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Kevin M. Tully
Wheaton College, kevin.tully@my.wheaton.edu

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Cover Page Footnote

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Decomposable Model Spaces and a Topological Approach to Curvature

By *Kevin Tully*

Abstract. This research investigates a model space invariant known as k-plane constant vector curvature, traditionally studied when $k = 2$, and introduces a new invariant, (m, k)-plane constant vector curvature. We prove that the sets of k-plane and (m, k)-plane constant vector curvature values are connected, compact subsets of the real numbers and establish several relationships between the curvature values of a decomposable model space and its component spaces. We also prove that every decomposable model space with a positive-definite inner product is k-plane constant vector curvature ϵ for some integer $k \geq 2$ and $\epsilon \in \mathbb{R}$. In two examples, we provide the first instance of a model space with (m, k)-plane constant vector curvature and leverage our theorems to efficiently calculate the k-plane constant vector curvature values of a decomposable model space. This research further characterizes model spaces by assigning new basis-independent values to its various subspaces and allows us to easily construct model spaces with prescribed curvature values.

1 Introduction

Differential geometry uses analysis and algebra to study curvature and other properties of smooth manifolds, which are topological spaces that are locally diffeomorphic to Euclidean space. (Formally, an n-dimensional smooth manifold is a Hausdorff, second-countable topological space that is locally Euclidean of dimension n, endowed with a maximal smooth atlas.) For example, the Gaussian curvature is an intrinsic property of a surface (a 2-dimensional manifold) independent of its isometric embedding in Euclidean space. Since a manifold locally resembles Euclidean space, we can use calculus and linear algebra to characterize its tangent space at any point. To extend the notion of intrinsic curvature to any manifold, one studies Riemannian manifolds, which are smooth manifolds equipped with a Riemannian metric (essentially a choice of inner product on each tangent space which varies smoothly from point to point). A Riemannian metric allows us to define familiar geometric notions on Riemannian manifolds,

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including lengths, angles, and distances. See [12] for further reading on Riemannian geometry.

In this paper, we study model spaces rather than manifolds. A *model space* $\mathscr{M} = (V, \langle\cdot,\cdot\rangle, R)$ is an n-dimensional real vector space V, a non-degenerate inner product $\langle\cdot,\cdot\rangle$ on V, and an algebraic curvature tensor R (defined below). Geometrically, given a manifold, a metric, and a point on the manifold, we can construct a model space from the tangent space, metric, and Riemannian curvature tensor at that point. Since a representative model space helps us glean certain information about a manifold at a point, our algebraic investigation has potential applications to the study of curvature, and we leave the geometric implications to future research.

An important tool for studying model spaces is an *algebraic curvature tensor*.

Definition 1.1. An **algebraic curvature tensor (ACT)** R is a multilinear function from ordered quadruples of tangent vectors to scalars,

$$R : V \times V \times V \times V \to \mathbb{R},$$

which satisfies the following properties for all $x, y, z, w \in V$:

1. Skew symmetry: $R(x, y, z, w) = -R(y, x, z, w)$,
2. Interchange symmetry: $R(x, y, z, w) = R(z, w, x, y)$, and
3. First Bianchi Identity: $R(x, y, z, w) + R(z, x, y, w) + R(y, z, x, w) = 0$.

If $\{e_1, \dots, e_n\}$ is a basis for V, we often shorten $R(e_i, e_j, e_k, e_l)$ to R_{ijkl}. The set of algebraic curvature tensors, denoted $\mathscr{A}(V)$, itself carries much internal structure. For example, $\mathscr{A}(V)$ is an $\frac{n^2(n^2-1)}{12}$-dimensional vector space [9], so

$$(\lambda R_1 + R_2)(x, y, z, w) = \lambda R_1(x, y, z, w) + R_2(x, y, z, w)$$

for all $\lambda \in \mathbb{R}$, $R_1, R_2 \in \mathscr{A}(V)$, and $x, y, z, w \in V$.

Let $\phi : V \times V \to \mathbb{R}$ be a symmetric, bilinear form. Following [1], we say that R_ϕ is a *canonical ACT* with respect to ϕ if it is of the form

$$R_\phi(x, y, z, w) = \phi(x, w)\phi(y, z) - \phi(x, z)\phi(y, w).$$

Based on Fiedler [6, 7], Gilkey proved that the set of canonical ACTs spans $\mathscr{A}(V)$ [8]. Since an inner product $\langle\cdot,\cdot\rangle$ is a symmetric, bilinear form, the ACT $R_{\langle\cdot,\cdot\rangle}$ is canonical. If $\langle\cdot,\cdot\rangle$ is a positive-definite inner product on V and $\{e_1, \dots, e_n\}$ is a $\langle\cdot,\cdot\rangle$-orthonormal basis for V, then $\langle e_i, e_j\rangle = \delta_{ij}$, and whenever $i \neq j$,

$$(R_{\langle\cdot,\cdot\rangle})_{ijji} = \delta_{ii}\delta_{jj} - \delta_{ij}^2 = 1.$$

Our research only concerns positive-definite inner products, but it could be extended to the non-degenerate setting. Indeed, curvature invariants have been studied in Lorentzian spaces [4].

The rich algebraic properties of canonical ACTs makes such tensors particularly promising for study. If ϕ is a symmetric, bilinear form on a vector space with a non-degenerate inner product, there is a unique self-adjoint linear transformation A such that $\phi(x, y) = \langle Ax, y\rangle$. Hence we can discuss algebraic aspects of ϕ, including eigenvalues, rank, and kernel, in terms of A. The *kernel* of an ACT is the set of vectors that produce a zero curvature value regardless of the other inputs:

$$\ker(R) = \{v \in V : R(v, y, z, w) = 0 \ \forall y, z, w \in V\}.$$

Using the properties in **definition 1.1**, it is known that vectors in the kernel are invariant in input; that is, $R = 0$ even if v is in the second, third, or fourth slot [5]. The following proposition (proved in [8]) demonstrates the close link between the kernel of a symmetric, bilinear form ϕ and that of its associated canonical ACT.

Proposition 1.2. *If* ϕ *is a symmetric, bilinear function and* $\operatorname{rank}(\phi) \geq 2$*, then*

$$\ker(\phi) = \{v \in V : \phi(v, w) = 0 \quad \forall w \in V\} = \ker(R_\phi).$$

Algebraic curvature tensors permit us to define and calculate various curvature invariants. One such invariant is the *sectional curvature*.

Definition 1.3. Let $\mathscr{M}$ be a model space, R an ACT, $x, y \in V$, and $\pi = \operatorname{span}\{x, y\}$ a non-degenerate 2-plane. The **sectional curvature** $\kappa : V \times V \to \mathbb{R}$ is defined by

$$\kappa(\pi) = \frac{R(x, y, y, x)}{\langle x, x\rangle\langle y, y\rangle - \langle x, y\rangle^2}.$$

We often shorten $\kappa(\operatorname{span}\{x, y\})$ to $\kappa(x, y)$. It is straightforward to check that $\kappa(\pi)$ is independent of the chosen basis for π (see [8]), so $\kappa(\pi)$ is indeed a curvature invariant. Note that if x and y are orthonormal, $\kappa(\pi) = R(x, y, y, x)$, so it is easier to calculate the sectional curvature when working with an orthonormal basis.

To clarify the previous definition, a 2-plane π is said to be *non-degenerate* if the inner product restricted to π is non-degenerate. Since this is always true if the inner product is positive-definite, we take all 2-planes (and similarly all k-planes) to be non-degenerate. However, more care is necessary if the inner product is merely non-degenerate, and we refer the reader to [11] for a thorough examination of curvature for 3-dimensional model spaces in such a setting.

A closely related measurement to the sectional curvature is the scalar curvature. Given a positive-definite inner product and an orthonormal basis $\{e_1, \dots, e_n\}$ for V, the *scalar curvature* τ is defined to be $\tau = \sum_{i,j} R_{ijji}$. In other words, τ is the mean of the sectional curvatures scaled by $n(n-1)$.

Our research uses two generalizations of sectional curvature to investigate constant curvature. Let us introduce two classical notions of constant curvature. A model space is *constant sectional curvature* ϵ, denoted $\mathrm{csc}(\epsilon)$, if $\kappa(\pi) = \epsilon$ for all non-degenerate 2-planes π. Since constant sectional curvature is a relatively strong condition, we consider a less restrictive property called constant vector curvature. A model space is *constant vector curvature* ϵ, denoted $\mathrm{cvc}(\epsilon)$, if every $v \in V \setminus \{0\}$ is contained in a non-degenerate 2-plane π with $\kappa(\pi) = \epsilon$. One can show that if a model space is $\mathrm{csc}(\epsilon)$, then it is $\mathrm{cvc}(\epsilon)$. Constant vector curvature was introduced by Schmidt and Wolfson in 2011 in their work on 3-dimensional manifolds [15].

While both csc and cvc are well understood in the 3-dimensional setting, little is known about model spaces of arbitrary finite dimension. For example, it has been shown that every 3-dimensional model space with a positive-definite inner product is $\mathrm{cvc}(\epsilon)$ for some scalar ϵ [17]. On the other hand, [11] shows this may not hold in Lorentzian model spaces. Until recently, research about curvature of higher-dimensional model spaces only considered 2-planes. In [1], Calle uses k-planes to investigate constant curvature, a concept first introduced in [3].

Definition 1.4. Let $\mathscr{M} = (V, \langle\cdot,\cdot\rangle, R)$ with $\{e_1, \dots, e_n\}$ an orthonormal basis for V. Define the model space $\mathscr{M}_L = (L, \langle\cdot,\cdot\rangle_L, R_L)$ with an orthonormal basis $\{E_1, \dots, E_k\}$ for $L \subseteq V$, where $\langle\cdot,\cdot\rangle_L = \langle\cdot,\cdot\rangle|_L$ and $R_L = R|_L \in \mathscr{A}(V)$. The ***k*-plane scalar curvature of** L is the function $\mathscr{K}_R : \mathrm{Gr}_k(V) \to \mathbb{R}$ given by

$$\mathscr{K}_R(L) = \sum_{j>i=1}^{k} \kappa(E_i, E_j).$$

Here, $\mathrm{Gr}_k(V)$ is *Grassmannian*, the space of all k-dimensional subspaces of V. If R is understood, we omit the subscript and simply write $\mathscr{K}(L)$, and we shorten

$$\mathscr{K}(\mathrm{span}\{e_1, \dots, e_k\}) \quad \text{to} \quad \mathscr{K}(e_1, \dots, e_k).$$

Although we technically evaluate $\mathscr{K}(L)$ with respect to $\mathscr{M}_L$, we usually discuss $\mathscr{K}(L)$ in terms of $\mathscr{M}$.

We pause for a moment to describe why the k-plane curvature is independent of the particular basis chosen. We know the scalar curvature is independent of the particular basis used to compute it since it is defined in terms of a basis-free construction (see, for example, [14]). If L is a k-plane in V, then one may restrict the positive definite inner product to get a positive definite inner product on L. One may also restrict the ACT R to L, and so the vector space L with these restricted objects forms a model space $\mathscr{M}_L$ on its own. The k-plane curvature of L is one-half of the scalar curvature of the model space $\mathscr{M}_L$ (see p. 6 of [1]). Therefore the k-plane curvature of L is independent of the basis chosen because the scalar curvature of any model space is independent of the basis chosen.

Based on [1], we can consider constant curvature conditions for model spaces of any finite dimension.

Definition 1.5. A model space $\mathcal{M}$ is k**-plane constant vector curvature** ϵ, denoted k-cvc(ϵ), if every $v \in \mathrm{V}$ is contained in a non-degenerate k-plane L with $\mathcal{K}(\mathrm{L}) = \epsilon$.

There is an analogous extension of csc to k-plane constant sectional curvature, but we do not discuss it here. We use the notation cvc and 2-cvc interchangeably.

Given an integer k, $\mathcal{C}_k$ denotes the set of all k-cvc values of $\mathcal{M}$. For brevity, if we say $\mathcal{M}$ is k-cvc($[\epsilon, \delta]$), we mean $\mathcal{M}$ is at least k-cvc($[\epsilon, \delta]$). If $\mathcal{C}_k$ is exactly $[\epsilon, \delta]$, we say so explicitly. Unless otherwise stated, we assume $2 \le k \le n-1$. Since κ and $\mathcal{K}$ are not defined for 1-dimensional subspaces, we do not consider $k = 1$. Also,

$$\mathcal{K}(\mathrm{V}) = \sum_{j>i=1}^{n} \kappa(e_i, e_j) = \sum_{j>i=1}^{n} \mathrm{R}_{ijji},$$

so the n-plane scalar curvature of V is $\tau/2$. Thus every model space is exactly n-cvc($\tau/2$), so we do not consider the case $k = n$ either.

Much like Calle generalizes cvc to k-cvc, we extend k-plane constant vector curvature to (m, k)-plane constant vector curvature. Instead of requiring each vector to be contained in a k-plane with some k-plane scalar curvature value, we examine the possibility that an m-plane is contained in such a k-plane. By considering different curvature measurements, we can better understand a model space and hence the local properties of the manifold it represents.

Definition 1.6. A model space $\mathcal{M}$ is (m, k)**-plane constant vector curvature** ϵ, denoted (m, k)-cvc(ϵ), if for all m-planes P, there exists a non-degenerate k-plane L containing P such that $\mathcal{K}(\mathrm{L}) = \epsilon$.

Analogous to standard k-plane scalar curvature, we let $\mathcal{C}_k^m$ denote the set of all (m, k)-plane constant vector curvature values of a given $\mathcal{M}$. In particular, $\mathcal{C}_k = \mathcal{C}_k^1$. We do not consider the case $m = k$, since if every k-plane is contained in a k-plane (i.e. itself) with curvature ϵ, then every k-plane has curvature ϵ.

In this paper, we investigate the properties of k-plane and (m, k)-plane constant vector curvature. Section 2 shows that $\mathcal{C}_k$ and $\mathcal{C}_k^m$ are compact intervals. Section 3 establishes several relationships between the curvature values of a decomposable model space and its component spaces. For example, **theorem 3.3** proves that k-cvc values of the component spaces give rise to $(k+1)$-cvc values for the composite space, and an important corollary demonstrates that every decomposable model space with a positive-definite inner product is k-cvc(ϵ) for some integer $k \ge 2$ and $\epsilon \in \mathbb{R}$ (see **corollary 3.6**). Section 4 explores k-cvc and (m, k)-cvc through two examples. The first example provides the first instance of a model space with (m, k)-plane constant vector curvature, and the second example uses our theorems to efficiently calculate the k-plane constant vector curvature values of a decomposable model space. This research further characterizes model spaces by assigning new basis-independent values to its various subspaces and allows one to construct model spaces with prescribed curvature values from simpler model spaces.

2 Topological Properties of $\mathscr{C}_k$ and $\mathscr{C}_k^m$

This section seeks to determine the topological invariants of the sets $\mathscr{C}_k$ and $\mathscr{C}_k^m$. Given a subspace U of V, let $\mathrm{U}^\perp$ be its *orthogonal complement*. Our first proof utilizes that SO(n), the *special orthogonal group*, is path-connected for $n \geq 2$ [16]. (The referee noted there is an elegant proof of **theorem 2.1** and **theorem 2.2** using flag manifolds and transitive Lie group actions, but we take a more elementary approach that is hopefully more approachable for those not familiar with smooth manifold theory.)

Theorem 2.1. *If* $\mathscr{M} = (\mathrm{V}, \langle\cdot,\cdot\rangle, \mathrm{R})$ *is a model space with* $n = \dim(\mathrm{V}) \geq 3$, *then* $\mathscr{C}_k^m$ *is connected for all* $m, k \in \mathbb{Z}$ *such that* $1 \leq m < k \leq n$. *In particular,* $\mathscr{C}_k$ *is connected for all* $2 \leq k \leq n$.

Proof. Let $1 \leq m < k \leq n$, and choose an m-plane P with an orthonormal basis $\mathscr{B}_\mathrm{P}$. Suppose $\mathscr{M}$ is (m,k)-cvc(ϵ) and (m,k)-cvc(δ). Then there are k-planes $\mathrm{L}_0, \mathrm{L}_1 \subset \mathrm{V}$ containing P with $\mathscr{K}(\mathrm{L}_0) = \epsilon$ and $\mathscr{K}(\mathrm{L}_1) = \delta$. Choose orthonormal bases $\mathscr{B}_0$ and $\mathscr{B}_1$ for L_0 and L_1 so that $\mathscr{B}_\mathrm{P} \subseteq \mathscr{B}_0$ and $\mathscr{B}_\mathrm{P} \subseteq \mathscr{B}_1$. Extend $\mathscr{B}_0$ and $\mathscr{B}_1$ to orthonormal bases $\mathscr{V}_0$ and $\mathscr{V}_1$ for V. Consider the following sets with $n - m$ elements:

$$\mathscr{V}_0' = \mathscr{V}_0 \setminus \mathscr{B}_\mathrm{P} \quad \text{and} \quad \mathscr{V}_1' = \mathscr{V}_1 \setminus \mathscr{B}_\mathrm{P}.$$

If $m = n-1$ and $k = n$, then $\mathscr{C}_k^m$ is connected, since $\mathscr{M}$ is exactly n-cvc($\tau/2$).

If $m < n-1$, then SO($n-m$) is path-connected, so there is a continuous deformation of orthonormal bases from $\mathscr{V}_0'$ to $\mathscr{V}_1'$. Restricting to the first $k - m$ vectors yields such a deformation from $\mathscr{B}_0 \setminus \mathscr{B}_\mathrm{P}$ to $\mathscr{B}_1 \setminus \mathscr{B}_\mathrm{P}$. But the vectors in $\mathscr{B}_\mathrm{P}$ are pairwise orthogonal to the vectors in $\mathscr{B}_0 \setminus \mathscr{B}_\mathrm{P}$ and $\mathscr{B}_1 \setminus \mathscr{B}_\mathrm{P}$, so adding the m vectors in $\mathscr{B}_\mathrm{P}$ to any intermediate basis gives an orthogonal, linearly independent set of k unit vectors. Hence, this rotation in the orthogonal complement of P is a continuous deformation of orthonormal bases from $\mathscr{B}_0$ to $\mathscr{B}_1$, and the span of the basis vectors defines a path from L_0 to L_1. Since the span of each intermediate basis contains P, the space of k-planes containing P is path-connected, and hence connected. Because $\mathrm{L} \mapsto \mathscr{K}(\mathrm{L})$ is continuous and ϵ, δ are arbitrary, $\mathscr{C}_k^m$ is connected. In particular, setting $m = 1$, $\mathscr{C}_k$ is connected for all $2 \leq k \leq n$. □

Because $\mathrm{Gr}_k(\mathrm{V})$ is compact [13] and $\mathrm{L} \mapsto \mathscr{K}(\mathrm{L})$ is continuous, $\mathscr{C}_k^m$ is compact and hence bounded. The following theorem shows $\mathscr{C}_k^m$ is also closed.

Theorem 2.2. *For any model space* $\mathscr{M} = (\mathrm{V}, \langle\cdot,\cdot\rangle, \mathrm{R})$, $\mathscr{C}_k^m$ *is closed for all* $m, k \in \mathbb{Z}$ *such that* $1 \leq m < k \leq n$. *In particular,* $\mathscr{C}_k$ *is closed for all* $2 \leq k \leq n$.

Proof. Let $\mathcal{K} : \mathrm{Gr}_k(\mathrm{V}) \to \mathbb{R}$ be the continuous map $\mathrm{L} \mapsto \mathcal{K}(\mathrm{L})$. It suffices to show that if $a \in \mathbb{R}$ is the limit of $(a_i) \subseteq \mathscr{C}_k^m$, then $a \in \mathrm{C}_k^m$. Let P be an m-plane

$$\mathrm{P} = \mathrm{span}\{v_1, \ldots, v_m\} \subset \mathrm{V}.$$

Then there is a k-plane $\mathrm{L}_i \supseteq \mathrm{P}$ with $\mathcal{K}(\mathrm{L}_i) = a_i$. Since $\mathrm{Gr}_k(\mathrm{V})$ is compact and $\mathcal{K}$ is continuous, there is a subsequence (L_{i_j}) of (L_i) converging to some $\mathrm{L} \in \mathrm{Gr}_k(\mathrm{V})$ with $\mathcal{K}(\mathrm{L}) = a$. But $v_1, \ldots, v_m \in \mathrm{L}_{i_j}$, so L contains $v_1, \ldots, v_m$ and hence P. Thus $a \in \mathscr{C}_k^m$, so $\mathscr{C}_k^m$ is closed. In particular, letting $m = 1$, $\mathscr{C}_k$ is closed. □

Theorem 2.1 and **theorem 2.2** imply that $\mathscr{C}_k^m \subseteq \mathbb{R}$ is connected and compact, so it is an interval of the form $[a, b]$. An important next step for future work is to develop sufficient conditions for demonstrating that certain $\epsilon \in \mathbb{R}$ cannote be in $\mathscr{C}_k^m$.

3 Decomposability and Curvature

Given a model space $\mathcal{M} = (\mathrm{V}, \langle\cdot,\cdot\rangle, \mathrm{R})$, one can ask whether $\mathcal{M}$ decomposes as the *direct sum* (denoted $\oplus$) of two or more model spaces. If so, V, $\langle\cdot,\cdot\rangle$, and R must decompose in a reasonable way. This section investigates the relationship between decomposability and curvature, leveraging the fact that $\mathscr{C}_k$ is connected. Note that these results only concern k-cvc, not (m, k)-cvc.

Definition 3.1. *A model space* $\mathcal{M} = (\mathrm{V}, \langle\cdot,\cdot\rangle, \mathrm{R})$ *is* **decomposable**, *written* $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$, *if* $\mathrm{V} = \mathrm{V}_1 \oplus \mathrm{V}_2$, $\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_1 \oplus \langle\cdot,\cdot\rangle_2$, $\mathrm{R} = \mathrm{R}_1 \oplus \mathrm{R}_2$, *and the following hold for all vectors* $v_1 \in \mathrm{V}_1$ *and* $v_2 \in \mathrm{V}_2$*:*

1. $\langle v_1, v_2\rangle = \langle v_1, 0\rangle_1 + \langle 0, v_2\rangle_2 = 0,$
2. $\mathrm{R}(v_1, v_2, \cdot, \cdot) = \mathrm{R}_1(v_1, 0, \cdot, \cdot) + \mathrm{R}_2(0, v_2, \cdot, \cdot) = 0.$

The vectors in (2) are unbiased in input, meaning this expression is zero even if v_1 or v_2 is in a different slot [5]. We denote the set of k-cvc values of $\mathcal{M}_1$ and $\mathcal{M}_2$ by ${}_1\mathscr{C}_k$ and ${}_2\mathscr{C}_k$. If $\mathcal{M}$ decomposes into three or more model spaces, we write

$$\mathcal{M} = \mathcal{M}_1 \oplus \cdots \oplus \mathcal{M}_j = \oplus_{i=1}^{j} \mathcal{M}_i.$$

We call $\mathcal{M}_i$ a *component space* and $\mathcal{M}$ the *composite space.* Since distinct V_i are orthogonal, we can uniquely write any $v \in \mathrm{V}$ as $v = a_1 v_1 + \cdots + a_j v_j$ for $a_i \in \mathbb{R}$ and unit vectors $v_i \in \mathrm{V}_i$. Also, given a nonzero vector v,

$$\hat{v} = \frac{v}{\|v\|}$$

denotes the associated unit vector of v. With this background, we can now examine how decomposabilty impacts $\mathscr{C}_k$, beginning with $k = 2$.

Proposition 3.2. *Suppose a model space $\mathscr{M} = (\mathrm{V}, \langle\cdot,\cdot\rangle, \mathrm{R})$ decomposes as $\mathscr{M} = \mathscr{M}_1 \oplus \mathscr{M}_2$ with $\dim(\mathrm{V}_i) \geq 2$. If there exists $\epsilon \in {}_1\mathscr{C}_2$ such that $-\epsilon \in {}_2\mathscr{C}_2$, then $\mathscr{M}$ is cvc(0).*

Proof. Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis for V. Suppose without loss of generality that $v \in \mathrm{V}$ is normalized. We must find a 2-plane containing v whose sectional curvature is zero. There are three cases: $v \in \mathrm{V}_1$, $v \in \mathrm{V}_2$, or v is a linear combination of vectors from V_1 and V_2. If $v \in \mathrm{V}_1$, take any unit vector $u_2 \in \mathrm{V}_2$. Then $\kappa(v, u_2) = 0$, and a similar argument applies if $v \in \mathrm{V}_2$.

Now, suppose there are unit vectors $v_1 \in \mathrm{V}_1$, $v_2 \in \mathrm{V}_2$ so that

$$v = \frac{v_1 + v_2}{\sqrt{2}}.$$

Because $\mathscr{M}_1$ is cvc(ϵ), there is a 2-plane π_1 containing v_1 with $\kappa(\pi_1) = \epsilon$. Choose a unit vector w_1 orthogonal to v_1 so that $\pi_1 = \mathrm{span}\{v_1, w_1\}$. Then

$$\kappa(v_1, w_1) = \mathrm{R}_1(v_1, w_1, w_1, v_1) = \epsilon. \tag{1}$$

Similarly, since $\mathscr{M}_2$ is cvc$(-\epsilon)$, there is a 2-plane π_2 containing v_2 with $\kappa(\pi_2) = -\epsilon$. Choose a unit vector w_2 orthogonal to v_2 so that $\pi_2 = \mathrm{span}\{v_2, w_2\}$. Then

$$\kappa(v_2, w_2) = \mathrm{R}_2(v_2, w_2, w_2, v_2) = -\epsilon. \tag{2}$$

Next, set $\pi_{12} = \mathrm{span}\{v, w\}$, where w is the unit vector

$$w = \frac{w_1 + w_2}{\sqrt{2}}.$$

Since v is orthogonal to w, $\{v, w\}$ is an orthonormal basis for π_{12}, so

$$\begin{aligned}
\kappa(\pi_{12}) &= \mathrm{R}(v, w, w, v) \\
&= \mathrm{R}\left(\frac{v_1 + v_2}{\sqrt{2}}, \frac{w_1 + w_2}{\sqrt{2}}, \frac{w_1 + w_2}{\sqrt{2}}, \frac{v_1 + v_2}{\sqrt{2}}\right) \\
&= \frac{1}{4}\mathrm{R}(v_1 + v_2, w_1 + w_2, w_1 + w_2, v_1 + v_2) \\
&= \frac{1}{4}\mathrm{R}_1(v_1, w_1, w_1, v_1) + \frac{1}{4}\mathrm{R}_2(v_2, w_2, w_2, v_2)
\end{aligned}$$

by **definition 3.1**. But these two terms cancel by (1) and (2), so $\kappa(\pi_{12}) = 0$. □

Proposition 3.2 only proves $\mathscr{M}$ is at least (not exactly) 2-cvc(0). Our first theorem shows how k-cvc values of a component space "lift" to the composite space.

Theorem 3.3. *If a model space $\mathscr{M} = (\mathrm{V}, \langle\cdot,\cdot\rangle, \mathrm{R})$ decomposes as $\mathscr{M} = \mathscr{M}_1 \oplus \mathscr{M}_2$ and $\mathscr{M}_1$ is k-cvc(ϵ) for an integer $2 \leq k \leq n-1$ and $\epsilon \in \mathbb{R}$, then $\mathscr{M}$ is $(k+1)$-cvc(ϵ).*

Proof. If $v \in \mathrm{V}$, then we can uniquely write

$$v = av_1 + bv_2.$$

for $a, b \in \mathbb{R}$ and unit vectors $v_1 \in \mathrm{V}_1$, $v_2 \in \mathrm{V}_2$. We consider three cases: $v \in \mathrm{V}_1$, $v \in \mathrm{V}_2$, or v is a linear combination of vectors in V_1 and V_2.

First suppose $v \in \mathrm{V}_1$. Since $\mathscr{M}_1$ is k-cvc(ϵ), there is a k-plane $\widetilde{\mathrm{L}} \subseteq \mathrm{V}_1$ containing v with $\mathscr{K}(\widetilde{\mathrm{L}}) = \epsilon$. Set $\mathrm{E}_1 = \hat{v} = v/\|v\|$, and complete it to an orthonormal basis $\{\mathrm{E}_1, \ldots, \mathrm{E}_k\}$ for $\widetilde{\mathrm{L}}$. Consider the $(k+1)$-plane

$$\mathrm{L} = \mathrm{span}\{\mathrm{E}_1, \ldots, \mathrm{E}_{k+1}\},$$

where $\mathrm{E}_{k+1} \in \mathrm{V}_2$ is a unit vector (clearly pairwise orthogonal to $\mathrm{E}_1, \ldots, \mathrm{E}_k$). Then

$$\begin{aligned}
\mathscr{K}(\mathrm{L}) &= \sum_{j>i=1}^{k+1} \kappa(\mathrm{E}_i, \mathrm{E}_j) \\
&= \sum_{j>i=1}^{k} \kappa(\mathrm{E}_i, \mathrm{E}_j) + \sum_{i=1}^{k} \kappa(\mathrm{E}_i, \mathrm{E}_{k+1}) \\
&= \epsilon + \sum_{i=1}^{k} \kappa(\mathrm{E}_i, \mathrm{E}_{k+1}),
\end{aligned} \tag{3}$$

since $\mathscr{K}(\widetilde{\mathrm{L}}) = \epsilon$. Because $\{\mathrm{E}_1, \ldots, \mathrm{E}_{k+1}\}$ is an orthonormal basis for L,

$$\kappa(\mathrm{E}_i, \mathrm{E}_{k+1}) = \mathrm{R}(\mathrm{E}_i, \mathrm{E}_{k+1}, \mathrm{E}_{k+1}, \mathrm{E}_i)$$

for $i = 1, \ldots, k$. But $\mathrm{E}_i \in \mathrm{V}_1$ for all i and $\mathrm{E}_{k+1} \in \mathrm{V}_2$, so

$$\mathrm{R}(\mathrm{E}_i, \mathrm{E}_{k+1}, \mathrm{E}_{k+1}, \mathrm{E}_i) = 0.$$

Therefore, each summand in (3) is zero. Thus

$$\mathscr{K}(\mathrm{L}) = \epsilon + \sum_{i=1}^{k} \mathrm{R}(\mathrm{E}_i, \mathrm{E}_{k+1}, \mathrm{E}_{k+1}, \mathrm{E}_i) = \epsilon,$$

so L is a $(k+1)$-plane containing v with $\mathscr{K}(\mathrm{L}) = \epsilon$. Since this construction works for any nonzero vector in V_1, $\mathscr{M}$ is $(k+1)$-cvc(ϵ) in the case $v \in \mathrm{V}_1$.

Next, suppose $v \in \mathrm{V}_2$. Since $\mathscr{M}_1$ is k-cvc(ϵ), there is a k-plane $\widetilde{\mathscr{L}} \subseteq \mathrm{V}_1$, with orthonormal basis $\{\mathrm{E}_1, \ldots, \mathrm{E}_k\}$, such that $\mathscr{K}(\widetilde{\mathscr{L}}) = \epsilon$. Set $\mathrm{E}_{k+1} = \hat{v}$, which is pairwise orthogonal to the vectors $\mathrm{E}_1, \ldots, \mathrm{E}_k$. Then

$$\mathscr{L} = \mathrm{span}\{\mathrm{E}_1, \ldots, \mathrm{E}_{k+1}\}$$

is a $(k+1)$-plane containing v, and $\mathscr{K}(\mathscr{L}) = \epsilon$ by a similar argument as above.

Now, suppose $a \neq 0 \neq b$, which means $v \in \mathrm{span}\{v_1, v_2\}$. Set

$$\mathrm{E}_1 = v_1 \quad \text{and} \quad \mathrm{E}_{k+1} = v_2.$$

Since $\mathcal{M}_1$ is k-cvc(ϵ), there is a k-plane $\widetilde{\mathfrak{L}} \subseteq V_1$, with orthonormal basis $\{E_1, \ldots, E_k\}$, so that $\mathcal{K}(\widetilde{\mathfrak{L}}) = \epsilon$. Then $v \notin \mathfrak{L}$, but E_{k+1} is pairwise orthogonal to the vectors $E_1, \ldots, E_k$ and $v \in \operatorname{span}\{E_1, E_{k+1}\}$, so

$$\mathfrak{L} = \operatorname{span}\{E_1, \ldots, E_{k+1}\}$$

is a $(k+1)$-plane containing v. Similarly as before, $\mathcal{K}(\mathfrak{L}) = \epsilon$. (Note that here and throughout the proof, we are strongly relying on the fact that the k-plane sectional curvatures are independent of the basis chosen.) □

The labeling of $\mathcal{M}_1$ and $\mathcal{M}_2$ is arbitrary, so **theorem 3.3** is equally valid if we swap $\mathcal{M}_1$ and $\mathcal{M}_2$. Using a similar argument, we can generalize **theorem 3.3** to include any finite direct sum decomposition $\mathcal{M} = \oplus_{i=1}^{j} \mathcal{M}_i$ with $j \geq 3$.

Corollary 3.4. *If a model space $\mathcal{M} = (V, \langle\cdot,\cdot\rangle, R)$ decomposes as $\mathcal{M} = \oplus_{i=1}^{j} \mathcal{M}_i$, where some $\mathcal{M}_i$ is k-cvc(ϵ) for $2 \leq k \leq \dim(V_i)$ and $\epsilon \in \mathbb{R}$, then $\mathcal{M}$ is $(k+1)$-cvc(ϵ).*

Proof. Write $\mathcal{M}$ as $(\oplus_{p \neq i} \mathcal{M}_p) \oplus \mathcal{M}_i$ and use **theorem 3.3**. □

Recall that every model space of dimension n is n-cvc($\tau/2$), where τ is the scalar curvature. This fact suggests some additional applications of **theorem 3.3**.

Corollary 3.5. *Suppose $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$ and $\dim(V_2) = 1$. Then either $\mathcal{M}_1$ is $(n-1)$-cvc($\tau/2$) or has no $(n-1)$-cvc($\tau/2$) values.*

Proof. Since $\mathcal{C}_n = \{\tau/2\}$, $\tau/2$ is the only possible $(n-1)$-cvc value of $\mathcal{M}_1$. □

Note that this result holds verbatim if we swap the roles of $\mathcal{M}_1$ and $\mathcal{M}_2$. By using a similar argument, we obtain the following important corollary.

Corollary 3.6. *Suppose $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$. Let τ_i be the scalar curvature of $\mathcal{M}_i$ and set $\dim(V_i) = n_i$. Then $\mathcal{M}$ is (n_1+1)-cvc($\tau_1/2$) and (n_2+1)-cvc($\tau_2/2$).*

Proof. Since $\mathcal{M}_1$ and $\mathcal{M}_2$ are n_1-cvc($\tau_1/2$) and n_2-cvc($\tau_2/2$), respectively, **theorem 3.3** implies that $\mathcal{M}$ is (n_1+1)-cvc($\tau_1/2$) and (n_2+1)-cvc($\tau_2/2$). □

Corollary 3.6 has a significant consequence: *every decomposable model space with a positive-definite inner product is k-cvc(ϵ) for some integer $k \geq 2$ and $\epsilon \in \mathbb{R}$.* This extends the known result that every 3-dimensional model space equipped with a positive-definite inner product is cvc(ϵ) for a unique value ϵ [17]. Thus, **corollary 3.6** significantly increases the number of known k-cvc model spaces.

Our next corollary uses the observation that the proof of **theorem 3.3** is independent of ${}_2\mathcal{C}_k$. Hence, if $\mathcal{M}_2$ is k-cvc(δ) for the same k as $\mathcal{M}_1$, the range of k-cvc values "lift" from both component spaces to the composite space.

Corollary 3.7. *Suppose a model space* $\mathscr{M} = (\mathrm{V}, \langle\cdot,\cdot\rangle, \mathrm{R})$ *decomposes as* $\mathscr{M} = \mathscr{M}_1 \oplus \mathscr{M}_2$*. If* $\mathscr{M}_1$ *is* k*-cvc*(ϵ) *and* $\mathscr{M}_2$ *is* k*-cvc*(δ) *for some integer* $2 \le k \le \min(\dim(\mathrm{V}_1), \dim(\mathrm{V}_2))$ *and* $\epsilon, \delta \in \mathbb{R}$*, then* $\mathscr{M}$ *is* $(k+1)$*-cvc*$([\epsilon, \delta])$.

Proof. Since $\mathscr{M}$ is $(k+1)$-cvc(ϵ) and $(k+1)$-cvc(δ) by **theorem 3.3**, if we use the convention that $[\epsilon, \delta] = \epsilon$ when $\epsilon = \delta$, **theorem 2.1** shows $\mathscr{M}$ is $(k+1)$-cvc$([\epsilon, \delta])$. $\square$

Corollary 3.7 gives an even stronger relationship between the $(k+1)$-cvc values of a decomposable model space and the k-cvc values of its component spaces. Naturally, this result generalizes to any finite direct sum decomposition.

Corollary 3.8. *Suppose a model space* $\mathscr{M} = (\mathrm{V}, \langle\cdot,\cdot\rangle, \mathrm{R})$ *decomposes as* $\mathscr{M} = \oplus_{i=1}^{j} \mathscr{M}_i$*, where each* $\mathscr{M}_i$ *is* k*-cvc*(ϵ_i) *for some integer* $2 \le k \le \min\{\dim(\mathrm{V}_i)\}_{i=1}^{j}$ *and* $\epsilon_i \in \mathbb{R}$*. Let* ϵ_m *and* ϵ_{M}*, respectively, be the minimum and maximum of the set* $\{\epsilon_i\}_{i=1}^{j}$*. Then* $\mathscr{M}$ *is* $(k+1)$*-cvc*$([\epsilon_m, \epsilon_{\mathrm{M}}])$.

Proof. By **corollary 3.7** applied to (3.4) to $\mathscr{M}_i$, $\mathscr{M}$ is $(k+1)$-cvc(ϵ_i) for all i. In particular, $\mathscr{M}$ is $(k+1)$-cvc(ϵ_m) and $(k+1)$-cvc(ϵ_{M}), where

$$\epsilon_m = \min\{\epsilon_i\}_{i=1}^{j} \quad \text{and} \quad \epsilon_{\mathrm{M}} = \max\{\epsilon_i\}_{i=1}^{j},$$

so $\mathscr{M}$ is $(k+1)$-cvc$([\epsilon_m, \epsilon_{\mathrm{M}}])$ by **theorem 2.1**. $\square$

In nearly all cases, **corollary 3.8** guarantees a range of $(k+1)$-cvc values for $\mathscr{M}$.

Corollary 3.9. *Let* $\mathscr{M} = \oplus_{i=1}^{j} \mathscr{M}_i$ *and fix an integer* $k \ge 2$*. Define*

$$\tau_m = \min\{\tau_i\}_{i=1}^{j} \quad \textit{and} \quad \tau_{\mathrm{M}} = \max\{\tau_i\}_{i=1}^{j}.$$

If $\dim(\mathrm{V}_i) = k$ *for all* i*, then* $\mathscr{M}$ *is* $(k+1)$*-cvc*$([\tau_m/2, \tau_{\mathrm{M}}/2])$.

Proof. Since $\dim(\mathrm{V}_i) = k$ for all i and $\mathscr{M}_i$ is k-cvc(τ_m) and $\mathscr{M}_l$ is k-cvc(τ_{M}) for some $i, l \in \{1, \ldots, j\}$, then $\mathscr{M}$ is k-cvc$([\tau_m/2, \tau_{\mathrm{M}}/2])$ by **corollary 3.8**. $\square$

Our final theorem provides further insight into the connection between the k-cvc values of a decomposable model space and its components.

Theorem 3.10. *Suppose a model space* $\mathscr{M} = (\mathrm{V}, \langle\cdot,\cdot\rangle, \mathrm{R})$ *decomposes as* $\mathscr{M} = \mathscr{M}_1 \oplus \mathscr{M}_2$*. If* $\mathscr{M}_1$ *is* i*-cvc*(ϵ) *and* $\mathscr{M}_2$ *is* j*-cvc*(δ) *for integers* i *and* j*, where*

$$2 \le i, j \le \max(\dim(\mathrm{V}_1), \dim(\mathrm{V}_2)),$$

and $\epsilon, \delta \in \mathbb{R}$*, then* $\mathscr{M}$ *is* $(i+j)$*-cvc*$(\epsilon + \delta)$.

Proof. Let $\{e_1,\dots,e_n\}$ be an orthonormal basis for V, $v \in V$, and write $v = av_1 + bv_2$ for $a, b \in \mathbb{R}$ and unit vectors $v_1 \in V_1$, $v_2 \in V_2$. Set $k = i + j$. We consider three cases: $v \in V_1$, $v \in V_2$, or v is a linear combination of vectors in V_1 and V_2.

First suppose $v \in V_1$. Since $\mathscr{M}_1$ is i-cvc(ϵ), there is an i-plane $I \subseteq V_1$ containing v such that $\mathscr{K}(I) = \epsilon$. Set $E_1 = \hat{v}$, and complete it to an orthonormal basis $\{E_1,\dots,E_i\}$ for I. Since $\mathscr{M}_2$ is j-cvc(δ), there is a j-plane $J \subseteq V_2$ with $\mathscr{K}(J) = \delta$. Let $\{E_{i+1},\dots,E_k\}$ be an orthonormal basis for J. Consider the k-plane

$$L = \operatorname{span}\{E_1,\dots,E_k\}.$$

Unwinding the definitions, we find

$$\begin{aligned}
\mathscr{K}(L) &= \sum_{t>s=1}^{k} \kappa(E_s, E_t) \\
&= \sum_{t>s=1}^{i} \kappa(E_s, E_t) + \sum_{t>i+1}^{k} \sum_{s=1}^{i} \kappa(E_s, E_t) + \sum_{t>s=i+1}^{k} \kappa(E_s, E_t) \\
&= \epsilon + \sum_{t>i+1}^{k} \sum_{s=1}^{i} \kappa(E_s, E_t) + \delta, \qquad (4)
\end{aligned}$$

since $\mathscr{K}(I) = \epsilon$ and $\mathscr{K}(J) = \delta$. Because $\{E_1,\dots,E_k\}$ is an orthonormal basis for I,

$$\kappa(E_s, E_t) = R(E_s, E_t, E_t, E_s)$$

for $s = 1,\dots,i$ and $t = i+1,\dots,k$. For each s and t,

$$R(E_s, E_t, E_t, E_s) = 0,$$

since $E_s \in V_1$ and $E_t \in V_2$. Thus, the summation in (4) is zero, so $\mathscr{K}(L) = \epsilon + \delta$. This construction works for any nonzero vector in V_1, so $\mathscr{M}$ is k-cvc($\epsilon + \delta$).

Next, suppose $v \in V_2$. Since $\mathscr{M}_1$ is i-cvc(ϵ), there is an i-plane $\mathscr{I} \subseteq V_1$, with some orthonormal basis $\{E_1,\dots,E_i\}$, such that $\mathscr{K}(\mathscr{I}) = \epsilon$. Set $E_{i+1} = \hat{v}$. Since $\mathscr{M}_2$ is j-cvc(δ), there is a j-plane $\mathscr{J} \subseteq V_2$ containing v, with orthonormal basis $\{E_{i+1},\dots,E_k\}$, so that $\mathscr{K}(\mathscr{J}) = \delta$. Then

$$\mathscr{L} = \operatorname{span}\{E_1,\dots,E_k\}$$

is a k-plane containing v and, by a similar argument as above, $\mathscr{K}(\mathscr{L}) = \epsilon + \delta$.

Now, if $a \neq 0 \neq b$, then $v \in \operatorname{span}\{v_1, v_2\}$. Set $E_1 = v_1$ and $E_{i+1} = v_2$. Because $\mathscr{M}_1$ is i-cvc(ϵ), there is an i-plane $I' \subseteq V_1$ containing v_1, with orthonormal basis $\{E_1,\dots,E_i\}$, such that $\mathscr{K}(I') = \epsilon$. Similarly, there is a j-plane $J' \subseteq V_2$ containing v_2, with orthonormal basis $\{E_{i+1},\dots,E_k\}$, so that $\mathscr{K}(J') = \delta$. Then

$$\mathfrak{L} = \operatorname{span}\{E_1,\dots,E_k\}$$

is a k-plane containing v, so $\mathscr{K}(\mathscr{L}) = \epsilon + \delta$ by a similar argument as the second paragraph. Since $k = i + j$, it follows that $\mathscr{M}$ is $(i+j)$-cvc($\epsilon + \delta$). $\square$

Let us make a few observations about this result. First, if neither $\mathscr{M}_1$ nor $\mathscr{M}_2$ is p-cvc(0) for any $p \geq 2$, but $\mathscr{M}_1$ is i-cvc(ϵ) and $\mathscr{M}_2$ is j-cvc($-\epsilon$) for integers

$$2 \leq i, j \leq \max(\dim(\mathrm{V}_1), \dim(\mathrm{V}_2))$$

and $\epsilon \in \mathbb{R}$, then $\mathscr{M}_1 \oplus \mathscr{M}_2$ would still be $(i+j)$-cvc(0). Second, if we define a model space $-\mathscr{M} = (\mathrm{V}, \langle\cdot,\cdot\rangle, -\mathrm{R})$ (meaning the ACT entries of $-\mathscr{M}$ are the negative of those of $\mathscr{M}$), then the previous observation implies that $-\mathscr{M}$ is k-cvc($-\epsilon$) whenever $\mathscr{M}$ is k-cvc(ϵ). It follows that the direct sum $\mathscr{M} \oplus (-\mathscr{M})$ is k-cvc(0).

As with **corollary 3.7**, we can easily generalize **theorem 3.10** to include any finite direct sum decomposition of a model space.

Corollary 3.11. *Suppose a model space $\mathscr{M} = (\mathrm{V}, \langle\cdot,\cdot\rangle, \mathrm{R})$ decomposes as $\mathscr{M} = \oplus_{i=1}^{j} \mathscr{M}_i$, where each $\mathscr{M}_i$ is k_i-cvc(ϵ_i) for some integer $2 \leq k_i \leq \max\{\dim(\mathrm{V}_i)\}_{i=1}^{j}$ and $\epsilon_i \in \mathbb{R}$. Define the scalars $k = \sum_{i=1}^{j} k_i$ and $\epsilon = \sum_{i=1}^{j} \epsilon_i$. Then $\mathscr{M}$ is k-cvc(ϵ).*

Proof. This follows from $j-1$ applications of **theorem 3.10**. Let

$$k = \sum_{i=1}^{j} k_i \quad \text{and} \quad \epsilon = \sum_{i=1}^{j} \epsilon_i.$$

Applying **theorem 3.10** to $\mathscr{M}_1$ and $\mathscr{M}_2$ shows $\mathscr{M}_1 \oplus \mathscr{M}_2$ is (k_1+k_2)-cvc($\epsilon_1+\epsilon_2$). Since $k_1 + k_2 \leq \dim(\mathrm{V}_1) + \dim(\mathrm{V}_2)$, we can apply **theorem 3.10** to $\mathscr{M}_1 \oplus \mathscr{M}_2$ and $\mathscr{M}_3$. Thus $\mathscr{M}_1 \oplus \mathscr{M}_2 \oplus \mathrm{M}_3$ is $(k_1+k_2+k_3)$-cvc($\epsilon_1+\epsilon_2+\epsilon_3$). Applying this process recursively a total of $j-1$ times, we conclude that $\mathscr{M}$ is k-cvc(ϵ). □

This section demonstrates the rich connection between the curvature values of the composite space and its component spaces. Armed with these results, one can easily construct a model space with certain k-cvc values by forming the direct sum of an appropriate collection of model spaces.

4 Examples

Now that we determine the topological structure of $\mathscr{C}_k$ and $\mathscr{C}_k^m$ and investigated the curvature properties of decomposable model spaces, we use two examples to illustrate these results. We provide the first instance of a model space with (m,k)-plane constant vector curvature and use our theorems to efficiently calculate the k-plane constant vector curvature values of a decomposable model space.

The rich connection between canonical ACTs and linear algebra makes such a tensor particularly appealing for our first example. Our method, inspired by [1], is to decompose v into vectors from the eigenspaces of ϕ. Recall that if $\mathrm{A}: \mathrm{V} \to \mathrm{V}$ is a linear transformation, then $v \in V \setminus \{0\}$ is called an *eigenvector* of A with *eigenvalue* $\lambda \in \mathbb{R}$ if $Av = \lambda v$. If λ_i is an eigenvalue, we use E_i to denote the *eigenspace* spanned by the associated eigenvectors.

Given a symmetric, bilinear function ϕ defined on a vector space with a non-degenerate inner product, there is a unique self-adjoint linear transformation $A : V \to V$ characterized by the equation $\phi(x, y) = \langle Ax, y\rangle$. Since ϕ is diagonalized, for any orthonormal eigenvectors E_i and E_j, $\langle E_i, E_j\rangle$ can be nonzero only if $i = j$. This observation suggests the following proposition, whose proof is in [1].

Proposition 4.1. *Let $\mathscr{M} = (V, \langle\cdot,\cdot\rangle, R)$ be a model space. If E_i, E_j are orthogonal unit vectors in the eigenspaces for λ_i, λ_j, respectively, then $\kappa(E_i, E_j) = \lambda_i\lambda_j$.*

Proposition 4.1 motivates us to use canonical ACTs in our first example. Knowing only the eigenvalues of ϕ, we can easily calculate sectional curvature values, and hence k-plane scalar curvature values. We have the following helpful bound on the sectional curvature values in terms of the products of eigenvalues.

Proposition 4.2. *[2] Let ϕ be a symmetric, bilinear function, and let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of ϕ, repeated according to multiplicity. Let m and M, respectively, be the minimum and maximum of the set $\{\lambda_i\lambda_j : i \neq j\}$. The set of sectional curvatures of R_ϕ is precisely the interval $[m, M]$.*

We are ready to introduce our example on (m, k)-plane constant vector curvature.

Example 4.3. Let $\mathscr{M} = (V, \langle\cdot,\cdot\rangle, R)$ be a model space with $\{e_1, \ldots, e_6\}$ an orthonormal basis for V, $\langle\cdot,\cdot\rangle$ a positive-definite inner product on V, and $R = R_\phi$, where ϕ is represented by the matrix

$$\phi = \begin{bmatrix} 0_3 & 0_3 \\ 0_3 & I_3 \end{bmatrix}.$$

Here, I_3 is the 3×3 identity matrix and 0_3 is the 3×3 zero matrix.

We defined the matrix in **example 4.3** so that $\phi(e_i, e_j)$ is the ij^{th} entry. Hence, the eigenvalues of ϕ are $\lambda_0 = 0$ and $\lambda_1 = 1$. The associated eigenspaces are $E_0 = \text{span}\{e_1, e_2, e_3\}$ and $E_1 = \text{span}\{e_4, e_5, e_6\}$, meaning $\dim(E_0) = \dim(E_1) = 3$. Since $\text{rank}(\phi) = 3 \geq 2$, we know $E_0 = \ker(R)$ by **proposition 1.2**. Also, E_0 and E_1 are orthogonal because $\{e_1, \ldots, e_6\}$ is an orthonormal basis for V. Therefore, we can write any $v \in V$ as $v = \alpha v_1 + \beta v_2$ for $\alpha, \beta \in \mathbb{R}$ and unit vectors $v_1, v_2 \in V$.

Proposition 4.4. *The model space $\mathscr{M}$ in **example 4.3** has the following properties:*

1. $\mathscr{C}_3^2 \subseteq \{0\}$, $\mathscr{C}_4^3 \subseteq \{0\}$, $C_5^4 \subseteq [0, 1]$
2. *$\mathscr{M}$ is $(2, 4)$-cvc(1),*
3. *$\mathscr{M}$ is $(2, 5)$-cvc$([1, 3])$.*

4. *$\mathscr{M}$ is $(m,6)$-cvc(3) and only $(m,6)$-cvc(3) for $1 \le m \le 5$.*

Proof. To prove (1), we first show $\mathscr{C}_3^2 \subseteq \{0\}$. Suppose $\mathscr{M}$ is $(2,3)$-cvc(ϵ) for some $\epsilon \in \mathbb{R}$. We narrow down possible values for ϵ by a careful choice of 2-plane. Let L be a 3-plane containing $\mathrm{P} = \mathrm{span}\{e_1, e_2\}$ with $\mathscr{K}(\mathrm{L}) = \epsilon$, and let $\mathscr{B} = \{f_1, f_2, f_3\}$ be an orthonormal basis for L. Since L contains P, we may suppose without loss of generality that $f_1 = e_1$ and $f_2 = e_2$. Since $f_1, f_2 \in \mathrm{E}_0 = \ker(\mathrm{R})$,

$$\mathscr{K}(\mathrm{L}) = \sum_{j>i=1}^{3} \kappa(f_i, f_j) = \lambda_0\lambda_0 + \mathrm{R}_{1331} + \mathrm{R}_{2332} = 0.$$

Therefore, if $\mathscr{M}$ is $(2,3)$-cvc(ϵ), then ϵ must be 0, so $\mathscr{C}_3^2 \subseteq \{0\}$.

A similar argument shows $\mathscr{C}_4^3 \subseteq \{0\}$. Suppose $\mathscr{M}$ is $(3,4)$-cvc(ϵ). Let

$$\mathrm{P}' = \mathrm{span}\{e_1, e_2, e_3\},$$

and let L' be a 4-plane with orthonormal basis $\{f_1', f_2', f_3', f_4'\}$. Suppose without loss of generality that $f_1' = e_1, f_2' = e_2$, and $f_3' = e_3$. Since $f_1', f_2', f_3' \in \ker(\mathrm{R})$,

$$\mathscr{K}(\widetilde{\mathrm{L}}) = \sum_{j>i=1}^{4} \kappa(f_i', f_j') = \kappa(f_1', f_4') + \kappa(f_2', f_4') + \kappa(f_3', f_4') = 0.$$

Therefore, if $\mathscr{M}$ is $(3,4)$-cvc(ϵ), then $\epsilon = 0$, so $\mathscr{C}_4^3 \subseteq \{0\}$.

Again, a similar argument shows $\mathscr{C}_5^4 \subseteq [0,1]$. Suppose $\mathscr{M}$ is $(4,5)$-cvc(ϵ). Let

$$\widetilde{\mathrm{P}} = \mathrm{span}\{e_1, e_2, e_3, e_4\},$$

and $\widetilde{\mathrm{L}}$ a 5-plane with orthonormal basis $\{\widetilde{f}_1, \widetilde{f}_2, \widetilde{f}_3, \widetilde{f}_4, \widetilde{f}_5\}$. Suppose without loss of generality that $\widetilde{f}_1 = e_1, \widetilde{f}_2 = e_2, \widetilde{f}_3 = e_3$, and $\widetilde{f}_4 = e_4$. Since $\widetilde{f}_1, \widetilde{f}_2, \widetilde{f}_3 \in \ker(\mathrm{R})$,

$$\begin{aligned}
\mathscr{K}(\widetilde{\mathrm{L}}) &= \sum_{j>i=1}^{5} \kappa(\widetilde{f}_i, \widetilde{f}_j) \\
&= \sum_{i=1}^{3} \kappa(\widetilde{f}_i, \widetilde{f}_4) + \sum_{i=1}^{3} \kappa(\widetilde{f}_i, \widetilde{f}_5) + \kappa(\widetilde{f}_4, \widetilde{f}_5) \\
&= \kappa(\widetilde{f}_4, \widetilde{f}_5).
\end{aligned}$$

But $0 \le \kappa(\widetilde{f}_4, \widetilde{f}_5) \le 1$ by **proposition 4.2**. Therefore, if $\mathscr{M}$ is $(4,5)$-cvc(ϵ), then $\epsilon \in [0,1]$, so $\mathscr{C}_5^4 \subseteq [0,1]$. This completes the proof of (1).

Consider (2). Let P be a 2-plane with orthonormal basis $\{x, y\}$. Write

$$x = ax_0 + bx_1 \quad \text{and} \quad y = cy_0 + dy_1$$

for $a, b, c, d \in \mathbb{R}$ and unit vectors $x_i, y_i \in \mathrm{E}_i$. Note that $x_0, y_0 \in x_1^{\perp} \cap y_1^{\perp}$ and $x_1, y_1 \in x_0^{\perp} \cap y_0^{\perp}$. To find a 4-plane L containing P with $\mathscr{K}(\mathrm{L}) = 1$, we perform a case-by-case study of the projections of x and y onto the eigenspaces E_0 and E_1.

First, suppose x_0 and y_0 are linearly independent. Since x_0 and y_0 span a 2-plane in E_0, there are orthogonal unit vectors $f_1, f_2 \in \mathrm{E}_0$ such that

$$\mathrm{span}\{f_1, f_2\} = \mathrm{span}\{x_0, y_0\}.$$

If $a = 0$, set $f_1 = u_0$ for a unit vector $u_0 \in y_0^{\perp} \cap \mathrm{E}_0$, $f_2 = y_0$. If $c = 0$, set $f_1 = x_0$ and $f_2 = v_0$ for a unit vector $v_0 \in x_0^{\perp} \cap \mathrm{E}_0$. If $a = c = 0$, let $f_1 = e_1$, $f_2 = e_2$.

Now, if x_0 and y_0 are linearly dependent, then any plane which contains x_0 also contains y_0, so we can choose $f_1 = x_0$ and $f_2 = v_0$ as above.

Similarly, suppose x_1 and y_1 are linearly independent. Since x_1 and y_1 span a 2-plane in E_1, there are orthogonal unit vectors $f_3, f_4 \in \mathrm{E}_1$ such that

$$\mathrm{span}\{f_3, f_4\} = \mathrm{span}\{x_1, y_1\}.$$

If $b = 0$, set $f_3 = u_1$ for a unit vector $u_1 \in y_1^{\perp} \cap \mathrm{E}_1$, $f_4 = y_1$. If $d = 0$, set $f_3 = x_1$ and $f_4 = v_1$ for a unit vector $v_1 \in x_1^{\perp} \cap \mathrm{E}_1$. If $b = d = 0$, let $f_3 = e_4$ and $f_2 = e_5$.

Now, if x_1 and y_1 are linearly dependent, then any plane which contains x_1 also contains y_1, so we can choose $f_1 = x_1$ and $f_2 = v_1$ as above.

In any case, we can construct an orthonormal basis $\{f_1, f_2, f_3, f_4\}$ for L. Since

$$\mathrm{span}\{x_0, x_1, y_0, y_1\} \subseteq \mathrm{L},$$

in particular, L contains $\mathrm{span}\{x, y\}$ and hence P. But $f_1, f_2 \in \mathrm{E}_0 = \ker(\mathrm{R})$, so

$$\mathscr{K}(\mathrm{L}) = \sum_{j>i=1}^{4} \kappa(f_i, f_j) = \kappa(f_3, f_4) = \lambda_1^2 = 1$$

by **proposition 4.1**. Since P is arbitrary, this proves that $\mathscr{M}$ is $(2,4)$-cvc(1).

Consider (3). Let P be a 2-plane with orthonormal basis $\{x, y\}$. We must find 5-planes L and L′ containing P such that $\mathscr{K}(\mathrm{L}) = 1$ and $\mathscr{K}(\mathrm{L}') = 3$. Proceeding exactly as in part (2), we obtain an orthonormal set $\{f_1, f_2, f_3, f_4\}$ whose span contains P. Since $\dim(\mathrm{E}_0) = 3$, there is a unit vector $f_5 \in \mathrm{E}_0$ orthogonal to each f_i, so $\{f_1, f_2, f_3, f_4, f_5\}$ is an orthonormal basis for L. Since $f_1, f_2, f_5 \in \mathrm{E}_0 = \ker(\mathrm{R})$,

$$\mathscr{K}(\mathrm{L}) = \sum_{j>i=1}^{5} \kappa(f_i, f_j) = \kappa(f_3, f_4) = \lambda_1^2 = 1$$

by **proposition 4.1**. Because P is arbitrary, this shows $\mathscr{M}$ is $(2,5)$-cvc(1).

Similarly, to obtain a fifth basis vector for L′, observe that there is a unit vector $f_5' \in \mathrm{E}_1$ orthogonal to each f_i, because $\dim(\mathrm{E}_1) = 3$. Set

$$f_i = f_i' \quad \text{for } i = 1, 2, 3, 4.$$

Then $\{f_1', f_2', f_3', f_4', f_5'\}$ is an orthonormal basis for L', and L' contains P. Since $f_1', f_2' \in E_0 = \ker(R)$, it follows from **proposition 4.1** that

$$\mathscr{K}(L) = \sum_{j>i=1}^{5} \kappa(f_i', f_j') = \kappa(f_3', f_4') + \kappa(f_3', f_5') + \kappa(f_4', f_5') = 3\lambda_1^2 = 3.$$

Thus $\mathscr{M}$ is $(2,5)$-cvc(3), so $\mathscr{M}$ is $(2,5)$-cvc$([1,3])$ by **theorem 2.1**.

To prove (4), choose an integer $1 \le m \le 5$ and an m-plane P. Since

$$\mathscr{K}(V) = \sum_{j>i=1}^{6} \kappa(e_i, e_j) = \kappa(e_4, e_5) + \kappa(e_4, e_6) + \kappa(e_5, e_6) = 3\lambda_1^2 = 3,$$

$\mathscr{M}$ is exactly 6-cvc(3). But V trivially contains every m-plane. □

We now present our second example. Given a model space, since we assume $\langle\cdot,\cdot\rangle$ is symmetric and bilinear, the ACT with respect to $\langle\cdot,\cdot\rangle$ is canonical with the associated matrix I_n. Hence, $\mathscr{M} = (V, \langle\cdot,\cdot\rangle, R_{\langle\cdot,\cdot\rangle})$ is csc(1). Letting $R_a = aR_{\langle\cdot,\cdot\rangle}$, this implies $\mathscr{M}_a = (V, \langle\cdot,\cdot\rangle, R_a)$ is csc(a), and hence cvc(a). We use this construction to illustrate our results about decomposable model spaces.

Example 4.5. Let $\mathscr{M} = (V, \langle\cdot,\cdot\rangle, R)$ be a model space with $\{e_1, \dots, e_6\}$ an orthonormal basis for V, $\langle\cdot,\cdot\rangle$ a positive-definite inner product on V, and $R = R_2 \oplus R_0 \oplus R_{-2}$. Define three model spaces $\mathscr{M}_i = (V_i, \langle\cdot,\cdot\rangle|_{V_i}, R_i)$ such that $\dim(V_i) = 2$ and $\mathscr{M} = \mathscr{M}_2 \oplus \mathscr{M}_0 \oplus \mathscr{M}_{-2}$.

Clearly, $\ker(R) = V_0$. Note that when we group components of $\mathscr{M}$ below, say $\mathscr{M} = (\mathscr{M}_2 \oplus \mathscr{M}_0) \oplus \mathscr{M}_{-2}$, we then view $\mathscr{M}_2 \oplus \mathscr{M}_0$ as a single model space.

Proposition 4.6. *The model space $\mathscr{M}$ in **example 4.5** has the following properties:*

1. *$\mathscr{M}$ is 3-cvc$([-2,2])$,*
2. *$\mathscr{M}$ is 4-cvc$([-2,2])$,*
3. *$\mathscr{M}$ is 5-cvc$([-2,2])$,*
4. *$\mathscr{M}$ is 6-cvc(0).*

Proof. Beginning with (1), consider the decompositon

$$\mathscr{M} = \mathscr{M}_2 \oplus \mathscr{M}_0 \oplus \mathscr{M}_{-2}.$$

Since $\mathscr{M}_2$ is csc(2), $\mathscr{M}_2$ is 2-cvc(2). Since $\mathscr{M}_0$ is 2-cvc(0) and $\mathscr{M}_{-2}$ is 2-cvc(-2), $\mathscr{M}_0 \oplus \mathscr{M}_{-2}$ is 2-cvc$([-2,0])$ (**corollary 3.7**), so $\mathscr{M}$ is 3-cvc$([-2,2])$ (**corollary 3.8**).

To prove (2), we view $\mathscr{M}$ as

$$\mathscr{M} = (\mathscr{M}_2 \oplus \mathscr{M}_0) \oplus \mathscr{M}_{-2}.$$

We claim $\mathscr{M}_2 \oplus \mathscr{M}_0$ is 2-cvc(0). To see this, take an arbitrary $v \in V_2 \oplus V_0$ and consider three cases. First, if $v \in V_2$, take $w \in V_0$. Then

$$\kappa(v, w) = R(v, w, w, v) = 0,$$

since v is orthogonal to w and $w \in \ker(R)$. Second, if $v \in V_0$, choose $w \in V_0$ orthogonal to v. This is possible since $\dim(V_0) = 2$. Then

$$\kappa(v, w) = R(v, w, w, v) = 0.$$

Third, if $v = v_2 + v_0$ for $v_2 \in V_2$ and $v_0 \in V_0$. Take $w \in V_0$ orthogonal to v. Then

$$\kappa(v, w) = R(v_2 + v_0, w, w, v_2 + v_0) = 0,$$

since $w \in \ker(R)$. Since $\mathscr{M}_{-2}$ is 2-cvc(−2), $\mathscr{M}$ is 4-cvc(−2) by **theorem 3.10**.

Now, if we view $\mathscr{M}$ as $\mathscr{M} = \mathscr{M}_2 \oplus (\mathscr{M}_0 \oplus \mathscr{M}_{-2})$, then a similar argument shows $\mathscr{M}_0 \oplus \mathscr{M}_{-2}$ is 2-cvc(0). Since $\mathscr{M}_2$ is 2-cvc(2), **theorem 3.10** implies that $\mathscr{M}$ is 4-cvc(2). Then $\mathscr{M}$ is 4-cvc([−2, 2]) by **theorem 2.1**, which proves (2).

To prove (3), consider the decomposition

$$\mathscr{M} = (\mathscr{M}_2 \oplus \mathscr{M}_{-2}) \oplus \mathscr{M}_0.$$

Because $\mathscr{M}_2$ is 2-cvc(2) and $\mathscr{M}_{-2}$ is 2-cvc(−2), $\mathscr{M}_2 \oplus \mathscr{M}_{-2}$ is 3-cvc([−2, 2]) by **corollary 3.7**. Since $\mathscr{M}_0$ is 2-cvc(0), $\mathscr{M}$ is 5-cvc(−2) and 5-cvc(2) by **theorem 3.10**. Then **theorem 2.1** implies that $\mathscr{M}$ is 5-cvc([−2, 2]).

Lastly, to prove (4), view $\mathscr{M}$ as $\mathscr{M} = \mathscr{M}_2 \oplus \mathscr{M}_{-2} \oplus \mathscr{M}_0$. Since $\mathscr{M}_2$ is 2-cvc(2), $\mathscr{M}_0$ is 2-cvc(0), and $\mathscr{M}_{-2}$ is 2-cvc(−2), $\mathscr{M}$ is 6-cvc(0) by **corollary 3.11**. □

5 Conclusion

This paper studies k-plane constant vector curvature in finite-dimensional model spaces and introduces a generalization called (m, k)-plane constant vector curvature. We prove that $\mathscr{C}_k$ and C_k^m are compact intervals. We also prove several theorems concerning decomposable model spaces and k-cvc. Most importantly, we show every decomposable model space with a positive-definite inner product is k-cvc(ϵ) for some integer $k \geq 2$ and $\epsilon \in \mathbb{R}$. Additionally, we give the first example of a model space with (m, k)-cvc and utilize our results to efficiently determine the k-cvc values of a decomposable model space. This research further characterizes model spaces by assigning new basis-independent values to its various subspaces and allows us to easily construct model spaces with prescribed curvature values.

There are several directions for future research. First, as mentioned in section 3, every 3-dimensional model space with a positive-definite inner product is cvc(ϵ) for a unique value ϵ [17]. ([4] provides a counterexample in the non-degenerate case.) By **corollary 3.6**, we now know every decomposable model space with a positive-definite inner product is k-cvc(ϵ) for some integer $k \geq 2$ and $\epsilon \in \mathbb{R}$. It is then natural to ask: is every model space k-cvc(δ) for some $k \geq 3$ and $\delta \in \mathbb{R}$? This broad question might be amenable to a careful case analysis. (See [11] for such an approach for 3-dimensional model spaces.) Naturally, we suggest thoroughly investigating the positive-definite setting first.

Second, one could generalize known notions of extremal curvature. A model space $\mathscr{M}$ has *extremal constant vector curvature*, denoted ecvc(ϵ), if ϵ is a bound (lower or upper) on the values in $\mathscr{C}_2$. Analogously, $\mathscr{M}$ has *k-plane extremal constant vector curvature*, written k-ecvc(ϵ), if ϵ is a bound (lower or upper) on the values in $\mathscr{C}_k$. What properties generalize from 2-planes to k-planes? Much is known about extremal constant vector curvature, but no one has studied k-ecvc.

Third, this paper often showed certain model spaces are at least (but not exactly) k-cvc($[\epsilon, \delta]$). Are there general methods to determine if a given model space is not k-cvc(γ) for certain γ? To begin, one could generalize the approach in [2] to find nontrivial bounds on $\mathscr{C}_k$. Another approach would be to search for converses (or partial converses) of the results in section 3. For example, if $\mathscr{M}$ decomposes into $\mathscr{M}_1$ and $\mathscr{M}_2$, do ${}_1\mathscr{C}_k$ and ${}_2\mathscr{C}_k$ completely determine the $(k+1)$-cvc values of $\mathscr{M}$?

Lastly, the referee suggested several geometric questions, including

1. For $k \geq 3$ and $m \geq 2$, are there manifolds with $\mathscr{C}_k$ or $\mathscr{C}_k^m$ nonempty other than product spaces? The referee noted that in [15], Schmidt and Wolfson provide examples of manifolds for which $\mathscr{C}_2$ is nonempty. Can their approach be generalized to find examples when $k \geq 3$ and $m \geq 2$?

2. Do complex projective space or quaternionic projective space satisfy any constant curvature conditions?

3. Given a manifold with k-plane or (m, k)-plane constant vector curvature, what does this tell us about its geometry?

These are merely a few possibilities for future inquiry.

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Kevin Tully
Wheaton College
kevin.tully@my.wheaton.edu