and the left endpoint of \( g_1 \) on \( I_{j+1} \). Thus by construction (4), the slope of the affine segment is \((-1)^{p_i}2^{li} \). Moreover, by (3), \( g_1 \) is \( \lambda \)-preserving. Hence, \( g_1 \in G \).

Construct \( f_1 \) as follows. Let \( f_1(0) = 0 \). For \( j = 1, 2, \ldots, 2n+1 \) and \( x \in I_j \), the slope of \( f_1 \) is set to 1 for odd \( j \) and to \( 2^{ki-li} \) for even \( j = 2i \). By construction, any breakpoint of \( f_1 \) is dyadic and any slope is an integer power of 2. To validate that \( f_1 \in F \), note that

\[
|I_{2i}| = |\mathcal{Y}| \cdot 2^{-ki} \implies |f_1(I_{2i})| = |\mathcal{Y}| \cdot 2^{-ki} \cdot 2^{ki-li} = |\mathcal{Y}| \cdot 2^{-li}
\]

By (3)

\[
\sum_{i=1}^{n} |f_1(I_{2i})| = \sum_{i=1}^{n} |I_{2i}| \implies \sum_{j=1}^{2n+1} |f_1(I_j)| = \sum_{j=1}^{2n+1} |I_j| = 1
\]

Finally, to show that \( g_1(f_1) = g \), note that by construction of \( f_1 \) and \( g_1 \), for \( j = 1, 2, \ldots, 2n+1 \),

\[
f_1(I_j) \approx g(I_j), g_1(I_j) \approx g(I_j) \implies g_1(f_1(I_j)) \approx g(I_j)
\]

Hence, \( g_1(f_1) = g \).

\[\square\]

**Corollary 3.2.** Let \( \{I_j' \} \) be a partition of \([0, 1]\). Let \( g \in G \). Suppose that for \( i = 1, \ldots, n \), \( g \) is a piecewise affine segment containing a single type I breakpoint \( A_i \) on \( I_{2i}' \). Suppose that \( \mathcal{Y} = g(I_{2i}') \) is the same and \( A_i, y \) is the same for all \( i = 1, \ldots, n \). Let \((-1)^{p_i}2^{li}\) be the slope of the affine segment on \( g^{-1}([\mathcal{Y}, A_i, y]) \cap I_{2i}' \) and \((-1)^{p_i}2^{ki}\) be the slope of the affine segment on \( g^{-1}([\mathcal{Y}, A_i, y]) \cap I_{2i}' \). If (3) holds, then \( f_1 \in F \), \( g_1 \in G \), and another partition of \([0, 1]\) \( \{I_j' \} \) exist such that the composition \( g_1(f_1(x)) = g(x) \) for any \( x \in [0, 1] \),

- for odd \( j \), \( |I_j'| = |I_j'| \) and \( g_1(I_j') \approx g(I_j) \);
- for even \( j \), \( g_1 \) on \( I_j' \) is an affine segment.

**Proof.** By Lemma 3.1, one can replace the affine segment of \( g \) on \( g^{-1}([A_i, y \cdot \mathcal{Y}]) \cap I_{2i}' \) with another one with slope \((-1)^{p_i}2^{li}\) for \( i = 1, 2, \ldots, n \), while keeping the slope of the affine segment unchanged on \( g^{-1}([\mathcal{Y}, A_i, y]) \cap I_{2i}' \). As a result, \( g_1 \) on \( [I_{2i}', \mathcal{Y}^{-1}] \) is an affine segment with no type I breakpoint inside.  

\[\square\]
Figure 2: Use of Lemma 3.1 and Corollary 3.2. The red segments in \( g \) are replaced by those in \( g_1 \), where \( g = g_1 \circ f_1 \). As a result, type I breakpoints \( A_1, A_2, A_3 \) of \( g \) are eliminated in \( g_1 \) because the left and the right derivatives are the same at \( A_1', A_2', A_3' \). The number next to an affine segment represents the absolute value of the slope.
Figure 2 illustrates the use of Lemma 3.1 and Corollary 3.2.

By Corollary 3.2, \( g \) can be constructed with \( g_1 \) by eliminating certain type I breakpoints of \( g \). In particular, the following corollary holds.

**Corollary 3.3.** Let \( \{I_i\} \) for \( i = 1, \ldots, m \) be a set of intervals with mutually disjoint interiors. Let \( g \in G \). If \( g \) is monotone on \( I_i \) and \( Y = g(I_i) \) is the same for all \( i \), and \( g^{-1}(Y) = \bigcup_{i=1}^{m} I_i \), then all the type I breakpoints in the interiors of \( \{I_i\} \) can be eliminated.

**Proof.** \( \forall y \in Y \), \( g^{-1}(Y) \) consists of one point on each \( I_i \). If the derivative exists at all points of \( g^{-1}(y) \), then (2) holds and so does (3) with a common set of \( \{l_i\} \) for any \( y \). The set of the affine segments in the neighborhood of \( g^{-1}(y) \), i.e., \( g^{-1}([y - \delta, y + \delta]) \) for some \( \delta > 0 \), with slopes of the absolute values equal to \( 2^{k(y)} \) on \( I_i \) can be replaced by the affine segments with slopes of the absolute values equal to \( 2^{k_i} \) with \( \sum_{i=1}^{m} 2^{-k_i} = 1 \). The set \( \{2^{k_i}\} \) remain the same for all \( y \in Y \). Thus, by Corollary 3.2, the monotone piecewise affine segment on each \( I_i \) is replaced by an affine segment and any type I breakpoint in the interior of \( I_i \) is eliminated.

Corollary 3.3 covers the case of \( g^{-1}(g(\bigcup_{i=1}^{m} I_i)) = \bigcup_{i=1}^{m} I_i \). The opposite case is where \( \forall y \in g(\bigcup_{i=1}^{m} I_i) \), \( g^{-1}(y) \notin \bigcup_{i=1}^{m} I_i \), which is addressed next.

**Lemma 3.4.** Let \( \{I_1 < \cdots < I_{2m+1}\} \) be a partition of \([0, 1]\). Suppose that \( g \in G \) is an affine segment on \( I_{2i} \) and \( Y = g(I_{2i}) \) is the same for \( i = 1, \ldots, m \). If \( \forall y \in Y \), \( g^{-1}(y) \notin \bigcup_{i=1}^{m} I_{2i} \), then \( f_1 \in F, g_1 \in G, \) and another partition of \([0, 1]\) \( \{I_1 < \cdots < I_{2m+1}\} \) exist such that the composition \( g_1(f_1(x)) = g(x) \) for any \( x \in [0, 1] \), \( g_1(I_{2i}) = g(I_{2i}) \) and \( g_1 \) is an affine segment on \( I_{2i} \) with slope of the absolute value equal to \( 2^{k_i} \) where \( \sum_{i=1}^{m} 2^{-k_i} = 2^{-K} \) for some positive integer \( K \).

**Proof.** Partition \( Y \) into intervals \( \{Y_j\} \), \( j = 1, 2, \ldots, n \), such that no breakpoint exists whose \( y \)-coordinate falls in the interior of any \( Y_j \), i.e., \( B \) breakpoint \( B \) such that \( B_y \in (Y_j^0, Y_j^1) \). Consider \( g^{-1}(Y_j) \). Let

\[
g^{-1}(Y_j) = \{I_{j,1}, I_{j,2}, \ldots, I_{j,m}, I_{j,m+1}, \ldots, I_{j,m+n_j}\}
\]

with mutually disjoint interiors, where the interval \( I_{j,i} \in I_{2i} \) for \( i = 1, \ldots, m \). By the hypothesis of the lemma that \( \forall y \in Y \), \( g^{-1}(y) \notin \bigcup_{i=1}^{m} I_{2i} \), it follows that \( n_j \geq 1 \). Because \( Y = \bigcup_{j=1}^{n} Y_j \), it follows that for \( i = 1, \ldots, m \),

\[
\bigcup_{j=1}^{n} I_{j,i} = I_{2i}.
\]

The graph of \( g \) is affine on every \( I_{j,i} \) because no breakpoint exists on \( Y_j^0 \). Let \( 2^{k_{j,i}} \) be the absolute value of the slope of the affine segment of \( g \) on \( I_{j,i} \). Because \( g \) is \( \lambda \)-preserving,

\[
\sum_{i=1}^{m+n_j} 2^{-k_{j,i}} = 1.
\]
Let
\[\sum_{i=1}^{m} 2^{-k_{j,i}} = L \cdot 2^{-K}, \quad (7)\]
where \(L\) is an odd integer. Because the graph of \(g\) is an affine segment on \(\mathcal{S}_{2i}\), \(k_{j,i}\) does not depend on \(j\) for any \(i = 1, \ldots, m\). For this reason, \(L, K\) do not have subscript \(j\) in (7).

Therefore, a subset of \(\Phi\) is an increasing order. Add such \(2\) terms one-by-one until the sum reaches \(1\). Therefore, a subset of \(\{1, 2, \ldots, m\}\), denoted by \(\Phi_{1}\), exists such that \(\sum_{i \in \Phi_{1}} 2^{-k_{j,i}} = 2^{-K}\). Because \(L - 1\) is even, let
\[\sum_{i=1, i \notin \Phi_{1}}^{m} 2^{-k_{j,i}} = (L - 1) \cdot 2^{-K} = L' \cdot 2^{-K'}, \]
where \(0 < K' < K\) and \(L' < L\). Similarly, a subset of \(\{1, 2, \ldots, m\} \setminus \Phi_{1}\), denoted by \(\Phi_{2}\), exists such that \(\sum_{i \notin \Phi_{2}} 2^{-k_{j,i}} = 2^{-K'}\). Now let
\[k'_{j,i} = \begin{cases} 
    k_{j,i} + K' + 1 - K, & i \in \Phi_{1}; \\
    k_{j,i} + 1, & i \in \Phi_{2}; \\
    k_{j,i}, & i \in \{1, 2, \ldots, m\} \setminus (\Phi_{1} \cup \Phi_{2}).
\end{cases} \quad (8)\]

Therefore,
\[\sum_{i=1}^{m} 2^{-k'_{j,i}} = \left(\sum_{i \in \Phi_{1}} + \sum_{i \in \Phi_{2}} + \sum_{i \in \{1, 2, \ldots, m\} \setminus (\Phi_{1} \cup \Phi_{2})}\right) 2^{-k'_{j,i}}\]
\[= 2^{-K} \cdot 2^{K-K'-1} + 2^{-K'} \cdot 2^{-1} + (L' - 1) \cdot 2^{-K'} = L' \cdot 2^{-K'}.\]

On the other hand,
\[\sum_{i=m+1}^{m+n} 2^{-k_{j,i}} = 1 - L \cdot 2^{-K}.\]

Similarly, a subset of \([m+1, m+2, \ldots, m+n]\), denoted by \(\Psi\), exists such that \(\sum_{i \in \Psi} 2^{-k_{j,i}} = 2^{-K}\). Now let
\[k'_{j,i} = \begin{cases} 
    k_{j,i} - 1, & i \in \Psi; \\
    k_{j,i}, & i \in \{m+1, m+2, \ldots, m+n\} \setminus \Psi.
\end{cases} \quad (9)\]

Therefore,
\[\sum_{i=m+1}^{m+n} 2^{-k'_{j,i}} = \left(\sum_{i \in \Psi} + \sum_{i \in \{m+1, m+2, \ldots, m+n\} \setminus \Psi}\right) 2^{-k'_{j,i}}\]
\[= 2^{-K} \cdot 2^{1} + (1 - L \cdot 2^{-K} - 2^{-K}) = 1 - L' \cdot 2^{-K'}.\]
From \{k_{j,i}\} to \{k'_{j,i}\}, L drops to L'. The above process continues until L = 1. Because k_{j,i} does not depend on j when \(i = 1, 2, \ldots, m\), neither does \(k'_{j,i}\) in (8). This property remains in the process.

When the process ends, for any \(j\)

\[
\sum_{i=1}^{m} 2^{-k'_{j,i}} = 2^{-K''}.
\]  

(10)

Because

\[
\sum_{i=1}^{m+n_j} 2^{-k''_{j,i}} = \sum_{i=1}^{m+n_j} 2^{-k_{j,i}} = 1,
\]  

apply Lemma 3.1 for all \(j\) one-by-one. Then \(f_1 \in F, g_1 \in G,\) and another partition of [0, 1] \{\mathcal{J}_1 < \cdots < \mathcal{J}_{2m+1}\} exist such that the composition \(g_1(f_1(x)) = g(x)\) for any \(x \in [0, 1]\). Just like (5), for \(i = 1, \ldots, m, \mathcal{J}_{2i}\) can be decomposed to \(\mathcal{J}_{2i} = \bigcup_{j=1}^{n_j} \mathcal{J}_{j,i}\) such that \(g_1(\mathcal{J}_{j,i}) \approx g(\mathcal{J}_{j,i}).\) Thus, \(g_1(\mathcal{J}_{2i}) \approx g(\mathcal{J}_{2i}).\) The graph of \(g_1\) is an affine segment on \(\mathcal{J}_{j,i}\) with slope of the absolute value equal to \(2^{k''_{j,i}},\) which does not depend on \(j\) when \(i = 1, 2, \ldots, m.\) Thus, \(g_1\) is an affine segment in every \(\mathcal{J}_{2i}.\) The absolute values of the slopes of these affine segments for \(i = 1, \ldots, m\) satisfies (10). This completes the proof. \(\Box\)

Figure 3 shows an example to illustrate the proof of Lemma 3.4.

Next, consider type II breakpoints.
Let $w_1$ be an $m$-fold window affine map, which is an $m$-fold window affine on an interval $I$. Let $2^{k_i}$ be the absolute value of the slope of the $i$-th leg of $w_1$ for $i = 1, 2, \ldots, m$. \( \sum_{i=1}^{m} 2^{-k_i} = 1 \) to be \( \lambda \)-preserving. Suppose that $g_1$ on $I$ is an affine segment, with slope of the absolute value of $2^{k_i}$. Then the composition $g_1(w_1)$ is identical to $g_1$ on $[0, 1)$. On $I$ the affine segment of $g_1$ is replaced by an $m$-fold window affine, whose $i$-th leg has an absolute value of the slope $2^{k_i}$. The ratio of the slopes of any two legs of $g_1(w_1)$ is the same as that of the corresponding legs of $g_1$. Lemma 3.5 follows.

**Lemma 3.5.** Suppose that $g$ on an interval $I$ is an $m$-fold window affine. Let \( 2^{k_i} \) be the absolute value of the slope of the $i$-th leg. If \( \sum_{i=1}^{m} 2^{-k_i} = 1 \), then $g_1 \in G$ and an $m$-fold window affine map $w_1$ exist such that the composition $g_1(w_1(x)) = g(x)$ for any $x \in [0, 1)$. $g_1([0, I]) = g([0, I])$, and $g_1$ is an affine segment on $I$ with slope of the absolute value of $2^{k_i}$.

Figure 4 shows the use of Lemma 3.5. The interval $I$ can be anywhere on $[0, 1)$ for odd $m$. For even $m$, $I$ must cover at least one endpoint; that is, $I = [0, 1]$.

A special case of Lemma 3.5 is when $K = 0$. Because \( \sum_{i=1}^{m} 2^{-k_i} = 1 \), \( \forall y \in g(I) \), \( g^{-1}(y) \in I \). Thus, \( g^{-1}(g(I)) = I \). This case is similar to what Corollary 3.3 covers. On the other hand, just like Lemma 3.4, Lemma 3.6 covers the case where \( \forall y \in g(\bigcup_{i=1}^{m} I_i) \), \( g^{-1}(y) \notin \bigcup_{i=1}^{m} I_i \). Specifically, in Lemma 3.4 when $I_{2i}$, for $i = 1, \ldots, m$, are adjacent, i.e., each of $I_3, I_5, \ldots, I_{2m-1}$ is reduced to a single point, the $m$ affine segments on $I_{2i}$ form a $m$-fold window affine on the interval $\bigcup_{i=1}^{m} I_{2i}$ and can be further simplified as stated in Lemma 3.6.

**Lemma 3.6.** Suppose that in Lemma 3.4, $\bigcup_{i=1}^{m} I_{2i}$ is an interval, denoted by $I$. That is, $g$ is an $m$-fold window affine on $I$. Then $f_1 \in F, g_1 \in G$, an $m$-fold window affine map.
Figure 5: Use of Lemma 3.6. The three solid red affine segments in the right figure represent \( g_1 \). The leftmost one is replaced by three dashed red segments due to a window affine map \( w_1 \) with \( m = 3 \). The resultant five red segments, dashed and solid, are horizontally adjusted by \( f_1 \) to arrive at \( g \) in the left figure. Two type II breakpoints of \( g \) are eliminated in \( g_1 \).

\[ w_1, \text{ and another partition of } [0, 1] \{ J_1 < \cdots < J_{2m+1} \} \text{ exist such that the composition } g_1(w_1(f_1(x))) = g(x) \text{ for any } x \in [0, 1], \text{ and } \bigcup_{i=1}^{m} J_{2i} \text{ is an interval on which } g_1 \text{ is an affine segment.} \]

**Proof.** The maps \( f_1, g_2 \) and the interval partition \( \{ J_1 < \cdots < J_{2m+1} \} \) are obtained by Lemma 3.4 such that \( g_2(f_1) = g \). Each of \( J_3, J_5, \ldots, J_{2m-1} \) is reduced to a single point just like \( J_3, J_5, \ldots, J_{2m-1} \). Thus, \( \bigcup_{i=1}^{m} J_{2i} \) is an interval, denoted by \( J \). In the proof of Lemma 3.4, \( g_2 \) on \( J \) is an \( m \)-fold window affine, just like \( g \) on \( J \). The difference between the two \( m \)-fold window affines is that the slopes of their legs are described in (7) and (10) respectively. Now let \( w_1 \) be an \( m \)-fold window affine map and an \( m \)-fold window affine on the interval \( J \). On \( J \) the slope of the \( i \)-th leg is \((-1)^{i+1}2^{2k''}j_i^{-2k''}\) for \( i = 1, 2, \ldots, m \). Modify \( g_2 \) to become \( g_1 \) by replacing the \( m \)-fold window affine of \( g_2 \) on \( J \) by an affine segment of \( g_1 \) with slope of the absolute value of \( 2^{2k''} \). The sign of the slope is such that \( g_1 \) is continuous at the endpoints of \( J \). By Lemma 3.5, \( g_2 = g_1(w_1) \). Therefore, \( g_1(w_1(f_1)) = g \). \qed

Figure 5 illustrates the use of Lemma 3.6. By Lemma 3.6, \( g \) can be constructed with \( g_1 \) by eliminating the \( m - 1 \) type II breakpoints of \( g \).

In Lemma 3.5 and Lemma 3.6, \( g \) must be an \( m \)-fold window affine on \( J \). The \( m \)-fold window affine consists of \( m \) legs each of which is an affine segment. One can generalize
the notion of \(m\)-fold window affine such that it consists of \(m\) legs each of which itself consists of piecewise affine segments. The precise definition is given below.

**Definition 3.7 (Generalized Window Affine).** A map \(g\) is a generalized \(m\)-fold window affine on an interval \(I\) if the interval \(I\) can be partitioned into \(\{I_1 < I_2 < \cdots < I_m\}\) such that for any \(i\) and \(j\), \(g(I_i) \equiv g(I_j)\) when \(i - j\) is an even number and \(g(I_i) \not\equiv g(I_j)\) when \(i - j\) is an odd number. Each \(I_i\) is referred to as a component interval of \(I\).

Figure 6 provides two examples of the generalized \(m\)-fold window affine, one for an even \(m\) and the other for an odd \(m\).

**Corollary 3.8.** Lemma 3.5 and Lemma 3.6 hold if \(g\) is a generalized \(m\)-fold window affine instead of an \(m\)-fold window affine on the interval \(I\), except that in the conclusion \(g_1\) is a piecewise affine segment, instead of an affine segment, on \(I\) where \(g_1(I) \equiv g(I_0)\) or \(g_1(I) \not\equiv g(I_0)\) with \(I_0\) being one component interval of \(I\). Whether \(g_1(I) \equiv g(I_0)\) or \(g_1(I) \not\equiv g(I_0)\) depends on the continuity of \(g_1\).

Recall that Corollary 3.3 and Lemma 3.4 consider two cases respectively: either \(\forall y \in g(I), g^{-1}(y) \subset I\) or \(\forall y \in g(I), g^{-1}(y) \not\subset I\). Corollary 3.9 addresses a mixed case using the notion of generalized window affine.

**Corollary 3.9.** Let \(g \in G\). Let the interval \(I \subset [0,1]\) and \(\mathcal{Y} = g(I)\). Suppose that \(I = \bigcup_{i=1}^{m} I_i\) where \(\{I_i\}\) are \(m\) intervals with mutually disjoint interiors and \(g\) is monotone on every \(I_i\). Suppose that \(c \in (\mathcal{Y}^0, \mathcal{Y}^1)\) exists such that \(\forall y \in (\mathcal{Y}^0, c), g^{-1}(y) \subset I\), and \(\forall y \in [c, \mathcal{Y}^1], g^{-1}(y) \not\subset I\). Then \(f_1 \in F, g_1 \in G\), an \(m\)-fold window affine map \(w_1\), and an interval \(I\) exist such that \(g = g_1(w_1(f_1))\), the graph of \(g_1\) on \(I\) consists of two affine segments connected by a type I breakpoint, and \(g_1([0, \mathcal{Y}^0]) \equiv g([0, \mathcal{Y}^0]), g_1([\mathcal{Y}^1, 1]) \equiv g([\mathcal{Y}^1, 1])\).

**Proof:** Because \(g\) is monotone on every \(I_i\), \(I_i\) can be partitioned into \(I_i = I_{i,0} \cup \mathcal{I}_{i,1}\) where \(g(I_{i,0}) = [\mathcal{Y}^0, c]\) and \(g(I_{i,1}) = [c, \mathcal{Y}^1]\). Combining Corollary 3.3 and Lemma 3.4, \(f_1 \in F, g_2 \in G\), an interval \(I = \bigcup_{i=1}^{m} (I_{i,0} \cup \mathcal{I}_{i,1})\) exist such that \(g_2(f_1) = g\), and \(g_2([0, \mathcal{Y}^0]) \equiv \).
g([0, x_0]), g_2([x_1, 1]) \cong g([x_1, 1]), \) where \( \mathcal{I}_{i,0} \) and \( \mathcal{I}_{i,1} \) are intervals with mutually disjoint interiors. For \( i = 1, 2, \ldots, m \), the graph of \( g_2 \) on \( \mathcal{I}_{i,1} \) is an affine segment with slope of the absolute value \( 2k_i^m \) with \( \sum_{i=1}^{m} 2^{-k_i^m} = 2^{-K''} \) for some positive integer \( K'' \) and the graph of \( g_2 \) on \( \mathcal{I}_{i,0} \) is an affine segment with the absolute value of the slope equal to \( 2k_i^m - K'' \). Recall that \( \sum_{i=1}^{m} 2^{-k_i^m} + K'' = 1 \). Hence, \( g_2 \) is a generalized \( m \)-fold window affine on \( \mathcal{I} \) where \( \mathcal{I}_{i,0} \cup \mathcal{I}_{i,1} \) is a component interval. By Corollary 3.8, an \( m \)-fold window affine map \( w \) exists such that \( w \) is an \( m \)-fold window affine on \( \mathcal{I} \) and \( g_2 = g_1(w) \). The interval \( \mathcal{I} \) can be partitioned to two intervals \( \mathcal{I} = \mathcal{I}_0 \cup \mathcal{I}_1 \) with mutually disjoint interiors such that the graph of \( g_1 \) on \( \mathcal{I}_0 \) is affine with slope of the absolute value equal to 1 and the graph of \( g_1 \) on \( \mathcal{I}_1 \) is affine with slope of the absolute value equal to \( 2^{K''} \). \( \square \)

Corollary 3.9 holds under a slightly modified hypothesis: \( \forall y \in [\mathcal{Y}_0, c], \ g^{-1}(y) \not\subset \mathcal{I}, \) and \( \forall y \in [c, \mathcal{Y}_1], \ g^{-1}(y) \not\subset \mathcal{I}. \) One can further extend the result to a scenario where \( c_1, c_2 \in (\mathcal{Y}_0, \mathcal{Y}_1) \) exist with \( c_1 < c_2 \) such that \( \forall y \in [\mathcal{Y}_0, c_1] \cup (c_2, \mathcal{Y}_1], \ g^{-1}(y) \not\subset \mathcal{I}, \) and \( \forall y \in [c_1, c_2], \ g^{-1}(y) \not\subset \mathcal{I}. \) The same conclusion as in Corollary 3.9 holds except that \( g_1 \) on \( \mathcal{I} \) consists of three affine segments connected by two type I breakpoints.

In Lemma 3.4 and Lemma 3.6 \( g \) is required to be an \( m \)-fold window affine on \( \mathcal{I} \). This requirement is relaxed in Lemma 3.10.

**Lemma 3.10.** Replacing “\( g \in \mathcal{G} \) is an affine segment on \( \mathcal{I}_{2i} \)” in Lemma 3.4 and replacing “\( g \) is an \( m \)-fold window affine on \( \mathcal{I} \)” in Lemma 3.6 by “let \( g \) be monotone on \( \mathcal{I}_{2i} \) for all \( i \)”, the conclusions in Lemma 3.4 and in Lemma 3.6 still hold.

**Proof.** The difference from Lemma 3.4 and Lemma 3.6 is that \( g \) is not necessarily an affine segment on \( \mathcal{I}_{2i} \). As in the proof of Lemma 3.4, partition the interval \( \mathcal{Y} \) into intervals \( \{\mathcal{Y}_j\}, \ j = 1, 2, \ldots, n \) such that no breakpoint exists whose \( y \)-coordinate falls in the interior of any \( \mathcal{Y}_j \). Consider \( g^{-1}(\mathcal{Y}_j) \). Let

\[
g^{-1}(\mathcal{Y}_j) = \{\mathcal{I}_{j,1}, \mathcal{I}_{j,2}, \ldots, \mathcal{I}_{j,m}, \mathcal{I}_{j,m+1}, \ldots, \mathcal{I}_{j,m+n} \}.
\]

By the hypothesis of the lemma, the interval \( \mathcal{I}_{j,i} \subset \mathcal{I} \) for \( i = 1, \ldots, m \), and the interval \( \mathcal{I}_{j,i} \cap \mathcal{I} = \emptyset \) for \( i = m + 1, \ldots, m + n \) with \( n \geq 1 \). Let \( 2^{k_{i,j}} \) be the absolute value of the slope of the affine segment on \( \mathcal{I}_{j,i} \).

Let \( j^* = \arg\min_{j=1,2,\ldots,n} n_j \). If \( \arg\min_{j=1,2,\ldots,n} n_j \) is not unique, then pick any one of them as \( j^* \). For \( j = 1, \ldots, n \), let

\[
k_{j,i}'' = \begin{cases} 
k_{j,i}' + i - m - n_j^* + 1, & i = m + n_j^*, m + n_j^* + 1, \ldots, m + n_j - 1; \\
k_{j,i}' + n_j - n_j^*, & i = m + n_j.
\end{cases}
\]

It is easy to verify that (11) holds. As in the proof of Lemma 3.4, one can find \( f_1 \in \mathcal{F}, \ g_1 \in \mathcal{G} \) and a partition of \( [0, 1] \) \( \{\mathcal{I}_1 < \cdots < \mathcal{I}_{2m+1}\} \), where \( \mathcal{I}_{2i} \) is further partitioned to \( \mathcal{I}_{2i} = \)
Theorem 3.11. Let $f_{1}(x) = g(x)$, $g_{1}(j_{i}, i) = g(j_{i}, i)$, and the absolute value of the slope changes from $2^{k_{i,j}}$ of $g$ on $j_{i,j}$ to $2^{k_{i,j}'}$ of $g_{1}$ on $j_{i,j}$ for $i = 1, \ldots, n_{j}$ and $j = 1, \ldots, n$ and remains unchanged from $g$ on $[0, 1] \setminus \bigcup_{i=1}^{m} j_{2,i}$ to $g_{1}$ on $[0, 1] \setminus \bigcup_{i=1}^{m} j_{2,i}$.

Note from the preceding construction (12) that $k_{i,j}'$ does not depend on $j$ when $i = 1, 2, \ldots, m$, because $n_{j'} \geq 1$. Thus, the graph of $g_{1}$ on $j_{2,i}$ is one affine segment. Lemma 3.4 is applicable to $g_{1}$ and the conclusion in Lemma 3.4 still holds.

Moreover, if $\bigcup_{i=1}^{m} j_{2,i}$ is an interval, then as in the proof of Lemma 3.6, $\bigcup_{i=1}^{m} j_{2,i}$ is an interval too, denoted by $J$. The graph of $g_{1}$ on $J$ is an $m$-fold window affine. Hence, Lemma 3.6 is applicable to $g_{1}$ and the conclusion in Lemma 3.6 still holds. □

In summary, Lemma 3.1 to Lemma 3.10 can be used to eliminate type I and type II breakpoints. Theorem 3.11 shows that for any $g \in G$, all interior breakpoints can be eliminated by repetitively applying these lemmas and corollaries. The only $G$ maps that have no interior breakpoints are the identity maps $g_{0,+}$ and $g_{0,-}$.

Theorem 3.11. Let $g \in G$. Then $g$ is equal to the composition of an identity map followed by a combination of $\mathbb{F}$ maps and window affine maps.

Proof. Suppose that Lemma 3.1 to Lemma 3.10 have been applied to eliminate the breakpoints of $g$ such that $g = g_{1} \circ f_{1} \circ w_{1} \circ f_{2} \circ w_{2} \circ \cdots$ where $g_{1} \in G$ and $f_{1} \circ w_{1} \circ f_{2} \circ w_{2} \circ \cdots$ represent a combination of $\mathbb{F}$ maps and window affine maps. Assume that no more interior breakpoints in $g_{1}$ can be eliminated using the preceding lemmas. Next we show that $g_{1}$ has no interior breakpoints.

Denote by $A_{0}$ the point of $g_{1}$ at $A_{0,x} = 0$. Without loss of generality, suppose that the derivative at $A_{0}$ is positive. As $x$ increases from 0, $g_{1}(x)$ increases until it reaches another point $A_{1}$ where $g_{1}(x)$ stops increasing. If $A_{1,x} = 1$, then $g_{1} = g_{0,+}$ and the proof is done. Otherwise, $A_{1}$ must be a type II breakpoint and the right derivative of $g_{1}$ is negative. As $x$ increases from $A_{1,x}$, $g_{1}(x)$ decreases until it reaches another type II breakpoint $A_{2}$.

First, suppose that $A_{2,y} \leq A_{0,y}$. Then a unique point $B_{2}$ exists where $A_{1,x} < B_{2,x} < A_{2,x}$ and $B_{2,y} = A_{0,y}$, as shown in Figure 7(a). Consider the following three cases illustrated respectively by the three dash lines coming out of point $A_{2}$ in Figure 7(a).

- Assume $\max(g_{1}([A_{2,x}, 1])) < A_{0,y}$. Then $\forall y \in [A_{0,y}, A_{1,y}]$, $\exists x \in [B_{2,x}, 1]$ such that $g_{1}(x) = y$. Thus, $g_{1}^{-1}(y)$ consists of two points $x_{1}, x_{2}$ where $x_{1} \in [A_{0,x}, A_{1,x}]$ and $x_{2} \in [A_{1,x}, B_{2,x}]$, and $A_{0}A_{1}$ and $A_{1}B_{2}$ must be affine segments with slopes $-2$ respectively. By Lemma 3.5, the breakpoint $A_{1}$ can be eliminated with a 2-fold window affine map. Contradiction.

- Assume $A_{0,y} \leq \max(g_{1}([A_{2,x}, 1])) \leq A_{1,y}$. Let $C_{1}, C_{2}$ be the points between the points $A_{0}, A_{1}$ and between the points $A_{1}, B_{2}$, respectively, such that $C_{1,y} = C_{2,y} = \max(g_{1}([A_{2,x}, 1]))$. Then $\forall y \in (\max(g_{1}([A_{2,x}, 1])), A_{1,y}], \exists x \in [B_{2,x}, 1]$ such that $g_{1}(x) = y$. Thus, $C_{1}A_{1}$ and $A_{1}C_{2}$ must be affine segments with slopes $2, -2$ respectively.

On the other hand, $\forall y \in [A_{0,y}, \max(g_1([A_{2,x}, 1]))]$, $\exists x \in (B_{2,x}, 1]$ such that $g_1(x) = y$. By Lemma 3.4, one can eliminate all type I breakpoints, if any, between $A_0, C_1$ and between $C_2, B_2$. By Lemma 3.1, make $A_0C_1$ and $C_2B_2$ affine segments with the same slope except for the sign. Therefore, the graph of $g_1$ on $A_0C_1A_2B_2$ is a generalized 2-fold window affine. By Corollary 3.8, the breakpoint $A_1$ can be eliminated with a 2-fold window affine map. Contradiction.

- Assume $\max(g_1([A_{2,x}, 1])) > A_{1,y}$. By Lemma 3.4, one can eliminate all type I breakpoints, if any, between $A_0, A_1$ and between $A_1, B_2$. By Lemma 3.1, make $A_0A_1$ and $A_1B_2$ affine segments with the same slope except for the sign. Therefore, the graph of $g_1$ on $A_0A_1B_2$ is a 2-fold window affine. By Lemma 3.5, the breakpoint $A_1$ can be eliminated with a 2-fold window affine map. Contradiction.

In the following, suppose that $A_{2,y} > A_{0,y}$. Then a unique point $B_1$ exists where $A_{0,x} < B_{1,x} < A_{1,x}$ and $B_{1,y} = A_{2,y}$. As $x$ increases from $A_{2,x}$, $g_1(x)$ increases until it reaches another type II breakpoint $A_3$. If $A_{3,y} \geq A_{1,y}$, then a unique point $B_3$ exists where $A_{2,x} < B_{3,x} < A_{3,x}$ and $B_{3,y} = A_{1,y}$, as shown in Figure 7(b). By Corollary 3.9 and Lemma 3.10, the type II breakpoints $A_1$ and $A_2$ can be eliminated. Contradiction.

Therefore, $A_{3,y} < A_{1,y}$. The process continues as shown in Figure 7(c). For odd $i$, the type II breakpoint $A_i$ is facing up and $A_{i,y} < A_{i+2,y}$. $A_{2i+1,y} > A_{2j,y}$ for any $i, j$. Suppose that $A_n$ is the endpoint where $A_{n,x} = 1$. $\min(A_{n-1,y}, A_{n-2,y}) < A_{n,y} < \max(A_{n-1,y}, A_{n-2,y})$. Therefore, a unique point $B_{n-1}$ exists where $A_{n-2,x} < B_{n-1,x} < A_{n-1,x}$ and $B_{n-1,y} = A_{n,y}$. By Lemma 3.4, one can eliminate all type I breakpoints, if any, between $B_{n-1}, A_{n-1}$ and between $A_{n-1}, A_n$. By Lemma 3.1, make $B_{n-1}A_{n-1}$ and $A_{n-1}A_n$ affine segments with the same slope except for the sign. Therefore, the graph of $g_1$ on $B_{n-1}A_{n-1}A_n$ is a 2-fold window affine. By Lemma 3.5, the breakpoint $A_{n-1}$ can be eliminated with a 2-fold window affine map. Contradiction.

Hence, we conclude that $g_1$ has no interior breakpoints. \hfill \square
Because any \( f \) map can be generated by the two generators defined in (1), it suffices to study how to construct the window affine maps thanks to Theorem 3.11.

Recall that Definition 2.12 defines an \( m \)-fold window affine map. We next define the notation of an \( m \)-fold window affine map. Denote by \( w_{m,f} \) an \( m \)-fold window affine map where \( f \subset [0,1] \) is the interval such that \( w_{m,f}(x) = x \) for \( x \in [0,1] \setminus f \) and \( w_{m,f}(x) \) is an \( m \)-fold window affine on the interval \( f \). Specifically, let \( \{f_1 < f_2 < \cdots < f_m\} \) be a partition of \( f \). The graph of \( w_{m,f} \) is an affine segment on each \( f_i \) with slope \((-1)^{i-1}2^{k_i}\) where \( \sum_{i=1}^{m} 2^{-k_i} = 1 \). The graph of \( w_{m,f} \) on \( f_i \) is referred to as the \( i \)-th leg of \( w_{m,f} \). The map \( w_{m,f} \) defined here is from the lower left corner to the upper right corner in the plane of \([0,1] \times [0,1] \). The map \( 1 - w_{m,f} \), which is from the upper left corner to the lower right corner, is given by \( g_{0,-}(w_{m,f}) \).

**Lemma 3.12.** Any \((m+2)\)-fold window affine map \( w_{m+2,f} \) on an interval \( f \) is equal to \( w_{m,f}(w(f)) \) where \( w \) is a 3-fold window affine map and \( f \in \mathbb{F} \).

**Proof.** Let \( w_{m,f} \) be an \( m \)-fold window affine map. Let \( f_m \) be the interval on which the \( m \)-th leg of \( w_{m,f} \) resides. Let \( w_{3,f_m} \) be a 3-fold window affine map on \( f_m \) with the absolute values of the slopes being \( 2^{q_j} \) for \( j = 1, 2, 3 \) on the three legs respectively. By definition, \( \sum_{j=1}^{3} 2^{-q_j} = 1 \).

By construction, \( w_{m,f}(w_{3,f_m}) \) is an \((m+2)\)-fold window affine map on the interval \( f \) where \( w_{m,f}(w_{3,f_m})(x) \) consists of three legs on \( f_m \) with slopes equal to \((-1)^{m+1+j}2^{k_m+q_j} \) for \( j = 1, 2, 3 \), and \( w_{m,f}(w_{3,f_m})(x) = w_{m,f}(x) \) for \( x \in [0,1] \setminus f_m \). The interval \( f_m \) is thus partitioned to three intervals \( f_{m,j} \) corresponding to the three legs.

Let \( f_1' < f_2' < \cdots < f_{m+2}' \) be the partition of \( f \) of any desired \((m+2)\)-fold window affine map \( w_{m+2,f} \). Recall that \( w_{m+2,f} \) is an affine segment on each \( f_i' \) with slope \((-1)^{i-1}2^{l_i} \). Because

\[
\sum_{i=1}^{m+2} 2^{-l_i} = \sum_{i=1}^{m-1} 2^{-k_i} + \sum_{j=1}^{3} 2^{-k_m+q_j} = 1,
\]

by Lemma 3.1, \( f \in \mathbb{F} \) exists to map \( f_i' \to f_j' \) for \( i = 1, \ldots, m-1 \) and \( f_{m+1+j} \) to \( f_{m,j} \) for \( j = 1, 2, 3 \) without altering anything on \([0,1] \setminus f \). Hence, \( w_{m+2,f} = w_{m,f}(w_{3,f_m}(f)) \). \( \square \)

From Lemma 3.12, any \( m \)-fold window affine map \( w_{m,f} \) can be constructed by repetitively applying 3-fold window affine maps on appropriate intervals of \( f \) to a 1-fold \( w_{1,f} \) for odd \( m \) or a 2-fold window affine map \( w_{2,f} \) for even \( m \). The 1-fold window affine map \( w_{1,f} \) is simply \( g_{0,+} \) or \( g_{0,-} \). Next we show that all 3-fold or 2-fold window affine maps can be constructed with a finite number of basic window affine maps.

Define the basic 3-fold window affine map \( \tilde{w}_{3,\frac{1}{3},\frac{1}{2}} \) as the special case of \( w_{3,\frac{1}{3},\frac{1}{2}} \) with the absolute values of the slopes being 2, 4, 4 on the three legs respectively. Lemma 3.14 states that almost any 3-fold window affine map can be constructed with the basic 3-fold window affine map. The remaining cases of 3-fold window affine maps are addressed in Lemma 3.16.
To arrive at Lemma 3.14, we first introduce the notion of a partition point in Lemma 3.13.

**Lemma 3.13.** Let \((x_1, y_1)\) and \((x_2, y_2)\) be two dyadic points where \(x_1 < x_2\) and \(y_1 < y_2\). Suppose that \(y_2 - y_1 \geq x_2 - x_1\). If \(\frac{y_2 - y_1}{x_2 - x_1} \neq 2^k\) for any integer \(k\), then a dyadic point \((x_3, y_3)\) exists with \(x_1 < x_3 < x_2\), \(y_1 < y_3 < y_2\) such that the slopes between \((x_1, y_1)\), \((x_3, y_3)\) and between \((x_2, y_2)\), \((x_3, y_3)\) are non-negative integer powers of 2.

**Proof.** Let 
\[
\frac{y_3 - y_1}{x_3 - x_1} = 2^{k_1}, \quad \frac{y_3 - y_2}{x_3 - x_2} = 2^{k_2}.
\]
Then
\[
x_3 = x_1 + \frac{2^{-k_2}(y_2 - y_1) - (x_2 - x_1)}{2^{k_1} - 1}.
\]
Two integer solutions are given by
\[
\begin{align*}
    k_2 &= \left\lfloor \log_2 \frac{y_2 - y_1}{x_2 - x_1} \right\rfloor \\
    k_1 &= \frac{k_2 + 1}{2}.
\end{align*}
\]
It is easy to verify that in either solution, \((x_3, y_3)\) is dyadic with \(x_1 < x_3 < x_2\) and \(y_1 < y_3 < y_2\). \(\square\)

Any of the two solutions \((x_3, y_3)\) in Lemma 3.13 is referred to as a **partition point** between the points \((x_1, y_1)\) and \((x_2, y_2)\).

**Lemma 3.14.** Any 3-fold window affine map \(w_{3, \mathcal{J}}\) is equal to \(f_1(\tilde{w}_{3, [\frac{1}{4}, \frac{1}{2}]}(f_2))\) for \(f_1, f_2 \in \mathbb{F}\) if \(0 < \mathcal{J}^0 < \mathcal{J}^1 < 1\).

**Proof.** We prove the lemma by construction as illustrated in Figure 8.

The map \(f_2\), which scales \(\tilde{w}_{3, [\frac{1}{4}, \frac{1}{2}]}\) horizontally to \(\tilde{w}_{3, [\frac{1}{4}, \frac{1}{2}]}(f_2)\), does the following.

(a) Map \([0, \mathcal{J}^0]\) to \([0, \frac{1}{2}]\). If \(\frac{1}{\mathcal{J}^0} - \frac{1}{\mathcal{J}^1}\) is in the form of \(2^k\) for some integer \(k\), then \(f_2\) on \([0, \mathcal{J}^0]\) is an affine segment; otherwise, \(f_2\) on \([0, \mathcal{J}^0]\) consists of two affine segments separated by a point \((x_1, y_1)\), the partition point between the points \((0,0)\) and \((\mathcal{J}^0, \frac{1}{4})\) by Lemma 3.13.

(b) Map \([\mathcal{J}^0, \mathcal{J}^1]\) to \([\frac{1}{2}, \frac{1}{4}]\). If \(\frac{1}{\mathcal{J}^0} - \frac{1}{\mathcal{J}^1}\) is in the form of \(2^k\) for some integer \(k\), then \(f_2\) on \([\mathcal{J}^0, \mathcal{J}^1]\) is an affine segment; otherwise, a point \((x_2, y_2)\) exists in Lemma 3.13 such that \(\frac{y_2 - \frac{1}{4}}{x_2 - \mathcal{J}^0}\) and \(\frac{2 - y_2}{x_2 - \mathcal{J}^0}\) are both in the form of \(2^k\). The graph of \(f_2\) on \([\mathcal{J}^0, \mathcal{J}^1]\) consists of six affine segments separated by the partition points \((x_2, \frac{1}{4} + \frac{y_2 - \frac{1}{2}}{2})\), \((\frac{\mathcal{J}^0 + \mathcal{J}^1}{2}, \frac{3}{8})\), \((x_2, \frac{3}{8} + \frac{1}{4} - y_2)\), \((\frac{\mathcal{J}^0 + 3\mathcal{J}^1}{4}, \frac{7}{16})\) and \((x_2, \frac{7}{16} + \frac{y_2 - \frac{1}{2}}{4})\) between the points.
Figure 8: Construction of $w_3, g$ with $\tilde{w}_{3, [\frac{1}{4}, \frac{1}{2}]}$. 
Hence, the conclusion follows.

**Lemma 3.15.** Any 2-fold window affine map $w_{2, \mathcal{J}}$ can be constructed with $w_{2, [\frac{3}{4}, 1]}$ if $\mathcal{J} \subset [0, 1]$.

**Proof.** As in Lemma 3.14, it can be shown with scaling construction that any 2-fold window affine map $w_{2, \mathcal{J}}$ can be constructed with $f_1(w_{2, [\frac{3}{4}, 1]}(f_2))$ for $f_1, f_2 \in F$ if $0 < \mathcal{J}^0 < \mathcal{J}^1 = 1$. On the other hand, if $0 = \mathcal{J}^0 < \mathcal{J}^1 < 1$, $w_{2, \mathcal{J}}$ can be constructed with $g_{0, -}(w_{2, \mathcal{J}^1}(g_{0, -}))$ where $\mathcal{J}^1 = [1 - \mathcal{J}^1, 1]$ and thus $w_{2, \mathcal{J}^1}$ can be constructed with $w_{2, [\frac{3}{4}, 1]}$. Hence, the conclusion follows.
Finally consider the special cases of 3-fold window affine maps that are not addressed in Lemma 3.14.

**Lemma 3.16.** Any 3-fold window affine map \( w_{3,J} \) with \( J^0 = 0 \) and/or \( J^1 = 1 \) is equal to the composition of 2-fold window affine maps and \( f \in F \).

**Proof.** Suppose \( J^1 = 1 \). The case of \( J^0 = 0 \) can be addressed similarly.

Consider the 3-fold window affine map \( \bar{w}_{3,J} \) as the special case of \( w_{3,J} \) with the absolute values of the slopes being 2, 4, 4 on the three legs respectively.

\[
\bar{w}_{3,J} = w_{2,J^1} \left( w_{2,J^0} \right)
\]

where \( J^1 = \left[ \frac{J^0 + 1}{2}, 1 \right] \). Then any \( w_{3,J} = \bar{w}_{3,J}(f) \) for some \( f \in F \) by Lemma 3.1.

Hence, Definition 3.17 summarizes the basic maps of \( G \) used in Lemmas 3.12, 3.14, 3.15, and 3.16.

**Definition 3.17 (Basic Maps of \( G \)).** The basic maps of the monoid \( G \) are \( g^0, +, g^0, -, \bar{w}_{3,J^0} \), \( w_{2,J^1} \), \( w_{2,[0,1]} \).

Figure 9 plots the basic maps in the monoid \( G \).

The following theorem follows from Theorem 3.11, Lemmas 3.12, 3.14, 3.15, and 3.16.

**Theorem 3.18.** Let \( g \in G \). Then \( g \) is equal to the composition of a combination of the basic maps of \( G \) defined in Definition 3.17 and the two generator maps of \( F \) defined in (1).

### 4 \( G \) is not finitely generated

Theorem 3.18 does not imply that \( G \) is finitely generated because the two generator maps of \( F \) defined in (1) are not the elements of \( G \). Indeed, Theorem 4.2 shows that unlike \( F \), \( G \) is not finitely generated. To this end, Lemma 4.1 studies the number of type II breakpoints of a composition map in \( G \). Denote by \( \#(g) \) the number of type II breakpoints of a map \( g \).
Lemma 4.1. Let \( g_1, g_2 \in G \). Then \( #(g_1 \circ g_2) \geq #(g_1) + #(g_2) \).

Proof. Consider two cases.

Case 1. Suppose that a point \( B \) on the graph of \( g_2 \) is a type II breakpoint. There exists a sufficiently small \( \delta > 0 \) such that the graph of \( g_2 \) is an affine segment on \( [B_x - \delta, B_x] \) and a different affine segment on \( [B_x, B_x + \delta] \) where the slopes of the two affine segments are of different signs. Either \( g_2(B_x - \delta) > g_2(B_x) \) and \( g_2(B_x + \delta) > g_2(B_x) \), or \( g_2(B_x - \delta) < g_2(B_x) \) and \( g_2(B_x + \delta) < g_2(B_x) \). The graph of \( g_1 \) is an affine segment on both \( [g_2(B_x), g_2(B_x - \delta)] \) and \( [g_2(B_x), g_2(B_x + \delta)] \) in the former case and on both \( [g_2(B_x - \delta), g_2(B_x)] \) and \( [g_2(B_x), g_2(B_x + \delta)] \) in the latter case. Therefore, \( g_1(g_2(B_x)) \) is a type II breakpoint on the graph of \( g_1 \circ g_2 \). That is, every type II breakpoint of \( g_2 \) corresponds to at least one type II breakpoint of \( g_1 \circ g_2 \).

Case 2. Suppose that a point \( A \) on the graph of \( g_1 \) is a type II breakpoint. By definition, \( 0 < A_x < 1 \). Because \( g_2 \) is a continuous map onto \([0, 1]\), a point \( C \) exists on the graph of \( g_2 \) such that \( C_x \in g_2^{-1}(A_x) \) and the point \( C \) is not a type II breakpoint of \( g_2 \). Thus, a sufficiently small \( \delta > 0 \) exists such that the graph of \( g_2 \) on \( [C_x - \delta, C_x + \delta] \) is monotone. Following the preceding argument in case 1, \( g_1(g_2(C_x)) \) is a type II breakpoint on the graph of \( g_1 \circ g_2 \). That is, every type II breakpoint of \( g_1 \) corresponds to at least one type II breakpoint of \( g_1 \circ g_2 \).

Note that because the point \( C \) in the case 2 is not a type II breakpoint of \( g_2 \), this type II breakpoint on the graph of \( g_1 \circ g_2, g_1(g_2(C_x))) \), is not included in the case 1. There is no double counting between the cases 1 and 2. Hence, \( #(g_1 \circ g_2) \geq #(g_1) + #(g_2) \).

Theorem 4.2. \( G \) is not finitely generated.

Proof. For any dyadic number \( \delta \in (0, 1) \), \( #(w_2, [\delta, 1]) = 1 \). If \( g_1, g_2 \in G \) exist such that \( w_2, [\delta, 1] = g_1 \circ g_2 \), then by Lemma 4.1, one of \( g_1 \) and \( g_2 \) is an identity map. Thus, \( w_2, [\delta, 1] \) cannot be constructed with \( w_2, [\delta', 1] \) with another dyadic number \( \delta' \neq \delta \) or other maps in \( G \) with two or more type II breakpoints. Because there are infinitely many dyadic numbers \( \delta \), the set of \( \{w_2, [\delta, 1]\} \) cannot be generated by a finite number of generators.

5 Equivalence Classes and Construction of a Finitely Generated Monoid \( H \)

Comparison of Theorem 3.18 and Theorem 4.2 indicates that the maps in \( F \) play an important role in allowing a finite number of basic maps to construct any map in \( G \). One idea is to “absorb” the maps in \( F \) in the construction. To study this idea formally, we next define equivalence relation and equivalence classes.

Definition 5.1 (Equivalence Relation). Define a binary relation \( \sim \) on \( G \) as follows. Suppose that \( g_1, g_2 \in G \). \( g_1 \sim g_2 \) if and only if \( f_1, f_2 \in F \) exist such that \( g_2 = f_1 \circ g_1 \circ f_2 \). The binary relation \( \sim \) is an equivalence relation because it is reflexive, symmetric, and transitive.
Definition 5.2 (Equivalence Class). The equivalence class of \( g \in G \), denoted by \([g]\), is the set \( \{ \hat{g} \in G | \hat{g} \sim g \} \).

To understand the effect of \( f_1 \) and \( f_2 \) on \( g \) in Definition 5.1, let \( g \in G \) and \( f \in \mathbb{F} \). Consider two intervals \( \mathcal{I}_0 \) and \( \mathcal{I}_1 \) where \( \mathcal{I}_0 = f^{-1}(\mathcal{I}_1) \). Suppose that the graph of \( f \) is an affine segment on the interval \( \mathcal{I}_0 \). The slope of the affine segment is \( s = \frac{|\mathcal{I}_1|}{|\mathcal{I}_0|} \). From the properties of \( \mathbb{F} \), a portion of the graph of \( g \) is scaled at a ratio of \( s \) to become part of \( g \circ f \) or \( f \circ g \). Specifically, to obtain \( g \circ f \), the graph of \( g \) on \( \mathcal{I}_1 \) is scaled horizontally to an interval \( \mathcal{I}'_0 \) with \( |\mathcal{I}'_0| = |\mathcal{I}_0| \). The exact location of \( \mathcal{I}'_0 \) on \([0,1]\) is such that the continuity is maintained in \( g \circ f \) and thus depends on the scaling of other portions. To obtain \( f \circ g \), the graph of \( g \) on \( g^{-1}(\mathcal{I}_0) \) is scaled vertically to an interval \( \mathcal{I}'_1 \) with \( |\mathcal{I}'_1| = |\mathcal{I}_1| \). The exact location of \( \mathcal{I}'_1 \) on \([0,1]\) is such that the continuity is maintained in \( f \circ g \). Figure 10 illustrates the scaling operation of an affine segment of \( f \).

It can be shown that for any \( f_1 \in \mathbb{F} \) and \( g \in G \), in general \( f_1 \circ g \circ f_1^{-1} \notin G \). However, one can prove the following lemma.

Lemma 5.3. Let \( f_1 \in \mathbb{F} \) and \( g \in G \). Then there exists \( f_2 \in \mathbb{F} \) such that \( f_1 \circ g \circ f_2 \in G \).

Proof. Partition \([0,1]\) into a set of intervals \( \{\mathcal{Y}_i\} \) such that the interiors of \( f_1^{-1}(\mathcal{Y}_i) \) contains no breakpoints of \( f_1 \) and the endpoints of \( \{\mathcal{Y}_i\} \) are all dyadic for all \( i \). Suppose that the derivative of \( f_1 \) on \( f_1^{-1}(\mathcal{Y}_i) \) is \( s_i \). From \( g \) to \( f_1 \circ g \), \( f_1 \) vertically scales the graph of \( g \) on \( g^{-1}(f_1^{-1}(\mathcal{Y}_i)) \) by a factor of \( s_i \).

Consider a map \( f \in \mathbb{F} \) that maps \( \mathcal{I}_i \) to \( \mathcal{J}_i \), i.e., \( \mathcal{J}_i = f(\mathcal{I}_i) \) for \( i = 1, 2, \ldots \), where \( \{\mathcal{I}_i\} \) and \( \{\mathcal{J}_i\} \) are two partitions of \([0,1]\). The map \( f \) is completely defined once the scaling factors from \( |\mathcal{I}_i| \) to \( |\mathcal{J}_i| \), for \( i = 1, 2, \ldots \), have been specified. We next construct \( f_2 \) by specifying the two partitions of \([0,1]\) for which \( f_2 \) is designed to map one to the other.

Let \( g^{-1}(f_1^{-1}(\mathcal{Y}_i)) = \bigcup_j \mathcal{I}_{i,j} \) where \( \mathcal{I}_{i,1}, \mathcal{I}_{i,2}, \ldots \) are intervals of mutually disjoint interiors. Let the map \( f_2 \) scale the graph of \( f_1 \circ g \) on the interval \( \mathcal{I}_{i,j} \) horizontally to the
Therefore, \( \mathcal{J}_{i,j} \) is a valid partition of \([0,1]\). Moreover, \( s_i \) is in the form of \( 2^k \) for an integer \( k \) and the endpoints of \( g^{-1}(f^{-1}(\mathcal{M}_i)) \) are dyadic because \( f_1 \in \mathcal{F} \) and \( g \in \mathcal{G} \). Therefore, \( f_2 \in \mathcal{F} \).

Second, from \( f_1 \circ g \) to \( f_1 \circ g \circ f_2 \), \( f_2 \) horizontally scales the graph of \( f_1 \circ g \) on \( g^{-1}(f^{-1}(\mathcal{M}_i)) \) by a factor of \( s_i \). Recall that from \( g \) to \( f_1 \circ g \), \( f_1 \) vertically scales the graph of \( g \) on \( g^{-1}(f^{-1}(\mathcal{M}_i)) \) by a factor of \( s_i \). Combining these two steps, from \( g \) to \( f_1 \circ g \circ f_2 \), the graph of \( g \) on \( g^{-1}(f^{-1}(\mathcal{M}_i)) \) is scaled horizontally and vertically by the same factor for all \( i \). Because \( g \in \mathcal{G} \), it follows that \( f_1 \circ g \circ f_2 \in \mathcal{G} \).

\( \square \)

**Lemma 5.4.** The equivalence classes defined in Definition 5.2 form a partition of the set \( \mathcal{G} \).

**Proof.** Because any map in \( \mathcal{F} \) is invertible, it follows that if \( \hat{g} \in [g] \), then \( g \in [\hat{g}] \) and \([g] = [\hat{g}]\), and if \( \hat{g}_1, \hat{g}_2 \in [g] \), then \([\hat{g}_1] = [\hat{g}_2]\). Therefore, any map in \( \mathcal{G} \) is in exactly one equivalence class. \( \square \)

However, the equivalence classes do not form a monoid. To construct a binary operation \( \circ \) on \([g]\), if \([\hat{g}_1 \circ \hat{g}_2]\) were identical for any \( \hat{g}_1 \in [g_1] \) and \( \hat{g}_2 \in [g_2] \), then one could define \([g_1] \circ [g_2] = [\hat{g}_1 \circ \hat{g}_2]\). However, as shown in Example 5.5, \([\hat{g}_1 \circ \hat{g}_2]\) can be different for different elements of \( \hat{g}_1 \in [g_1] \) and \( \hat{g}_2 \in [g_2] \).

**Example 5.5.** Let \( g_1 = w_{2,[\frac{1}{2},1]} \) and \( g_2 = w_{2,[0,1]} \). Recall that \( w_{2,[\frac{3}{4},1]} \) and \( w_{2,[0,1]} \) are two basic maps of \( \mathcal{G} \) and are illustrated in Figure 9. Let \( \hat{g}_2 = w_{2,[0,1]} \in [g_2] \). Consider two elements in equivalence class \([g_1]\): \( \hat{g}_{1,1} = w_{2,[\frac{1}{2},1]} \in [g_1] \) and \( \hat{g}_{1,2} = w_{2,[\frac{3}{4},1]} \in [g_1] \). Figure 11 compares \( \hat{g}_{1,1} \circ \hat{g}_2 \) and \( \hat{g}_{1,2} \circ \hat{g}_2 \). Clearly they are not in the same equivalence class.

To avoid the technical difficulty of working with equivalence classes directly, consider the notion of sets of equivalence classes instead.

**Definition 5.6 (Set of Equivalence Classes).** Let \( \Phi \subseteq \mathcal{G} \). Let \( [\Phi] \) be the set of equivalence classes \([g]\), \( \forall g \in \Phi \). That is, \([\Phi] = \{[g] \mid g \in \Phi\}\). Define a binary operation \( \circ \) on \([\Phi]\) as follows: \([\Phi_1] \circ [\Phi_2] \) is the set of all equivalence classes \([\hat{g}_1 \circ \hat{g}_2]\) where \( \hat{g}_1 \in [g_1] \) and \( \hat{g}_2 \in [g_2] \) for some \( g_1 \in \Phi_1 \) and \( g_2 \in \Phi_2 \).
Proof. By Definition 5.6, if \( g \in ([\Phi_1] \circ [\Phi_2]) \circ [\Phi_3] \), then \( f_1, f_2, \ldots, f_{10} \in \mathbb{F} \) exist such that for some \( g_1 \in \Phi_1, g_2 \in \Phi_2, g_3 \in \Phi_3, \)
\[
g = f_1 \circ ((f_2 \circ ((f_3 \circ g_1 \circ f_4) \circ (f_5 \circ g_2 \circ f_6)) \circ f_7) \circ (f_8 \circ g_3 \circ f_9)) \circ f_{10}.
\]
Let
\[
g = f'_1 \circ ((f'_2 \circ g_1 \circ f'_3) \circ (f'_4 \circ ((f'_5 \circ g_2 \circ f'_6) \circ (f'_7 \circ g_3 \circ f'_8)) \circ f'_9)) \circ f'_{10},
\]
where \( f'_1, \ldots, f'_{10} \in \mathbb{F} \) are determined such that
\[
\begin{align*}
f_1 \circ f_2 \circ f_3 &= f'_1 \circ f'_2, \\
f_4 \circ f_5 &= f'_4 \circ f'_3, \\
f_6 \circ f_7 \circ f_8 &= f'_6 \circ f'_7, \\
f_9 \circ f_{10} &= f'_9 \circ f'_{10},
\end{align*}
\]
and \( f'_2 \circ g_1 \circ f'_3, f'_5 \circ g_2 \circ f'_6, f'_7 \circ g_3 \circ f'_8, \) and \( f'_4 \circ ((f'_5 \circ g_2 \circ f'_6) \circ (f'_7 \circ g_3 \circ f'_8)) \circ f'_9 \) are all in \( \mathbb{G} \).
Let \( f'_1 = g_{0,+} \) and \( f'_2 = f_{1} \circ f_{2} \circ f_{3} \). From Lemma 5.3, there exists \( f'_3 \) to make \( f'_2 \circ g_1 \circ f'_3 \in \mathbb{G} \). Next, let \( f'_4 = g_{0,+} \) and \( f'_5 = (f'_6)^{-1} \circ f_4 \circ f_5 \). There exists \( f'_6 \) to make \( f'_5 \circ g_2 \circ f'_{6} \in \mathbb{G} \). Let
\[
f'_7 = (f'_6)^{-1} \circ f_6 \circ f_7 \circ f_8.
\]
There exists \( f'_8 \) to make \( f'_7 \circ g_3 \circ f'_9 \in \mathbb{G} \). Finally, there exists \( f'_9 \) such that
\[
f'_4 \circ ((f'_5 \circ g_2 \circ f'_6) \circ (f'_7 \circ g_3 \circ f'_8)) \circ f'_9 \in \mathbb{G}.
\]
Let \( f'_{10} = (f'_6)^{-1} \circ f_9 \circ f_{10} \). Hence, by Definition 5.6, \( g \in ([\Phi_1] \circ ([\Phi_2] \circ [\Phi_3])) \). This completes the proof.

Let
\[
\Phi_d = \{g_{0,+}\}, \Phi_b = \{g_{0,-}\}, \Phi_c = \{w_{3,1/2}\}, \Phi_d = \{w_{2,3/4}\}, \Phi_e = \{w_{2,0,1}\}.
\]
Construct a collection of sets of equivalence classes, each of which is equal to \([\Phi_1] \circ [\Phi_2] \circ \cdots \) where \( \Phi_i \) for any \( i \) is one of \( \Phi_d, \Phi_b, \Phi_c, \Phi_d, \Phi_e \). Denote by \( \mathbb{H} \) the collection.
Theorem 5.8. \( \mathbb{H} \) is a monoid and is finitely generated. The union of all the elements of the collection is a set of equivalence classes, the union of which is \( \mathbb{G} \).

Proof. By Lemma 5.7, associativity holds for the elements in the collection. The equivalence class set \( [\Phi_a] \) is the identity element. The collection is thus a monoid. By construction, \( [\Phi_a], [\Phi_b], [\Phi_c], [\Phi_d], \) and \( [\Phi_e] \) are the generators of the monoid. By Theorem 3.18, any map \( g \in \mathbb{G} \) is equal to the composition of a combination of maps, each of which is in one of equivalence classes \( [g]_{g \in \Phi_a} = [g_0, +], [g]_{g \in \Phi_b} = [g_0, -], [g]_{g \in \Phi_c} = [\bar{w}_3, (\frac{1}{2}, 1)], \)
\( [g]_{g \in \Phi_d} = [w_2, (\frac{3}{4}, 1)], \) and \( [g]_{g \in \Phi_e} = [w_2, (0, 1)]. \) Therefore, the last part of the theorem holds. \( \square \)

6 Conclusion

This paper has defined a new monoid, \( \lambda \)-preserving Thompson monoid \( \mathbb{G} \), modeled on the Thompson group \( \mathbb{F} \). The paper shows that any element of the monoid \( \mathbb{G} \) is the composition of a finite number of basic elements of the monoid \( \mathbb{G} \) and the generators of the Thompson group \( \mathbb{F} \). However, unlike the Thompson group \( \mathbb{F} \), the monoid \( \mathbb{G} \) is not finitely generated. The paper then defines equivalence classes of the monoid \( \mathbb{G} \), use them to construct a monoid \( \mathbb{H} \) that is finitely generated, and shows that the union of the elements of the monoid \( \mathbb{H} \) is a set of equivalence classes, the union of which is \( \mathbb{G} \).

References


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