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Irrational Philosophy? Kronecker's Constructive Philosophy and Finding the Real Roots of a Polynomial

By *Richard B. Schneider*

Abstract. The prominent mathematician Leopold Kronecker (1823 – 1891) is often relegated to footnotes and mainly remembered for his strict philosophical position on the foundation of mathematics. He held that only the natural numbers are intuitive, thus the only basis for all mathematical objects. In fact, Kronecker developed a complete school of thought on mathematical foundations and wrote many significant algebraic works, but his enigmatic writing style led to his historical marginalization. In 1887, Kronecker published an extended version of his paper, "On the Concept of Number," translated into English in 2010 for the first time by Edward T. Dean, who confirms that Kronecker is "notoriously difficult to read." In his paper, Kronecker proves that a so-called "algebraic number," meaning any root of a polynomial with integer coefficients, can be isolated from the other roots of that polynomial, as Dean says, "using solely talk of natural numbers." To ease the reader's comprehension of Kronecker's prose, here we explicate in detail the argument contained in that paper.

1 Introduction

Irrationality was first observed by the Ancient Greeks when they proved the diagonal of a square to be "incommensurable" with its side. Since then the concept of irrational numbers has been studied. They had defined the natural numbers and from them recognized the rational numbers, but then assumed every quantity could be represented as either a natural or rational number, until the diagonal of the unit square (equal to $\sqrt{2}$) was shown to be incommensurable with its side in about 430 B.C.E. — meaning that the square root of two cannot be expressed as a rational number [13, p. 39]. How to handle irrational numbers in mathematics has since been a question. Today, we call the set of all rational and irrational numbers the real numbers. Surrounding this

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topic are multiple schools of thought that developed in the 1800s, when there were attempts to establish a strong foundation for calculus, and to formalize mathematics into irrefutable philosophy. Modern mathematician Harold M. Edwards (1936 - 2020) has published much on mathematician Leopold Kronecker's contribution to these matters. Kronecker's philosophy and historical weight will be examined alongside the explication of a translated work on irrational numbers. Kronecker developed a complete school of thought on mathematical foundations and wrote many significant algebraic works, but his enigmatic writing style led to his historical marginalization.

This paper was suggested to me by my instructor, Dr. Richard Delaware, after he read remarks about Kronecker in "The Role of History in the Study of Mathematics" by Prof. Edwards [10]. We were then lucky enough to find a 2010 English translation by Edward T. Dean of an excerpt of the extended version of Kronecker's "Über den Zahlbegriff" ("On the Concept of Number") [6]. I have explained section III of that translation here using the techniques described in Dr. Delaware's 2019 MAA *Convergence* article [7]. Much of this paper is spent on a close reading of Kronecker's work in order to show his thought process and how he developed the field of mathematics while adhering to his school of thought. Section 2 helps establish the historical context of Kronecker's paper and philosophy; then in Section 3 we move to the close examination of his argument that one can constructively define irrational numbers; and finally Section 4 brings the background and analysis together for a better understanding of Kronecker's school of thought. This explanation of finding real roots of polynomials with integer coefficients extends our knowledge about constructive proofs and how Kronecker's views mathematics.

2 Leopold Kronecker's Philosophy on the Foundation of Mathematics

Leopold Kronecker (1823 - 1891) studied both philosophy and mathematics in Berlin, and in 1845 he published his dissertation on algebraic number theory under the supervision of Peter Gustav Lejeune Dirichlet (1805 - 1859). He went on to manage his family's estate in Liegnitz for several years [11, p. 941]. In 1855, Kronecker returned to Berlin, and around the same time at the Academy of Berlin, Ernst Eduard Kummer (1810 - 1893) and Karl Weierstrass (1815 - 1897) were appointed to chairs in mathematics. In 1861, Kronecker was elected to the Berlin Academy, where he taught and eventually succeeded Kummer [11, pp. 941-942]. Upon Kronecker's death, the Berlin Academy published his collected works, but that second publication did not make the works any more polished or increase their acceptance [9, p. 131]. Edwards believes that while Kronecker is regarded as a great mathematician whose works are still studied, "his work is obscured by his unfamiliar style, by the fact that he published mostly research papers rather than expository treatises, and by the lack of secondary sources and disciples to develop and expand upon his ideas" [9, pp. 131–132].

One example of the weak acceptance of Kronecker's publications is his presentation of a generalization to arbitrary number fields of Kummer's "ideal prime factors" for cyclotomic fields. Kummer announced in 1859 that Kronecker would soon have this work published, but that didn't happen for twenty-two years [9, p. 136]. Richard Dedekind's (1831 - 1916) first version of "Ideal Theory" had been published eleven years prior to Kronecker's, and because of Kronecker's late publication and "forbidding form", Dedekind's theory would become the standard version of ideal theory — even though, as Edwards claims, Kronecker's theory was superior [9, p. 136]. Another example reveals the difficulty of reading Kronecker's research. Dedekind had questions about parts of Kronecker's work, but when Adolf Hurwitz (1859 - 1919) wanted to respond, he had to consult the notes of Kronecker's lectures in order to discover the reasoning behind the conclusions presented [9, p. 137]. Furthermore, Kronecker's writing was difficult for his students to understand; Hulmut Hasse (1898 - 1979) described Kronecker's lectures as notoriously difficult, and Dmitry Fyodorovich Seliwanoff (1855 - 1932), a young mathematician in Berlin at the time, said, "Und wenn die Vorlesung aus ist, wirr rrufen alle 'wunderfoll' und habben nicht verstanden" ("And when the lecture was over, we all exclaimed 'Wonderful' but have not understood a thing") [12, p. 115-116]. This makes apparent that Kronecker's writing style required much previous knowledge and familiarity with his thinking.

Kronecker is also known superficially for his mathematical philosophy, but written support for his views are scarce [11, p. 942]. Even his famous quote, "Die ganzen Zahlen hat der liebe Gott gemacht, alles andere ist Menschenwerk" ("God created the integers; everything else is the work of man"), was published secondhand [11, p. 942]. During the late 1800s there was much focus on establishing a rigorous foundation for calculus and the idea of infinity within mathematics [11, p. 941]. At this time, Georg Cantor (1845 - 1918) introduced set theory, which allowed the usage of infinite collections. This theory allowed purely arithmetical theories of real numbers that answered questions of infinitesimals, limits, derivatives, and infinite series which had been formulated and used for more than two centuries; however, still unaddressed were the problems originally raised by George Berkeley in *The Analyst* (1734), in which he referred satirically to the notion of infinitesimals (infinitely small quantities) as the "ghosts of departed quantities" [11, p. 941; 2, p. 59]. Yet, these theories introduced new problems known as set-theory paradoxes [11, p. 941].

Kronecker's philosophical view on mathematics was that only the natural numbers exist [11, p. 942]. From a physical view of the world and of arithmetic, one could argue that we can only have a positive number of countable objects, 1, 2, 3, and so on. Likewise, Kronecker held that all other mathematical objects must be constructed from the natural numbers in an "algorithmic" approach that takes only a finite number of steps [11, p. 942]. Kronecker wrote his mathematics entirely in expressions involving natural numbers. Edward Dean, the translator of Kronecker's paper on the foundations of mathematics which will be explicated here below, explains this style with the following example. Someone who has never conceptualized negative numbers, someone at the very beginning of an understanding of mathematics, would not understand the equation $7-9=3-5$ because the two sides do not result in a natural number [6, p. 2]. Developing the concept of a negative number, like -2 here, would require creating a new *object*, which is not allowed in Kronecker's school of thought [3, p. 147]. To avoid creating such a new object, Kronecker's idea was that $7-9=3-5$ can be recast as $7+9x=3+5x$ provided $x+1=0$, where x is an indeterminate. Finally, he used Gauss's idea of modular congruence to provide a more complex representation of the previous equation:

$$7+9x \equiv 3+5x \pmod{(x+1)},$$

which is equivalent to $7-9=3-5$ and avoided the use of negative integers. Similarly, Kronecker avoided creating the concept of fractional numbers [6, p. 6]. Reading mathematics using this style would be hard for someone unacquainted with the replacement of familiar concepts by methods or algorithms stemming from the natural numbers. One begins to understand the difficulty in reading Kronecker's work.

Kronecker asserted that mathematical objects that were declared as complete and infinite or were defined in a non-constructive manner were then inherently mathematically incorrect [11, p. 942]. This means that one cannot start with the concept of all real numbers as a foundation for mathematics, nor can the irrational numbers be represented as a complete entity, since both are non-constructive infinite sets.

Unfortunately, holding such a position on the foundational philosophy of mathematics put him in conflict with some prominent contemporaries [11, p. 942]. Described as "Verbotsdiktator", or forbidding dictator, by David Hilbert (1862 - 1943), Kronecker was often at odds with Weierstrass and Cantor because of these views [9, p. 130; 10, p. 942]. An unpublished transcript of Hilbert's lectures quotes him as declaring: "the Kroneckerian tendency towards a far-reaching restriction of mathematical concept-formation and inference is still frequently advocated by figures of authority, and his method of erecting prohibitions is still very popular" [9, p. 945]. One anecdote of those conflicts tells that when Kronecker nominated Dedekind for membership in the Berlin Academy, his report covering Dedekind's work failed to mention Dedekind's paper "Continuity and the irrational numbers" (1872) [4], which made use of completed infinite sets of rational numbers [11, p. 942].

Kronecker's peers found themselves at odds with him over the usage of declared infinite sets, and these conflicts would delay the publication of results and limit the work of others. Historical papers that discuss these topics or conflicts also tend to portray Kronecker as the antagonist since he was prominent in the mathematical establishment and had very conservative views on how proofs should be constructed and what they should contain. His strict requirements, personal attacks, and delaying publication of papers with which he disagreed easily allows for this portrayal, but it also discounts the philosophy behind which Kronecker justified these actions. Edwards argument in stating that Kronecker is marginalized in the history of mathematics is not that he didn't make an impact or was not prominent, but that historic accounts often discount his mathematical philosophy and focus on the conflicts. Joseph Dauben, in "Georg Cantor and the Battle for Transfinite Set Theory", describes the creation and defense of different "levels" of infinity, a concept with which Kronecker disagreed on philosophical grounds. Here Kronecker is described as one of Cantor's teachers and lists personal attacks on Cantor such as "scientific charlatan" [5, p. 1]; however, he also notes that the early biographers presented Cantor as the "hapless victim of the persecutions of his contemporaries" [5, p. 2]. Cantor's major proof that the real numbers are non-denumerable was published under the title "Ueber eine Eigenschaft des Inbegriffes aller reellen algebraischen Zahlen" ("On a Property of the Collection of All Real Algebraic Numbers") in 1874 [5, p. 4]. This title does not appropriately describe the content, and here Dauben agrees with Walter Purkert and Hans Ilgauds on their assessment that Cantor's article was published in *Crelle's Journal* under this title because of the "conditions in Berlin" [5, p. 4-5]. These "conditions" were the opposition of Kronecker, a member of the editorial board, who was against founding analysis on results like the Bolzano-Weierstrass Theorem and the existence of upper and lower limits of certain infinite sets of numbers [5, p. 5-6]. Additionally, Cantor would probably have known about Kronecker's involvement in delaying Heinrich E. Heine's (1821 - 1881) article on trigonometric series [5, p. 6]. The delay of publication due to conservative beliefs and Cantor's intentional understatement of his claims show that Kronecker's philosophy led to conflicts, and we also see how the historical accounts could paint Kronecker as the "villain" of these stories.

In discussions about the foundations of mathematics and their corresponding underlying philosophies, Kronecker is often cast as a villain because he "advocated in all of mathematics an algorithmic approach that was unattractive" [9, p. 137]. His opposition to the now widely-accepted set theory places him in a preconceived box, but Edwards points out that this must be inaccurate because it only values works that fit modern prejudices [9, p. 137].

3 Kronecker's "On The Concept of Number"

Leopold Kronecker wrote "Über den Zahlbegriff" ("On the Concept of Number", 1887) to express his philosophical views about mathematics and to show that everything done in mathematics stems from the natural numbers. After its publication, Kronecker extended this article to firmly show the usage of such concepts by counting the roots of a polynomial function. His goal was to construct such roots, also known as "algebraic numbers", through a finite number of steps from the natural numbers. In particular, his claim was that "The so-called existence of the real irrational roots of algebraic equations is grounded solely in the existence of intervals with the specified quality; the legitimacy of calculating with the individual roots of an algebraic equation is based completely upon the possibility of isolating them" [6, p. 13]. This means that the irrational numbers exist only through a prior construction, such as the well-known ones for $\sqrt{2}$ and π. These two values can be uniquely identified as the diagonal of the unit square and as the ratio of circumference to diameter for any circle, respectively.

We will now start the close examination of Kronecker's text. The original work is contained in the indented sections with my comments in square brackets. For the reader's convenience the work is divided into several "Excerpts" so that we can examine Kronecker's omitted reasoning between sections.

> I have shown in an earlier article that the introduction and application of *algebraic* numbers is expendable whenever the isolation of conjugates [distinct roots of a polynomial with integer coefficients] among themselves is not necessary. However, this isolation can also be done without the introduction of new concepts, and the essence of the matter can clearly emerge only if it is carried out in this way, which will be presented here in the same manner as I have done it for ten years in my university lectures. Thereby, the "more precise analysis of the concept of real roots of algebraic equations" which I have announced at the end of the first part of "Grundzüge einer arithmetischen Theorie der algebraischen Grössen" will simultaneously be given.
>
> [Preliminary Note] If $f(x)$ is an entire integral function of x [a polynomial with integer coefficients] which has no common factor with its derivative $f'(x)$, then there are entire integral functions $\phi(x),\ \phi_1(x)$ for which the equation:
>
> $$\phi(x)f(x)+\phi_1(x)f'(x)=\mathrm{D} \qquad \text{(A)}$$
>
> holds. Here D denotes the absolute value of the discriminant of $f(x)$, hence a positive whole number.

Kronecker began with a function defined entirely in terms of integers so as to not introduce "new" concepts. From this he established (A) to describe the the function in relation to its discriminant. Recall that two integers n, m are "relatively prime" if they share no prime factors. (For example, $6 = 2\cdot 3$ and $35 = 5\cdot 7$ are relatively prime.) It is simple to show using the Euclidean Algorithm that if m and n are relatively prime, there exist integers a, b such that:

$$an + bm = 1.$$

The set of all polynomials in the same "indeterminate" x with integer coefficients has the same properties as the set of integers. In particular, two such polynomials, here $f(x)$ and its derivative $f'(x)$, are relatively prime if they share no irreducible polynomial factors; this is analogous to the concept of relative primality of integers. In that case, there exist polynomials $a(x)$ and $b(x)$ with integer coefficients such that:

$$a(x)f(x)+b(x)f'(x)=1.$$

If we multiply this equation by a positive integer, say the absolute value D of the discriminant of $f(x)$, which is defined below, then we can write $\mathrm{D}\cdot a(x)=\phi(x)$ and $\mathrm{D}\cdot b(x)=\phi_1(x)$ so that the equation becomes Kronecker's equation:

$$\phi(x)f(x)+\phi_1(x)f'(x)=\mathrm{D}.$$

The well-known discriminant of a quadratic polynomial $a_0+a_1x+a_2x^2$ is the number $a_1^2-4a_2a_0$, which is an integer here since a_0, a_1, a_2 are constrained to be integers. It "discriminates" between the types of possible roots the quadratic polynomial can have:

$$a_1^2-4a_2a_0\begin{cases}=0 & \text{two equal real roots}\\ >0 & \text{two distinct real roots}\\ <0 & \text{two (conjugate) non-real complex roots}\end{cases}$$

The more general expression defines the discriminant as the resultant of $f(x)$ and $f'(x)$ which can be found as the determinant of a Sylvester matrix (which is of dimension $2n-1\times 2n-1$), whose entries consists of zeros and the integer coefficients of the polynomial; thus, we can conclude that the discriminant is an integer [8].

The discriminant can also be expressed in terms of its roots, so this is used to provide a relationship between the integer coefficients and the real roots. Specifically, for a polynomial $f(x)=a_0+a_1x+\cdots+a_nx^n$ of degree $n\geq 2$ with integer coefficients, $a_n\neq 0$, and roots $r_1,\ldots,r_n$, the discriminant $\mathrm{Disc}_x(f)$ can be defined in the terms of its roots [8]:

$$\mathrm{Disc}_x(f)=(-1)^{n(n-1)/2}\cdot a_n^{2n-2}\cdot\prod_{i\neq j}(r_i-r_j).$$

It is clear that $f(x)$ has multiple roots if and only if its discriminant is zero. Since in Kronecker's argument he assumed $f(x)$ and $f'(x)$ are relatively prime, then $f(x)$ could not have multiple roots. For example, if $f(x)=(x-r_1)^2$, having r_1 as a multiple root, then $f'(x)=2(x-r_1)$ would share the factor $(x-r_1)$ with $f(x)$, and $f(x)$ and $f'(x)$ would not be relatively prime. Thus, here $\mathrm{Disc}_x(f)\neq 0$, hence $\mathrm{D}=|\mathrm{Disc}_x(f)|\geq 1$.

Now let:

$$f(x)=a_0+a_1x+a_2x^2+\cdots+a_nx^n$$

and let a_g be the absolute greatest of the n coefficients $a_0, a_1,\ldots,a_{n-1}$. If one then designates the rational fraction $\frac{|a_g|+|a_n|}{|a_n|}$ with $\mathbf{r}$, then:

$$\left|\frac{f(x)}{a_n}-x^n\right|<(\mathbf{r}-1)\frac{|x|^n-1}{|x|-1}$$

> [This is a statement of Cauchy's Bound Theorem (1839) [14]]. Hence for any value of x not lying between $-\mathbf{r}$ and $\mathbf{r}$
>
> $$|f(x) - a_n x^n| < |a_n x^n| \text{ and consequently: } \mathrm{sgn}.f(x) = \mathrm{sgn}.a_n x^n$$
>
> So then $f(x)$ changes its sign only within the interval $(-\mathbf{r}, \mathbf{r})$. If one sets the abbreviation:
>
> $$f(x+\sigma) - f(x) = \sigma f_1(x,\sigma), \qquad (f_1(x,\sigma) - f'(x))\phi_1(x) = \sigma\psi(x,\sigma),$$
>
> then $f_1(x,\sigma)$ and $\psi(x,\sigma)$ are entire integral functions of x and σ, and if one understands $\bar{f}_1(x,\sigma)$, $\bar{\phi}(x)$, $\bar{\phi}_1(x)$, $\bar{\psi}(x,\sigma)$, respectively, to be those functions which develop from $f_1(x,\sigma)$, $\phi(x)$, $\phi_1(x)$, $\psi(x,\sigma)$ by replacing the coefficients with their absolute values, then the inequalities:
>
> $$|f_1(x,\sigma)| < \bar{f}_1(\mathbf{r},1), \qquad |\phi(x)| < \bar{\phi}(\mathbf{r}),$$
> $$|\phi_1(x)| < \bar{\phi}_1(\mathbf{r}), \qquad |\psi(x,\sigma)| < \bar{\psi}(\mathbf{r},1)$$
>
> obviously hold, so long as the value of x lies between $-\mathbf{r}$ and $\mathbf{r}$ [$|x| < \mathbf{r}$] and σ lies between -1 and 1.

Kronecker used Cauchy's Bound Theorem to focus on the interval in which the roots of the polynomial could exist. We provide a proof of this inequality below. Note that Kronecker misstates the inequality as strict and that necessarily $|x| \neq 1$, but this does not affect his argument below.

Proof:

$$\begin{aligned}
f(x) &= a_0 + a_1 x + \cdots + a_{n-1}x^{n-1} + a_n x^n. \\
\text{Recall: } |x|^n - 1 &= (|x|-1)(|x|^{n-1} + \cdots + |x| + 1). \\
\text{So, } \frac{f(x)}{a_n} &= \frac{a_0}{a_n} + \frac{a_1}{a_n}x + \cdots + \frac{a_{n-1}}{a_n}x^{n-1} + x^n, \text{ and} \\
\left|\frac{f(x)}{a_n} - x^n\right| &= \left|\frac{a_0}{a_n} + \frac{a_1}{a_n}x + \cdots + \frac{a_{n-1}}{a_n}x^{n-1}\right| \\
&\le \left|\frac{a_0}{a_n}\right| + \left|\frac{a_1}{a_n}\right||x| + \cdots + \left|\frac{a_{n-1}}{a_n}\right||x|^{n-1} \\
&\qquad \text{by the triangle inequality} \\
&\le \left|\frac{a_g}{a_n}\right| \cdot \left[1 + |x| + \cdots + |x|^{n-1}\right] \\
&= (\mathbf{r}-1) \cdot \frac{|x|^n - 1}{|x| - 1}. \qquad \blacksquare
\end{aligned}$$

This result eliminates the section of the domain where the leading term dominates the expression from our search for real roots. Outside of the interval $(-\mathbf{r}, \mathbf{r})$, that is, when

$|x| \geq \mathbf{r} > 1$, which can be rewritten as $\frac{1}{|x|-1} \leq \frac{1}{\mathbf{r}-1}$, we have

$$\left|\frac{f(x)}{a_n} - x^n\right| \leq (\mathbf{r}-1)\frac{|x|^n - 1|}{|x|-1} \leq (\mathbf{r}-1)\frac{|x|^n - 1}{\mathbf{r}-1} = |x|^n - 1 < |x|^n.$$

Multiplication by $|a_n|$ yields $|f(x) - a_n x^n| < |a_n x^n|$, which means that the value of $f(x)$ can never be further away from that of its leading term $a_n x^n$ than 0 is (sgn. $f(x)$ = sgn. $a_n x^n$). Kronecker then introduced new functions to establish more information about the roots of the polynomial. For example, because $|x| < \mathbf{r}$ and $|\sigma| < 1$, meaning $x \leq |x| < \mathbf{r}$ and $x + \sigma \leq |x| + |\sigma| < \mathbf{r} + 1$,

$$|f_1(x,\sigma)| \leq |\bar{f}_1(x,\sigma)| < |\bar{f}_1(\mathbf{r},1)| = \bar{f}_1(\mathbf{r},1).$$

Kronecker's other three inequalities likewise follow.

> If s denotes now a whole number which exceeds the greatest of the four rational values:
>
> $$\frac{\bar{f}_1(\mathbf{r},1)}{\mathrm{D}}, \qquad \frac{\bar{\phi}(\mathbf{r})}{\mathrm{D}}, \qquad \frac{\bar{\phi}_1(\mathbf{r})}{\mathrm{D}}, \qquad \frac{\bar{\psi}(\mathbf{r},1)}{\mathrm{D}},$$
>
> by at least one unit, and if one then sets:
>
> $$\phi(x) = (s-1)\mathrm{D}\theta(x), \;\; \phi_1(x) = (s-1)\mathrm{D}\theta_1(x),$$
> $$\psi(x,\sigma) = (s-1)\mathrm{D}\mathrm{H}(x,\sigma),$$
>
> then the equation (A) is transformed into the following:
>
> $$\theta(x)f(x) + \theta_1(x)\cdot\frac{f(x+\sigma)-f(x)}{\sigma} = \sigma\mathrm{H}(x,\sigma) + \frac{1}{s-1} \tag{B}$$
>
> and the values of the functions $\theta(x),\ \theta_1(x),\ \mathrm{H}(x,\sigma)$ are absolutely less than 1 for the values of x and σ which are restricted by the inequalities:
>
> $$-\mathbf{r} < x < \mathbf{r}, \qquad\qquad -1 < \sigma < 1 \qquad\qquad .$$
>
> If σ is absolutely less than $\frac{1}{s}$, then the inequality:
>
> $$|f(x)| + \left|\frac{f(x+\sigma)-f(x)}{\sigma}\right| > \frac{1}{s(s-1)}$$
>
> follows from the equation (B), and the inequality:
>
> $$|f(x')| + \left|\frac{f(x'')-f(x')}{x''-x'}\right| > \frac{1}{s(s-1)} \tag{C}$$
>
> thence obtains for any two values x', x'' lying in the interval $(-\mathbf{r},\mathbf{r})$ whose difference, taken absolutely, is less than $\frac{1}{s}$.

Kronecker constructed a useful inequality (C) that offers information about the polynomial $f(x)$, namely, that any interval of length less than $\frac{1}{s}$ between two arbitrary points, x', x'' lying within $(-\mathbf{r}, \mathbf{r})$ has the property that the absolute value of the function at one endpoint, $|f(x')|$, added to the absolute value of the slope of the secant through the points on the graph at both endpoints, $\left|\frac{f(x'')-f(x')}{x''-x'}\right|$, has a lower bound.

For this key derivation of equation (C), Kronecker first introduced a number $s > max\left\{\frac{\bar{f}_1(\mathbf{r},1)}{D}, \frac{\bar{\phi}(\mathbf{r})}{D}, \frac{\bar{\phi_1}(\mathbf{r})}{D}, \frac{\bar{\psi}(\mathbf{r},1)}{D}\right\}+1$. We thus see that $s > 1$ which implies $\frac{1}{s} < 1$. Using $|\sigma| < \frac{1}{s}$ then we derive equation (B) from equation (A) in terms of s:

$$\begin{aligned}
\theta(x)f(x) + \theta_1(x)\cdot\frac{f(x+\sigma)-f(x)}{\sigma} \\
&= \frac{\phi(x)f(x)}{(s-1)D} + \frac{\phi_1(x)}{(s-1)D}\cdot\frac{f(x+\sigma)-f(x)}{\sigma} \\
&= \frac{\phi(x)f(x)+\phi_1(x)f_1(x,\sigma)}{(s-1)D} \\
&= \frac{\phi(x)f(x)+\phi_1(x)(f_1(x,\sigma)+f'(x)-f'(x))}{(s-1)D} \\
&= \frac{(\phi(x)f(x)+\phi_1(x)f'(x))+\phi_1(x)(f_1(x,\sigma)-f'(x))}{(s-1)D} \\
&= \frac{D+\sigma\psi(x,\sigma)}{(s-1)D}, \text{ using equation (A)} \\
&= \frac{D}{(s-1)D} + \frac{\sigma\psi(x,\sigma)}{(s-1)D} \\
&= \frac{1}{s-1} + \sigma H(x,\sigma)
\end{aligned}$$

Reviewing our statement here and recalling that $-\mathbf{r} < x < \mathbf{r}$, $-1 < \sigma < 1$, and

$$s-1 > max\left\{\frac{\bar{f}_1(\mathbf{r},1)}{D}, \frac{\bar{\phi}(\mathbf{r})}{D}, \frac{\bar{\phi_1}(\mathbf{r})}{D}, \frac{\bar{\psi}(\mathbf{r},1)}{D}\right\},$$

so that

$$\frac{1}{s-1} < \frac{1}{max\left\{\frac{\bar{f}_1(\mathbf{r},1)}{D}, \frac{\bar{\phi}(\mathbf{r})}{D}, \frac{\bar{\phi_1}(\mathbf{r})}{D}, \frac{\bar{\psi}(\mathbf{r},1)}{D}\right\}} \leq \frac{1}{\left(\frac{\bar{\phi}(\mathbf{r})}{D}\right)} = \frac{D}{\bar{\phi}(\mathbf{r})},$$

we have

$$|\theta(x)| = \frac{|\phi(x)|}{(s-1)D} < \frac{\bar{\phi}(\mathbf{r})}{(s-1)D} < \frac{D}{\bar{\phi}(\mathbf{r})}\cdot\frac{\bar{\phi}(\mathbf{r})}{D} = 1.$$

Similarly, $|\theta_1(x)| < 1$ and $|H(x,\sigma)| < 1$. The restrictions of equation (B) are then

$$-\mathbf{r} < x < \mathbf{r}, \qquad -1 < \sigma < 1, \qquad |\theta(x)| < 1, \qquad |\theta_1(x)| < 1, \qquad \text{and} \qquad |H(x,\sigma)| < 1.$$

These inequalities allowed Kronecker to establish (C), which is the key result in the introduction of his argument:

$$\begin{aligned} |f(x)| + \left|\frac{f(x+\sigma)-f(x)}{\sigma}\right| &= |f(x)| + |f_1(x,\sigma)| \\ &> |\theta(x)||f(x)| + |\theta_1(x)||f_1(x,\sigma)|, \\ &\qquad \text{since } |\theta(x)| < 1,\ |\theta_1(x)| < 1 \\ &\geq |\theta(x)f(x) + \theta_1(x)f_1(x,\sigma)| \\ &= \left|\sigma\mathrm{H}(x,\sigma) + \frac{1}{s-1}\right| \qquad\qquad \text{by (B)} \\ &> \frac{1}{s(s-1)}, \end{aligned}$$

where, since $|\sigma| < \frac{1}{s}$ and $|\mathrm{H}(x,\sigma)| < 1$, then $|\sigma\mathrm{H}(x,\sigma)| < \frac{1}{s}$, so in particular $\sigma\mathrm{H}(x,\sigma) > -\frac{1}{s}$, meaning the last inequality above follows, because

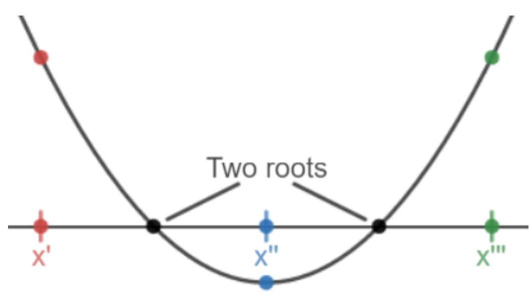
$$|\sigma\mathrm{H}(x,\sigma) + \frac{1}{s-1}| \geq \sigma\mathrm{H}(x,\sigma) + \frac{1}{s-1} > -\frac{1}{s} + \frac{1}{s-1} = \frac{1}{s(s-1)}.$$

[Claim] It will be shown now that while x remains in an interval of size $\frac{1}{s}$, the function $f(x)$ will change its sign either not at all or only *one* time, i.e. if:

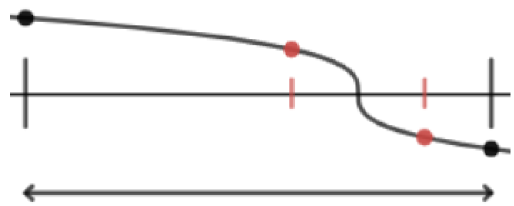
$$x' < x'' < x''' \text{ and } x''' - x' \leq \frac{1}{s},$$

then it cannot be that:

$$\text{sgn.} f(x') = -\text{sgn.} f(x'') = \text{sgn.} f(x''').$$



[Case 1:] If the value of $f(x)$ has at the beginning of an interval the opposite sign as it does at the end of the interval, which interval we would like to designate with



(J) and which is no larger than $\frac{1}{s}$, then for any division of (J) into subintervals, the same must also be the case for at least one of the subintervals.
[Proof] Now let r be an arbitrary whole number, and think of the interval (J) as divided into rD equal parts [where D is as above]. Then let (J′) be one such subinterval in which the initial and final values of $f(x)$ have opposite signs. Finally let x', x'' be two arbitrary values of x lying in the interval (J′) [so $x'' - x' < \frac{1}{r\mathrm{D}} \cdot \frac{1}{s}$], for which:

$$x' < x'', \qquad \text{sgn.} f(x') = -\text{sgn.} f(x'').$$

Since now:

$$f(x'') - f(x') = (x'' - x') f_1(x', x'' - x')$$

and also:

$$|f(x'') - f(x')| < (x'' - x') \bar{f}_1(\mathbf{r}, 1) \leq (x'' - x')(s-1)\mathrm{D}, \tag{D}$$

with consideration of the inequality: $x'' - x' \leq \frac{1}{rs\mathrm{D}}$, it follows that:

$$|f(x'') - f(x')| < \frac{1}{r}$$

and therefore, since $f(x')$ and $f(x'')$ have opposite signs, also:

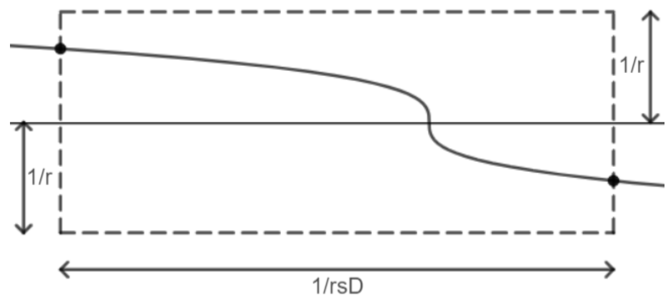
$$|f(x')| < \frac{1}{r}, \qquad |f(x'')| < \frac{1}{r} \tag{E}$$

must hold.
[Summary:] *On any interval of size $\frac{1}{s}$, whose initial and final points have $f(x)$ with opposite signs, if one selects an arbitrary whole number r, then one can find at least one interval of size $\frac{1}{rs\mathrm{D}}$ whose initial and final points likewise have $f(x)$ with opposite signs, and in which all values of $f(x)$ are absolutely less than $\frac{1}{r}$.*
[End of Case 1]

This first case is apparent to a modern reader familiar with the concept of continuity, but Kronecker's mathematical philosophy required him to provide a construction of the subinterval where the endpoints evaluate to values with opposite signs. He did this by applying the definitions $f(x+\sigma)-f(x)=\sigma f_1(x,\sigma)$, $|\sigma|<1$ and the inequalities $f_1(x',x''-x')<\bar{f}_1(\mathbf{r},1)$, $s>\frac{\bar{f}_1(\mathbf{r},1)}{\mathrm{D}}+1$ so that inequalities (D) provided an upper bound to the slope within the interval,

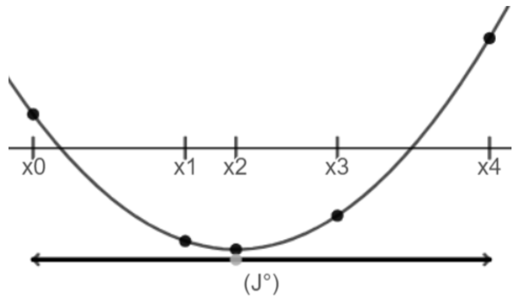
$$|f(x'')-f(x')|<(x''-x')(s-1)\mathrm{D}\le\frac{1}{rs\mathrm{D}}(s-1)\mathrm{D}=\frac{s-1}{rs}<\frac{1}{r}.$$

Since $f(x'')$ and $f(x')$ have opposite signs, then $|f(x'')-f(x')|=|f(x'')|+|f(x')|$, whence follows the inequalities (E). This limits the absolute value of the endpoints of the graph of $f(x)$ over this interval to a box of width $\frac{1}{rs\mathrm{D}}$ and height $\frac{1}{r}$, as displayed below. This idea



will be used to estimate roots later.

> [Case 2] If $f(x)$ has the *same* sign at the initial point as at the endpoint of an interval which is no larger than $\frac{1}{s}$, then $f(x)$ retains just this sign within the entire interval.
> [Proof by Contradiction] In particular, if one designates the interval with (J^0), its initial point with x_0, its final point with x_4, and one assumes that for a value x_2 lying between x_0 and x_4 the function $f(x)$ has a different sign as $f(x_0)$ than [*sic*] as $f(x_4)$
> [Claim] then there would also be two values x_1 and x_3 on either side of x_2 and still lying within the interval (J^0), determined by the equations:
>
> $$x_1=x_2-\frac{|f(x_2)|}{(s-1)\mathrm{D}},\qquad x_3=x_2+\frac{|f(x_2)|}{(s-1)\mathrm{D}},\tag{F}$$
>
> for which:
>
> $$\mathrm{sgn.}f(x_0)=-\mathrm{sgn.}f(x_1)=\mathrm{sgn.}f(x_4)=-\mathrm{sgn.}f(x_3).$$
>
> [Proof of Claim that x_1,x_3 are in (J^0)] First of all, that the values x_1 and x_3 still lie within the interval (J^0), i.e. that the inequalities:
>
> $$x_2-x_0>\frac{|f(x_2)|}{(s-1)\mathrm{D}},\qquad x_4-x_2>\frac{|f(x_2)|}{(s-1)\mathrm{D}}$$



obtain, can be inferred from the inequalities:

$$|f(x_2) - f(x_0)| < (x_2 - x_0)(s-1)\mathrm{D}, \qquad |f(x_4) - f(x_2)| < (x_4 - x_2)(s-1)\mathrm{D},$$

which follow from the above inequality (D), by further considering that according to the hypothesis:

$$\mathrm{sgn.} f(x_2) = -\mathrm{sgn.} f(x_0) = -\mathrm{sgn.} f(x_4).$$

[Proof of Claim that sgn. $f(x_0) = -$sgn. $f(x_1)$ and sgn. $f(x_4) = -$sgn. $f(x_3)$] Secondly, we now have according to the above inequality (D):

$$|f(x_2) - f(x_1)| < (x_2 - x_1)(s-1)\mathrm{D},$$
$$|f(x_3) - f(x_2)| < (x_3 - x_2)(s-1)\mathrm{D},$$

and these inequalities necessitate that both $f(x_1)$ and $f(x_3)$ have the same sign as $f(x_2)$ thus the opposite of the function values $f(x_0)$ and $f(x_4)$.

Kronecker first had to show that the two points x_1, x_3 are in the interval J^0, meaning both subintervals are strictly greater than the distance between x_2 and either of x_1, x_3. These results came directly from the inequalities stated in Case 1 where the change between two points of an interval is constrained. For example, for the first inequality involving points x_2, x_0, by (D) and because $f(x_2)$, $f(x_0)$ have opposite signs, we have $|f(x_2)| \le |f(x_2)| + |f(x_0)| = |f(x_2) - f(x_0)| < (x_2 - x_0)(s-1)\mathrm{D}$, so $\frac{|f(x_2)|}{(s-1)\mathrm{D}} < x_2 - x_0$. The second claim of this Case showed that all of the defined points in this interval have the same sign. To further examine this conclusion, if in the first inequality expressed, $f(x_1)$ and $f(x_2)$ have different signs,

$$x_2 - x_1 > \frac{|f(x_2) - f(x_1)|}{(s-1)\mathrm{D}} = \frac{|f(x_2)| + |f(x_1)|}{(s-1)\mathrm{D}} > \frac{|f(x_2)|}{(s-1)\mathrm{D}},$$

which is contradictory to the equation in (F). These two claims are required in order to further tease out the properties of such an interval J^0 that Kronecker had defined.

After this, both the interval (x_0, x_1) and the interval (x_3, x_4) would be such that $f(x)$ has opposite signs at the start and finish, and so according to that, as was proven above, values x', x'' could be determined for which:

$$x_0 < x' < x_1, \qquad x_3 < x'' < x_4, \qquad |f(x')| < \frac{1}{r}, \qquad |f(x'')| < \frac{1}{r},$$

where r is hypothetically arbitrary [any positive integer]. Now, however, according to the inequality (C):

$$|f(x')| + \left|\frac{f(x'') - f(x')}{x'' - x'}\right| > \frac{1}{s(s-1)}$$

would have to hold, and thus, since:

$$|f(x')| < \frac{1}{r}, \qquad |f(x'') - f(x')| < |f(x'')| + |f(x')| < \frac{2}{r},$$

also:

$$\frac{1}{r} + \frac{2}{r(x'' - x')} > \frac{1}{s(s-1)}$$

holds, and finally, since:

$$x'' - x' > x_3 - x_1 = \frac{2|f(x_2)|}{(s-1)\mathrm{D}}$$

holds, also:

$$\frac{1}{r} + \frac{(s-1)\mathrm{D}}{r|f(x_2)|} > \frac{1}{s(s-1)}\ ,$$

or:

$$r < s(s-1)\left(1 + \frac{(s-1)\mathrm{D}}{|f(x_2)|}\right).$$

Since, however, the number r can be selected arbitrarily, this inequality cannot hold, and thus it is indeed to be concluded that in an interval which is no larger than $\frac{1}{s}$, as soon as one knows the function $f(x)$ has only one sign at both of the endpoints, the same is the case throughout the interval.
[End of Case 2]

Note that Kronecker, whose school of thought required proofs to be done constructively, just used a proof by contradiction here. This is not incongruent with his philosophy on mathematics. He required proof by construction to show the existence of something, but Case 2 shows that if the sign of both endpoints is the same, then the polynomial $f(x)$ did not cross the x-axis, meaning that there was no root is this interval. There was nothing to construct. While this school of thought seemed to constrict mathematicians to only that which could be constructed through finitely many operations, it did not do away with logic but only limited nonconstructive proofs of existence.

Case 2 applied our understanding of our constrained function to the key inequality (C) so as to produce an upper bound of an arbitrary integer r. This ruled out the possibility of an interval J^0 containing more than one root and limited each such interval to at most one root. This was the goal of Kronecker's work: to identify any real algebraic number through only arithmetic with natural numbers, by enumerating the roots of a polynomial. The construction of such a polynomial was necessary since it would be clear to a modern reader familiar with analysis and set theory that the real numbers were not denumerable; nonetheless the concept of real numbers was not a concern to him since the rationals can be derived from the natural numbers and numbers that are not rational, like $\sqrt{2}$, are only acceptable if we can construct them.

> By now it follows immediately that in an interval of size $\frac{1}{s}$, $f(x)$ cannot change sign more than once. For, were it the case that
>
> $$\mathrm{sgn.} f(x_0) = -\mathrm{sgn.} f(x_1) = \mathrm{sgn.} f(x_2)$$
>
> for three values x_0, x_1, x_2, with $x_0 < x_1 < x_2$, lying in the interval, then the interval (x_0, x_2) would be one such whose size is less than $\frac{1}{s}$, and at whose initial and final points $f(x)$ would have the same sign. However, as was just proven [by Case 2], in such an interval $f(x)$ cannot change its sign; thus it cannot be that
>
> $$\mathrm{sgn.} f(x_0) = -\mathrm{sgn.} f(x_1).$$

Therefore within an interval of size less than $\frac{1}{s}$, $f(x)$ changes sign either once, indicating that interval contains a *single* root, or not at all, in which the interval contains *no* roots. So each root can be uniquely identified by its surrounding interval, and by choice of r, as shown in Case 1, you can get arbitrarily close to any real root of the function by finding a subinterval of size $\frac{1}{rs\mathrm{D}}$ which encloses it whose endpoints have opposite signs.

Kronecker went on to offer a procedure (which we omit here) to “calculate" the real roots of a given function. It locates the real roots of the function and serves as a way to constructively identify specific irrational numbers as a countable collection of integers. This was the underlying purpose of this proof and it represented his mathematical philosophy.

To finally make clear the role that D plays in these inequalities, recall that $\mathrm{D} \geq 1$. Suppose by way of contradiction that $r_i,\ r_j \in (\mathrm{J}')$, a subinterval of length $< \frac{1}{rs\mathrm{D}}$, where r is any positive integer. Then since $\frac{1}{s} < 1$ and $\frac{1}{\mathrm{D}} \leq 1$,

$$|r_i - r_j| < \frac{1}{r\mathrm{D}} \frac{1}{s} < \frac{1}{r}.$$

But, there are only finitely many choices for $r_i - r_j$, while r can be chosen so large that $\frac{1}{r}$ is as small as desired. Contradiction. So (J′) cannot contain two or more roots of $f(x)$.

After this, if the whole number t is determined by the inequality condition:

$$s(|a_g| + |a_n|) \leq t|a_n| < |a_n| + s(|a_g| + |a_n|),$$

the function $f(x)$ can change sign only in an interval $(\frac{k-1}{s}, \frac{k}{s})$, in which k has one of the values: $-t+1, -t+2, \ldots, t-1, t$. Therefore one needs to determine only the signs of the $2t$ values:

$$f\left(\frac{k}{s}\right) \qquad (k = -t+1, -t+2, \ldots, t-1, t)$$

in order to determine in which of those $2t-1$ intervals of size $\frac{1}{s}$ the function $f(x)$ changes its sign — and then only one time. The Anzahl [total number] of these intervals is at the same time that which one designates as the Anzahl of the real roots of the equation $f(x) = 0$, and the former will totally replace the latter via the specified procedure, which Sturm's theorem [References 1 and 15] provides. But also the so-called calculation with the real roots itself will be replaced through the given procedure; for if it is shown for a particular number k that:

$$\text{sgn.} f\left(\frac{k-1}{s}\right) f\left(\frac{k}{s}\right) = -1,$$

then one needs only calculate the initial and final values of $f(x)$ in the subintervals of size $\frac{1}{rs\mathrm{D}}$, i.e. thus the $r\mathrm{D}+1$ values:

$$f\left(\frac{k}{s} - \frac{h}{rs\mathrm{D}}\right) \qquad (h = 0, 1, \ldots, r\mathrm{D})$$

and to determine that number h for which:

$$\text{sgn.} f\left(\frac{k}{s} - \frac{h}{rs\mathrm{D}}\right) f\left(\frac{k}{s} - \frac{h-1}{rs\mathrm{D}}\right) = -1,$$

in order to consequently infer that, in the interval:

$$\frac{k}{s} - \frac{h}{rs\mathrm{D}} \leq x < \frac{k}{s} - \frac{h-1}{rs\mathrm{D}},$$

the function $f(x)$ changes its sign and remains absolutely less than $\frac{1}{r}$ throughout.

[Conclusion] The so-called existence of the real irrational roots of algebraic equations is grounded solely in the existence of intervals with the specified quality;

> the legitimacy of calculating with the individual roots of an algebraic equation is based completely upon the possibility of isolating them, hence upon the possibility of determining a number, which we designated with s above. If such a number s is determined which has the property that the intervals of size $\frac{1}{s}$ are sufficiently small to isolate the distinct roots of the same equation, then “greater than" and “less than" for arbitrary irrational algebraic numbers is also determined after this, if — as is obviously permissible — one thinks of the two algebraic numbers, which are to be ordered, as two roots of one and the same equation. The true essence of the matter, however, becomes completely clear in the above deduction only when one also avoids the use of fractions therein and makes use exclusively of whole numbers.

By taking the ‘at most one root’ property of the defined intervals of size less than $\frac{1}{s}$, the procedure defined above expanded the result to develop a method to identify if a root exists within an interval. The number of intervals that meet the criteria sgn.$f\left(\frac{k-1}{s}\right)f\left(\frac{k}{s}\right) = -1$ where a sign change occurs is the number (Anzahl) of real roots of the given polynomial. Repeatedly implementing Kronecker’s procedure to identify intervals with real roots would then allow for the real roots to be approximated with arbitrary accuracy. This is a result of the constraint $|f(x)| < \frac{1}{r}$ as described in Case 1 and for a choice of r, we can be confident that the interval that contains the root is within $\frac{1}{rs\mathrm{D}}$ of the actual value.

4 Conclusion

Kronecker proved that he could count the finitely many real roots, rational and irrational, of a given polynomial so that they could be estimated with arbitrary accuracy. This underscored and illustrated his mathematical philosophy by showing that the introduced concepts that others freely used were not necessary, and that his conclusion was not further abstraction but a desire to develop algebraic representations of the concept of a real number. This proof served as a model for Kronecker’s style of constructive argumentation that derived all statements through arithmetic with the natural numbers. It placed him in a distinct category because of his philosophy, whereas his contemporaries were developing the foundation of modern analysis using concepts to which he objected, on the grounds that they were not epistemologically verifiable.

Harold Edwards, in his 1991 remembrance speech for Leopold Kronecker, noted that a discussion of philosophy is inevitable when dealing with mathematics [9, p. 138]. For Kronecker the foundation of mathematics that was philosophically ‘safe’ was arithmetic; thus for him every new concept should be constructed from arithmetic in order to be grounded in acceptable mathematics [9, p. 138]. Hilbert attempted to remove this philosophical foundation from mathematical discussions, but Edwards observed that, "when a modern mathematician wants to avoid philosophical considerations, he puts everything strictly in terms of set theory, which, except for the theory of finite sets, lies far outside Kronecker's 'sicheren Hafen'" (safe harbor) [9, p. 138]. The main point of contention with set theory for Kronecker was that it was unnecessary — as displayed in this constructive identification of the real roots of a polynomial.

Kronecker's philosophy of mathematics is much more developed than the marginal presentation given by historians. Jacqueline Boniface of the Université de Nice-Sophia Antipolis details Kronecker's school of thought and how it compares to other prominent philosophies that are given more attention [3, p. 143]. She highlights the three key parts of Kronecker's doctrine: conceiving of mathematics as a natural science, his brand of realism, and the manner of mathematics.

Treating the conceptualization of mathematics as a natural science would require the foundation of mathematics to be *phenomena*, the basic concepts observed intuitively by everyone through experience [3, pp. 144-145]. For Kronecker, the most basic observable phenomenon was the set of natural numbers, which arose from the ability of one to count objects [3, p. 145]. On the other hand, Frege (1845 - 1925) wanted to use logic to give a foundation for mathematics, while Hilbert wanted to "base arithmetic and logic on sensitive intuition", and Brouwer (1881 - 1966) wanted to start with pure intuition [3, p. 146]. Instead, from the natural numbers Kronecker developed a foundation for mathematics, and then concerned himself with the necessary implications in his study of mathematics.

Kronecker's realism discussed the idea of mathematical 'discoveries' rather than 'inventions', since that which already exists can only be discovered. However, Kronecker was not an empiricist, because he applied the idea of discoveries only to the objects within mathematics [3, p. 147]. The objects and their properties are discovered within mathematics through the creation of new methods. This is akin to the approach to natural sciences as given in Kronecker's analogy of mathematicians to geographers, where he describes the field of mathematics as a static entity that exists outside of observation and it is mathematicians who map and understand the relations and properties present [3, pp. 147 - 148]. These considerations move Kronecker's philosophy from an association with Brouwer, through intuition, towards Frege, through a brand of realism [3, p. 148]. Kronecker was more restrictive than Frege, who expanded the concept of numbers through logic, whereas Kronecker required constructive definitions to find the specific objects being discussed [3, p. 149].

Finally, the differentiation of realism between objects and methods moved Kronecker from the foundations of mathematics to the manner in which mathematics is done. Kronecker's realism allowed for the creation of methods that could use objects, but did not allow for *objects* to be created, unless absolutely necessary [3, p. 150]. This objection to the creation of objects and abstraction of the mathematical environment was a characteristic of the conviction that, to Kronecker, shows that the abstract concept of real numbers was not required. In the creation of infinite collections, their lack of necessity was the criticism that Kronecker often advanced.

The constructive proof that finds the real roots, rational and irrational, of a polynomial function illustrates Kronecker's philosophy of mathematics. Kronecker approached the search for a foundation of mathematics in the attempt to find the smallest, most intuitive kernel. From this kernel, the natural numbers, Kronecker built arithmetic and then concerned himself with the philosophy of approach, or rather, Kronecker was concerned with how mathematics could be constructed and proved. Unfortunately, this restrictive realism led to the caricature of "Verbotsdiktator" that diminished the importance of Kronecker's doctrine, and, the difficulty associated with reading his work prevented a wider readership.

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