Rose-Hulman Undergraduate Mathematics Journal

Volume 24 | Issue 1 Article 3

# Optimal Monohedral Tilings of Hyperbolic Surfaces

Leonardo DiGiosia
*Rice University*, lsd2@rice.edu

Jahangir Habib
*Williams College*, jih1@williams.edu

Jack Hirsch
*Yale University*, jack.hirsch@yale.edu

Lea Kenigsberg
*Columbia University*, lea@math.columbia.edu

Kevin Li
*Yale University*, k.li@yale.edu

*See next page for additional authors*



## Recommended Citation

DiGiosia, Leonardo; Habib, Jahangir; Hirsch, Jack; Kenigsberg, Lea; Li, Kevin; Pittman, Dylanger; Petty, Jackson; Xue, Christopher; and Zhu, Weitao (2023) "Optimal Monohedral Tilings of Hyperbolic Surfaces," *Rose-Hulman Undergraduate Mathematics Journal*: Vol. 24: Iss. 1, Article 3.
Available at: https://scholar.rose-hulman.edu/rhumj/vol24/iss1/3

Cover Page Footnote
The authors would like to thank Frank Morgan for his guidance and mentorship over the years. We would also like to thank the Yale University, Williams College, SUMRY, and SMALL.

Authors
Leonardo DiGiosia, Jahangir Habib, Jack Hirsch, Lea Kenigsberg, Kevin Li, Dylanger Pittman, Jackson Petty, Christopher Xue, and Weitao Zhu

ℝose-Hulman Undergraduate Mathematics Journal
Volume 24, Issue 1, 2023

# Optimal Monohedral Tilings of Hyperbolic Surfaces

By *Leonardo Di Giosia, Jahangir Habib, Jack Hirsch, Lea Kenigsberg, Kevin Li, Dylanger Pittman, Jackson Petty, Christopher Xue*, and *Weitao Zhu*

**Abstract.** The hexagon is the least-perimeter tile in the Euclidean plane for any given area. On hyperbolic surfaces, this "isoperimetric" problem differs for every given area, as solutions do not scale. Cox conjectured that a regular $k$-gonal tile with 120-degree angles is isoperimetric. For area $\pi/3$, the regular heptagon has 120-degree angles and therefore tiles many hyperbolic surfaces. For other areas, we show the existence of many tiles but provide no conjectured optima. On closed hyperbolic surfaces, we verify via a reduction argument using cutting and pasting transformations and convex hulls that the regular 7-gon is the optimal $n$-gonal tile of area $\pi/3$ for $3 \le n \le 10$. However, for $n > 10$, it is difficult to rule out non-convex $n$-gons that tile irregularly.

## 1 Introduction

In 2001 Hales [11] proved that the regular hexagon is the least-perimeter, unit-area tile of the plane, and further that no such tiling of a flat torus can do better. Efforts to generalize this result to hyperbolic surfaces have to date been unsuccessful (see section 5). We focus on monohedral tilings (by a single prototile) and address the conjecture that a regular $k$-gon with 120° angles is optimal. Unfortunately, regular polygonal tiles of the hyperbolic plane $\mathbf{H}^2$ cover only a countable set of areas. We prove that *equilateral* $2n$-gonal tiles ($n \ge 2$) cover large intervals of areas; for example, there are equilateral 12-gonal tiles for all of the possible areas from 0 to $10\pi$, except possibly the interval $(4\pi, 5\pi]$ (see section 4).

The regular polygons of 120° angles tile many closed hyperbolic surfaces, where we address the following conjecture:

**Conjecture 1.1.** *Any non-equivalent tile of area* $\pi/3$ *of a closed hyperbolic surface has more perimeter than the regular heptagon* $\mathrm{R}_7$.

Our theorem 8.3 proves by direct casework that the regular 7-gon with 120° angles is optimal in comparison with all $n$-gons of area $\pi/3$ for $n \le 10$. section 9 demonstrates

*Mathematics Subject Classification.* Subject classification.
*Keywords.* Place keywords here.

the initial steps of this approach in comparison with 11-gons, which proves difficult. Following a general approach, a subsequent paper [12] proves more: for $k \geq 7$, the regular $k$-gon with 120° angles is optimal in comparison with any $n$-gon of area $(k-6)\pi/3$, for all $n$. section 10 shows the application of this approach to the classical Euclidean case.

**Methods**

To obtain equilateral $2n$-gonal tiles of $\mathbf{H}^2$, it suffices by Margulis and Mozes [14] (Proposition 4.10) to construct equilateral $2n$-gons with angles summing in various combinations to $2\pi$. Proposition 4.11 actually shows there is an equilateral $2n$-gon with any repeated sequence of angles (so that opposing angles are equal) as long as the exterior angles sum to less than $2\pi$. The careful induction argument considers the effects as the constant sidelength $\ell$ approaches 0 and $\infty$.

To prove $R_7$ is the optimal tile of an appropriate closed hyperbolic surface, Proposition 3.5 first verifies that among $n$-gons of given area, the regular one minimizes perimeter. It follows easily that $R_7$ has less perimeter than all other $n$-gonal tiles for $n \leq 7$. For $n > 7$, we show that in an $n$-gonal tiling there are on average at least $n-7$ vertices of degree 2 per tile. In particular for $n \geq 8$, an $n$-gonal tile has a concave angle. This means that the convex hull of an octogonal tile (see section 6) has at most 7 sides with generally more area and perimeter than $R_7$. Similarly, if a 9-gonal tile (see section 7) has two or more concave angles, it has more perimeter than $Q_7$. If it instead has one concave angle, a flattening argument that fills in the concave angle and truncates the corresponding convex angle which fits into it preserves area, reduces perimeter, and yields a heptagon which generally has more perimeter than $R_7$. Finally, for a 10-gonal tiling (see section 8), there may be many concave angles filled by many different convex angles, perhaps nested inside one another, complicating the flattening procedure. Proposition 8.2 reduces the analysis to six substantive cases and shows that each may be flattened without resulting in self-intersecting shapes.

Hales [11] remarks that Fejes Tóth, who proved the honeycomb conjecture for *convex* cells [8], predicted that general cells would involve considerable difficulties [7, p. 183] and said that the conjecture had resisted all attempts at proving it [9]. Removing the convexity hypothesis is the major advance of Hales's work and of ours, although we consider just polygonal cells.

## 2 Definitions

**Definition 2.1** (Tiling)**.** Let M be a closed Riemannian surface. A tiling of M is an embedded multigraph on M with no vertices of degree 0 or 1. A tiling is *polygonal* if

1. every edge is a geodesic;

2. every face is an open topological disk.

The oriented boundary of a face of a polygonal tiling is called a polygon. A tiling is *monohedral* if all faces are congruent.

**Remark 2.1.** All polygonal tilings are *connected* multigraphs. When tiling a closed surface with a tile Q, one copy $Q^*$ might be edge-to-edge with itself. An example is tiling a hyperbolic two-holed torus with a single hyperbolic octagon, which has all eight of its vertices coinciding at one point, and each edge coinciding with another edge. A second example is tiling a Euclidean torus by tiling the square fundamental region with thin vertical rectangles. The rectangle is edge-to-edge with itself at top and bottom, and the two vertices of a vertical edge coincide. This is consistent because a tiling is defined as a *multi*graph.

It is often useful to consider $m$-gons which "look like" $n$-gons because of angles of measure $\pi$, such as a rectangle which appears to be a triangle because it has three angles of measure $2\pi/3$ and one angle of measure $\pi$. To clarify this situation, we introduce a notion of equivalence to polygons.

**Definition 2.2** (Equivalent)**.** Two polygons Q and $Q'$ are *equivalent* $Q \sim Q'$ if they are equal after the removal of all vertices of measure $\pi$.

**Remark 2.2.** We can't in general define away vertices of measure $\pi$; a vertex in a tiling could, for example, have angles $\pi, \pi/2, \pi/2$, so the vertex has to be there because of the $\pi/2$ angles.

**Definition 2.3** (Convex Hull)**.** Let R be a polygonal region in a closed hyperbolic surface M. The convex hull H(R) is taken in the hyperbolic plane (with the minimal number of vertices). The convex hull of an $n$-gonal region R is a $k$-gonal region for some $k \leq n$. The convex hull has no less area and no more perimeter.

**Remark 2.3** (Existence)**.** By standard compactness arguments, there is a perimeter-minimizing tiling for prescribed areas summing to the area of the surface, except that polygons may bump up against themselves and each other, possibly with angles of measure 0 and $2\pi$, in the limit. We think that no such bumping occurs, but we have no proof.

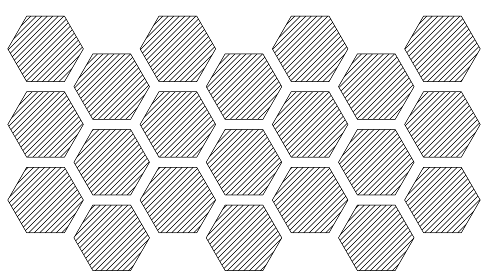

Figure 1: Hales (2001) proved that regular hexagons provide the least-perimeter equal-area tiling of the plane.

## 3 Hyperbolic Geometry

We begin with some basic results of hyperbolic geometry. Of particular interest are formulae concerning the area and perimeter of polygons in hyperbolic space. **corollary 3.1** proves that the regular heptagon is optimal (Conjecture 1.1) among polygons with seven or fewer sides.

**Proposition 3.1.** *By the Gauss-Bonnet Theorem, an $n$-gon in the hyperbolic plane with interior angles $\theta_1, \ldots, \theta_n$ has area $(n-2)\pi - \sum \theta_i$. In particular, a regular $n$-gon with interior angle $\theta$ has area*

$$A(n,\theta) = (n-2)\pi - n\theta. \tag{1}$$

**Proposition 3.2** (Law of Cosines)**.** *If $l$ is the length of the side opposing angle $\theta_3$ in a triangle with interior angles $\theta_i$, then*

$$\cos\theta_3 = \sin\theta_1 \sin\theta_2 \cosh l - \cos\theta_1 \cos\theta_2.$$

*In particular, for right triangle $\triangle ABC$ with legs $a, b$,*

$$\cosh(a) = \cos(\angle A)/\sin(\angle B).$$

**Proposition 3.3.** *A regular $n$-gon with interior angle $\theta$ has perimeter*

$$P(n,\theta) = 2n \cosh^{-1}\left(\frac{\cos(\pi/n)}{\sin(\theta/2)}\right). \tag{2}$$

*Proof.* Connect the center of $Q_n$ to each of its vertices to form $n$ isosceles triangles. Bisect the $n$ congruent triangles into $2n$ right triangles by connecting the center of the polygon to the bisector of each side of the polygon. Each triangle has interior angles $\pi/2, \pi/n$, and $\theta/2$. By Proposition 3.2, the length of the leg on the polygonal side of each of the $2n$ right triangles is $\cosh^{-1}(\cos(\pi/n)/\sin(\theta/2))$. ■

**Definition 3.4.** For $k \geq 7$, let $A_k = A(k, 2\pi/3) = (k-6)\pi/3$ and $P_k = P(k, 2\pi/3)$ denote the area and perimeter of the regular $k$-gon $R_k$ with angles $2\pi/3$.

Regular $n$-gons are isoperimetric among $n$-gons.

**Proposition 3.5** ([12], Prop. 3.7)**.** *In the hyperbolic plane, the regular $n$-gon $Q_n$ has less perimeter than any other $n$-gon $Q$ of the same area.*

**Corollary 3.1.** *Tile a closed hyperbolic surface by polygons of area $\pi/3$ with 7 or fewer sides. Then each of those tiles has perimeter greater than or equal to that of the regular heptagon of area $\pi/3$.*

*Proof.* This corollary follows immediately from Proposition 3.5. ■

Intuitively, the perimeter of regular $n$-gons of a fixed area should decrease as $n$ increases to approach the limiting bound of a circle, the most efficient way to enclose a given area. Instead of performing computations of the perimeter to prove this, we appeal to the fact that regular $n$-gons are more efficient than any other $n$-gons to show that this is, in fact, the case.

**Proposition 3.6.** *The perimeter of a regular $n$-gon for a fixed area is decreasing as a function of $n$.*

*Proof.* Let $Q_n$ and $Q_{n+1}$ be the regular polygons of a fixed area with $n$ and $n+1$ sides. Let $Q^*_{n+1}$ be an $(n+1)$-gon formed by adding a vertex of measure $\pi$ to $Q_n$. By Proposition 3.5,

$$P(Q_{n+1}) < P(Q^*_{n+1}) = P(Q_n).$$ ■

**Remark 3.2.** As expected, the perimeter of a regular $n$-gon of area A is increasing as a function of A, for $0 < A < (n-2)\pi$. By Proposition 3.1 and Proposition 3.3, the perimeter of the $n$-gon is

$$2n\cosh^{-1}\left(\frac{\cos(\pi/n)}{\sin(((n-2)\pi - A)/2n)}\right),$$

and it is increasing because $\cosh^{-1}$ and sin are increasing over $(0,\infty)$ and $(0,\pi/2)$ respectively.

**Corollary 3.3.** *The regular $k$-gon has less perimeter than any other $n$-gon of equal or greater area for $3 \le n \le k$.*

*Proof.* The corollary follows immediately from Proposition 3.5 and Proposition 3.6. ■

The following corollary is an easy step toward Conjecture 1.1.

**Proposition 3.7.** *Consider an $n$-gon $Q$ of area $A_k = (k-6)\pi/3$. If the convex hull $H(Q)$ has $k$ or fewer vertices, then $P(Q) \ge P_k = P(R_k)$, with equality only if $Q \sim H(Q) = R_k$.*

*Proof.* Recall $H(Q)$ has no less area and at least as much perimeter as $Q$. **corollary 3.3** finishes the proof. ■

**Corollary 3.4.** *If an $n$-gon $Q$ of area $A_k = (k-6)\pi/3$ has at least $n-k$ concave angles, then $P(Q) \ge P_k$ with equality if and only if there are exactly $n-k$ such angles and they are all exactly $\pi$, and hence $Q \sim R_k$.*

*Proof.* The corollary is immediate from Proposition 3.7, because if $Q$ has at least $n-k$ concave angles, then the convex hull $H(Q)$ has $k$ or fewer vertices, with equality as claimed. ■

## 4 Monohedral Tilings of the Hyperbolic Plane

We seek a least-perimeter tile of $\mathbf{H}^2$ of given area. For area $(n-6)\pi/3$ we conjecture that the regular $n$-gon with 120° angles is best. For other areas there is no natural candidate. After Goodman-Strauss [10] and Margulis and Mozes [14] we prove the existence of equilateral even-gonal tiles for wide ranges of areas.

**Conjecture 4.1.** *In $\mathbf{H}^2$, the regular $n$-gon with 120° angles has less perimeter than any non-equivalent tile of equal area.*

The following proposition provides a necessary and sufficient condition for a regular polygon to tile the hyperbolic plane.

**Proposition 4.2.** *A regular polygon of interior angle $\theta$ tiles $\mathbf{H}^2$ if and only if $\theta$ divides $2\pi$.*

*Proof.* Of course if Q tiles, $\theta$ divides $2\pi$. Conversely, as long as $\theta$ divides $2\pi$, you can form a tiling by surrounding one copy of Q with layers of additional copies. Alternatively, this proposition follows directly from Proposition 4.10. ■

**Remark 4.1.** Similarly, if each angle of a triangle divides $\pi$, then the triangle tiles the hyperbolic plane.

**Corollary 4.2.** *An isosceles triangle T with angle $\theta_1$ dividing $2\pi$ and angles $\theta_2 = \theta_3$ dividing $\pi$ tiles $\mathbf{H}^2$.*

*Proof.* For such a T, form a regular polygon Q with interior angle $2\theta_2$ by attaching $2\pi/\theta_1$ copies of T at the vertex of measure $\theta_1$. By Proposition 4.2, Q tiles $\mathbf{H}^2$. Thus T tiles $\mathbf{H}^2$. ■

**Remark 4.3.** The preceding propositions suggest several immediate but important observations.

1. The areas of regular polygonal tiles are discrete except at the integer multiples of $\pi$. This follows from the fact that for bounded area, $n$ is bounded above for a regular $n$-gonal tile of that area, and the areas of regular $n$-gons approach $(n-2)\pi$.

2. There are only finitely many regular polygonal tiles of given area.

3. If a polygon tiles the hyperbolic plane, then each angle is included in some positive integer linear combination that equals $2\pi$.

4. The converse of (3) is false. For instance, if a triangle T has angles $\theta_1, \theta_2, \theta_3$ satisfying a unique equation $\theta_1 + 3\theta_2 + 5\theta_3 = 2\pi$, T does not tile $\mathbf{H}^2$. This remark is a corollary of the following theorem of Goodman-Strauss.

**Theorem 4.3** (Goodman-Strauss [10], Thm. 6.2)**.** *Suppose a hyperbolic triangle* $\mathrm{T}$ *with vertex angles* $\alpha_i$ *satisfies exactly one equation of the form* $\sum k_i\theta_i = 2\pi$ *with nonnegative integral coefficients. Then* $\mathrm{T}$ *tiles* $\mathbf{H}^2$ *if and only if all the coefficients are at least* $2$ *and congruent to one another modulo* $2$*.*

We now relax the "exactly one" hypothesis of theorem 4.3.

**Lemma 4.4.** *For any triangle* $\mathrm{T}$ *with angles* $\alpha_1, \alpha_2, \alpha_3$ *satisfying* $\sum k_i\alpha_i = 2\pi$ *for nonnegative integers* $k_i$*, there exists a scalene triangle* $\mathrm{T}'$ *whose angles satisfy this equation and no other nonnegative linear combination that sums to* $2\pi$*.*

*Proof.* The constraint $\sum k_i\theta_i = 2\pi$ determines a plane $\Pi$ which intersects the region of possible hyperbolic triangle angles

$$\mathrm{B} = \left\{ \sum_{0<\theta_i} \theta_i < \pi \right\}$$

of the first octant of $\theta_1\theta_2\theta_3$ space. For integers $(k'_1, k'_2, k'_3) \neq (k_1, k_2, k_3)$, the collection of affine subspaces

$$\Pi \cap \left\{\sum k'_i\theta_i = 2\pi\right\}$$

is a countable set of lines and empty sets in $\Pi$. Choose $(\alpha'_1, \alpha'_2, \alpha'_3) \in \Pi \cap \mathrm{B}$ lying on no such line. The triangle $\mathrm{T}'$ with angles $\alpha'_i$ is scalene because

$$(k_1 - 1)\alpha'_1 + (k_2 + 1)\alpha'_2 + k_3\alpha'_3 \neq k_1\alpha'_1 + k_2\alpha'_2 + k_3\alpha'_3$$

implies $\alpha'_1 \neq \alpha'_2$. A similar argument shows each $\alpha'_i$ is distinct. ■

**Remark 4.5.** Denote $\mathrm{T}'/m$ as the triangle with angles $1/m$ times those of $\mathrm{T}'$. The statement in **lemma 4.4** can be strengthened so that $\mathrm{T}'$ satisfies the given equation and no other *rational* combination of its angles sums to $2\pi$. Then, by theorem 4.3, if $k_i \geq 1$, $\mathrm{T}'/m$ tiles for all even $m$. If the coefficients are at least 2 and congruent modulo 2, then $\mathrm{T}'/m$ tiles for all positive integers $m$.

**Proposition 4.4** (cf. Thm. 4.5 of [10])**.** *Consider a triangle* $\mathrm{T}$ *and a tile* $\mathrm{T}'$*. Suppose that every nonnegative integral linear combination* $\sum k_i\theta_i = 2\pi$ *satisfied by the angles of* $\mathrm{T}'$ *is also satisfied by the angles of* $\mathrm{T}$*. Then* $\mathrm{T}$ *tiles in the same way.*

*Proof.* First consider the case where $\mathrm{T}'$ is scalene. Then the triangle T tiles in exactly the same way as $\mathrm{T}'$. The angles still sum to $2\pi$ around every vertex, and the edges match because a tiling by the scalene triangle $\mathrm{T}'$ always matches an edge to itself.

Now suppose $\mathrm{T}'$ is isosceles with angles $\alpha_1, \alpha_2 = \alpha_3$. Since $\mathrm{T}'$ tiles, some linear combination $\sum k_i\alpha_i = 2\pi$ with $k_2 \neq 0$. If $k_2 = k_3$, decrease $k_2$ and increase $k_3$ by 1. Then $\alpha_i$ must also satisfy $k_1\alpha_1 + k_3\alpha_2 + k_2\alpha_3 = 2\pi$. Since T must satisfy these two equations and $k_2 \neq k_3$, T must be isosceles. Therefore T tiles in exactly the same way as $\mathrm{T}'$. Angles still sum to $2\pi$ around every vertex, and the edges match since both triangles are isosceles. ■

**Proposition 4.5.** *A triangle* T *tiles with every angle at every vertex if its angles* $\alpha_i$ *satisfy* $\sum k_i\alpha_i = 2\pi$ *for* $k_i \geq 2$ *congruent modulo* 2.

*Proof.* Suppose T satisfies $\sum k_i\theta_i = 2\pi$ with $k_i \geq 2$ congruent modulo 2. By **lemma 4.4**, there exists a scalene triangle T′ that satisfies $\sum k_i\theta_i = 2\pi$ for those $k_i$ and no other nonnegative integers. By theorem 4.3, T′ tiles with every angle at every vertex because each $k_i$ is positive, and by Proposition 4.4, T tiles in the same way. ■

We can now use Proposition 4.5 to obtain certain tilings in the hyperbolic plane.

**Proposition 4.6.** *A triangle* T *with angles* $\theta_i$ *such that* $2k\theta_1 + \theta_2 + \theta_3 = \pi$ *for some positive integer* $k$ *tiles* $\mathbf{H}^2$.

*Proof.* This proposition follows immediately from Proposition 4.5. ■

**Proposition 4.7.** *There is a non-equilateral isosceles triangular tile* T *of* $\mathbf{H}^2$ *for all possible triangular areas* A*, i.e., for* $0 < A < \pi$.

*Proof.* Let T be the hyperbolic isosceles triangle with angles

$$\theta_1 = \frac{A}{2k-1},$$
$$\theta_2 = \theta_3 = \pi/2 - k\theta_1,$$

for some integer $k > \pi/(2\pi - 2A)$ large enough to make $\theta_1 < \theta_2 = \theta_3$. By Gauss-Bonnet, T has area A. By Proposition 4.6, T tiles. ■

**Corollary 4.6.** *There is a* $n$*-gonal tile of* $\mathbf{H}^2$ *for any given area* $0 < A < \pi$.

*Proof.* By Proposition 4.7, there exists a non-equilateral triangular tile T of area A. Choose a side of distinct length, and add $n-3$ equally spaced vertices to get a degenerate $n$-gonal tile of area A. ■

**Remark 4.7.** Margulis and Mozes [14, Thm. 5] explicitly construct strictly convex $n$-gonal tiles of every possible area $0 < A < (n-2)\pi$ for $n \geq 5$. The tiling is generically nonperiodic, although invariant under a discrete group of symmetries. Their Theorem 4 constructs some equilateral tiles for all $n \geq 3$ by perturbing the regular $n$-gon and using Proposition 4.10 below.

**Proposition 4.8.** *There is a rhombic tile of* $\mathbf{H}^2$ *for all possible quadrilateral areas* A*, i.e. for* $0 < A < 2\pi$.

*Proof.* By Proposition 4.7, there exists a non-equilateral isosceles triangular tile of area A/2. Consider a tiling by this isosceles triangle. Pair tiles connected by the side of distinct length. Each pair of isosceles triangles forms the same rhombus of area A, and this rhombus tiles. ■

**Remark 4.8.** Margulis and Mozes [14, Thm. 4] construct some rhombic tiles, but only for some areas, by perturbing the regular 4-gon.

**Conjecture 4.9.** *A quadrilateral with distinct angles $\theta_i$ tiles if and only if there is some combination*

$$\Sigma k_i \theta_i = 2\pi,$$

*for integers $k_i > 1$ congruent modulo 2.*

The following proposition of Margulis and Mozes [14] gives a sufficient condition for equilateral $n$-gonal tiles, $n \geq 4$, which Margulis and Mozes use to construct some aperiodic tiles. Our Proposition 4.11 provides a general construction of equilateral $n$-gons, and then our Proposition 4.12 constructs equilateral even-gonal tiles of a wide range of areas.

**Proposition 4.10** (Margulis and Mozes [14], Prop. 2.2)**.** *Let Q be a convex equilateral polygon in $\mathbf{H}^2$ with $n \geq 4$ vertices and angles $\theta_1, \ldots, \theta_n$ at most $\pi/2$. Assume that any three angles (allowing repetition) may be complemented by more (allowing repetition) to sum to $2\pi$. Then Q tiles $\mathbf{H}^2$.*

To prove the existence of many $2n$-gonal *tiles* in Proposition 4.12, we need Proposition 4.11 about the existence of equilateral $2n$-gons (fig. 2). Note that by Gauss-Bonnet, as the sum of half the angles approaches $(n-1)\pi$, the area goes to 0.

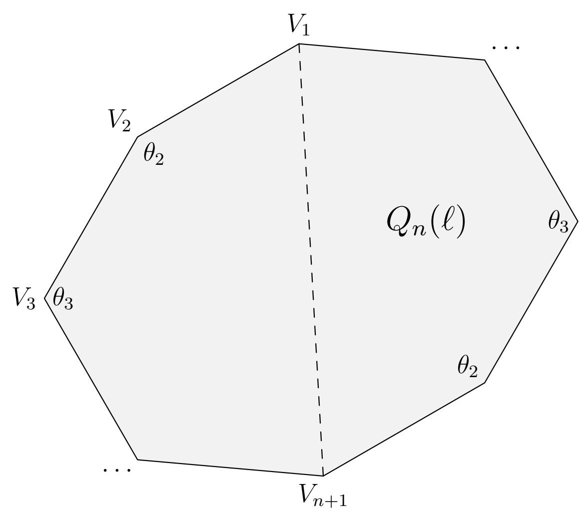


Figure 2: Construction of equilateral $2n$-gon with angles $\theta_1, \ldots, \theta_n, \theta_1, \ldots, \theta_n$.

**Proposition 4.11.** *Consider $\theta_1, \ldots, \theta_n$ such that $0 < \theta_i \leq \pi$ and $\sum \theta_i < (n-1)\pi$. Then there is a convex equilateral $2n$-gon in $\mathbf{H}^2$ with angles $\theta_1, \ldots, \theta_n, \theta_1, \ldots, \theta_n$.*

First we need two lemmas.

**Lemma 4.9.** *Consider* $\theta_2,\dots,\theta_n$ *with* $0<\theta_i<\pi$. *Let* $Q_n(\ell)$ *denote the* $(n+1)$*-gon* $V_1\dots V_{n+1}$ *of edge lengths* $\ell$, *with the interior angle of vertex* $V_i$ *having measure* $m(V_i)=\theta_i$ *for* $i\in\{2,\dots,n\}$. *Then*

(1) *for large* $\ell$, $Q_n(\ell)$ *is embedded;*

(2) $m(V_1), m(V_{n+1})\to 0$ *as* $\ell\to\infty$*;*

(3) $d(V_1,V_{n+1})\to\infty$ *as* $\ell\to\infty$.

*Proof.* For a proof by induction, first consider the case $n=2$. For any $\ell$, the triangle $Q_2(\ell)$ is trivially embedded since $\theta_2<\pi$. The rest follows by induction and the hyperbolic Law of Cosines. For the induction step, for $\ell$ large, since by induction $Q_{n-1}(\ell)$ is embedded and $m(V_n)$ in $Q_{n-1}(\ell)$ is small, therefore $Q_n(\ell)$ is embedded. Again the rest follows by the Law of Cosines. ■

**Lemma 4.10.** *Let* $L$ *be the supremum of* $\ell$ *such that* $Q_n(\ell)$ *is not embedded. If* $L>0$, *then for some* $\ell_0>L$, $m(V_1)+m(V_{n+1})>\pi$ *in* $Q_n(\ell_0)$.

*Proof.* It follows from **lemma 4.9** that for large enough $\ell>L$, $V_{n+1}$ is outside $Q_{n-1}(\ell)$ since $m(\angle V_1V_nV_{n-1})\to 0$ as $\ell\to\infty$, and symmetrically, $V_1$ is outside polygon $V_2\dots V_{n+1}$. $Q_n(\ell)$ varies continuously with $\ell$, and as $\ell$ decreases, $Q_n(\ell)$ is embedded as long as $V_{n+1}$ is outside $Q_{n-1}(\ell)$ and $V_1$ is outside polygon $V_2\dots V_{n+1}$. Since $L>0$, we may assume that for some $\ell_0>L$, $V_{n+1}$ is arbitrarily close to $Q_{n-1}(\ell)$, at which point

$$\text{Area}(V_1V_nV_{n+1})<\epsilon/2$$

and

$$m(\angle V_1V_nV_{n+1})<\epsilon/2,$$

with $0<\epsilon<m(\angle V_2V_1V_n)$ on $Q_n(\ell')$. Note that $Q_n(\ell_0)$ is embedded since $\ell_0>\ell\geq L$, and

$$m(V_1)+m(V_{n+1})=\pi-\text{Area}(V_1V_nV_{n+1})-m(\angle V_1V_nV_{n+1})+m(\angle V_2V_1V_n)>\pi.$$

■

*Proof of Proposition 4.11.* Consider the polygonal chain $V_1\cdots V_{n+1}$ where each edge is of length $\ell$ and each angle $V_i$ has measure $\theta_i$ for $2\leq i\leq n$. By **lemma 4.9**, the $(n+1)$-gon $Q$ with vertices $V_1,\dots,V_{n+1}$ is embedded for sufficiently large $\ell$. Furthermore, $m(V_1)+m(V_{n+1})$ continuously approaches 0 as large $\ell$ goes to infinity.

Let $L$ be the supremum of $\ell$ such that $Q_n(\ell)$ is not embedded. Suppose $L>0$. By **lemma 4.10**, there exists an $\ell_0>L$ such that $m(V_1)+m(V_{n+1})>\pi$. Since $m(V_1)+m(V_{n+1})\to 0$ as $\ell\to\infty$ and $\theta_1<\pi$, there must exist an $\ell\in(\ell_0,\infty)$ such that $m(V_1)+$

$m(V_{n+1}) = \theta_1$ on $Q_n(\ell)$. If $L = 0$, then $Q_n(\ell)$ is embedded for every $\ell > 0$, so $m(V_1) + m(V_{n+1})$ attains every value from 0 to the Euclidean limit $(n-1)\pi - \sum_{i=2}^{n}\theta_i$. In either case, for some $\ell > 0$, $m(V_1) + m(V_{n+1}) = \theta_1$ on $Q_n(\ell)$. Therefore adjoining two copies of the chain $V_1 \cdots V_{n+1}$ with length $\ell$ yields the desired $2n$-gon. ■

**Proposition 4.12.** *For even $n \geq 6$, there is a strictly convex equilateral $n$-gonal tile $Q$ of $\mathbf{H}^2$ of area $A$ for $(n-2)\pi/2 < A < (n-2)\pi$.*

*Proof.* Note first what turns out to be one exceptional case: the regular 6-gon with $\pi/6$ angles tiles by Proposition 4.2. It has area $3\pi$.

Let $\sigma = (n-2) - A/\pi$. By the hypothesis on $A$, $0 < \sigma < (n-2)/2$. If $\sigma < 2(n-4)/(n-2)$, there is an integer $m$ such that

$$\frac{4}{(n-2)\sigma} < m < \frac{2}{\sigma} \tag{3}$$

because the length of the interval is greater than 1. Otherwise let $m = 4/(n-2)$. Note that

$$m = \frac{4}{n-2} > \frac{4}{(n-2)\sigma}$$

and

$$m = \frac{4}{n-2} < \frac{2}{\sigma},$$

so $m$ satisfies the same inequalities as eq. (3) in this case. The sharp inequality in the lower bound holds because $\sigma \geq 2(n-4)/(n-2) \geq 1$, with equality only for the already handled case $n = 6, A = 3\pi$.

Let $\theta_1 = (\pi/m(n-4))(2 - m\sigma)$. Note that $0 < \theta_1 < 2\pi/m(n-2)$ by eq. (3) (and $\theta_1 < \pi/2$). Finally, let $\theta$ be such that

$$(n-2)(\theta_1 + \theta) = 2\pi/m. \tag{4}$$

Note that $0 < \theta < \pi/2$. By Proposition 4.11, there exists an equilateral $n$-gon $Q$ with two angles of measure $\theta_1$ and the rest of measure $\theta$. Since the angles are all less than $\pi/2$, $Q$ is strictly convex. By Proposition 3.1, eq. (4), the definition of $\theta_1$, and the definition of $\sigma$,

$$\begin{aligned} \text{Area}(Q) &= (n-2)\pi - (2\theta_1 + (n-2)\theta) \\ &= (n-2)\pi - (2\pi/m - (n-4)\theta_1) \\ &= (n-2)\pi - \pi\sigma \\ &= A. \end{aligned}$$

If $m$ is integral, by eq. (4) and Proposition 4.10, $Q$ tiles. In the case $m = 4/(n-2)$,

$$4(\theta_1 + \theta) = 2\pi,$$

and again by Proposition 4.10, $Q$ tiles. ■

**Remark 4.11.** For $n = 6$, as the area approaches $3\pi$ from below, $\theta$ approaches 0 and $\theta_1$ approaches $\pi/2$, and as the area approaches $3\pi$ from above, $\theta_1$ approaches 0 and $\theta$ approaches $\pi/4$. Fortunately, this exceptional case is covered by a regular 6-gon.

**Corollary 4.12.** *For even $n \geq 4, k \geq 2$, there is a (degenerate) equilateral $kn$-gonal tile* Q *of* $\mathbf{H}^2$ *of area* A *for any* $(n-2)\pi/2 < \mathrm{A} < (n-2)\pi$.

*Proof.* Add $k-1$ equally-spaced vertices to each edge of the $n$-gonal tile guaranteed by Proposition 4.8 and Proposition 4.12. ∎

**Corollary 4.13.** *For any $n \equiv 2 \pmod 4$ at least* 6 *and any $k \geq 1$, there is a nondegenerate equilateral $kn$-gonal tile* Q *of* $\mathbf{H}^2$ *of area* A *for any* $(n-2)\pi/2 < \mathrm{A} < (n-2)\pi$.

*Proof.* Consider the tiling by the equilateral $n$-gon of Proposition 4.12 with angles

$$\theta_1, \theta_2, \ldots, \theta_{n/2}, \theta_1, \theta_2, \ldots, \theta_{n/2}$$

and desired area. The case $k = 1$ is already done, so assume $k \geq 2$. Let Q be the $kn$-gon constructed by deforming the edges of the $n$-gon: add, in alternating fashion, an indent or an outdent to the edges of the $n$-gon, which evidently preserves area. The indents and outdents are congruent equilateral polygonal chains of $k$ edges, and can be made arbitrarily small to guarantee that Q does not intersect itself. We claim that Q tiles. Note that, as $n/2$ is odd, the edges between angles $\theta_i, \theta_{i+1}$ and $\theta_{i+n/2}, \theta_{i+1+n/2}$ are dented differently: one has an outdent, and the other an indent.

Consider the graph for the $n$-gonal tiling; we use it to generate an analogous tiling for Q. There are no odd cycles, because a cycle bounds a collection of even-gons and the unused (interior) edges are paired up. Hence the graph is bipartite, consisting of two sets C and C′.

For a vertex of C, arrange all the dents clockwise about the vertex; for a vertex of C′, arrange them counter-clockwise. Since the dentings alternate, every face of this new graph is congruent to Q. ∎

**Remark 4.14.** Consider 12-gons for example. Proposition 4.12 provides equilateral 12-gonal tiles from the largest possible area $10\pi$ down to $5\pi$ (excluding the endpoints). There are regular 12-gonal tiles for areas of the form $(10 - 24/k)\pi$ (Proposition 3.1 and Proposition 4.2), including for example $2\pi$ and $4\pi$. Adding triangular dents to the edges of equilateral 6-gonal tiles as in **corollary 4.13** yields equilateral 12-gonal tiles for areas from $4\pi$ down to $2\pi$. Evenly placing two vertices as in **corollary 4.12** on each of the edges of a rhombic tile (Proposition 4.8) yields (degenerate) tiles for areas from $2\pi$ down to 0. The only missing cases are areas in the interval $(4\pi, 5\pi]$. *Non-equilateral* tiles are provided for all possible areas by **corollary 4.6**.

We consider the possibility of *curvilinear* edges, that is, non-self-intersecting smooth curves which are not necessarily geodesics.

**Proposition 4.13.** *An isoperimetric curvilinear triangular tile of the hyperbolic plane must be convex.*

*Proof.* Assume that there is a non-convex isoperimetric curvilinear triangular tile. If every edge contains the same area as a geodesic, replacing the edges with geodesics maintains area and reduces perimeter, contradiction. In the case that one contains more and another contains less, a similar contradiction is obtained. Hence either two edges contain more and one contains less, or two contain less and one contains more. Then around a vertex of the tiling one type must match up against the other type, so that all outside edges are of the same type, which leads to a contradiction around an outside vertex. ■

**Remark 4.15.** Proposition 4.13 is easier in closed hyperbolic surfaces, because the number of edges bulging out must equal the number bulging in, while in $\mathbf{H}^2$ such a discrepancy might be pushed off to infinity. Even in closed surfaces an extension to higher curvilinear $k$-gons remains conjectural, because straightening one edge of a tile might cause it to intersect another part of the tile.

## 5 Monohedral Tilings of Closed Hyperbolic Surfaces

In 2005 Cox [2, 3] and subsequently Šešum [15] proposed generalizing Hales's hexagonal isoperimetric inequality to prove that a regular $k$-gons $\mathrm{R}_k$ ($k \geq 7$) with 120° angles provides a least-perimeter tiling of an appropriate closed hyperbolic surface for given area. Carroll et al. [1] showed that the proposed polygonal isoperimetric inequality fails for $k > 66$. Our theorem 8.3 proves the result for $\mathrm{R}_7$ among monohedral tilings by a polygon of at most 10 sides. Although theorem 8.3 applies even if the regular polygon does not tile, Proposition 5.1 notes that there are many closed hyperbolic surfaces which it does tile. It is possible for many-sided polygons to tile, but Proposition 5.3 shows that as $n$ increases, $n$-gonal tiles necessarily have many concave angles. **corollary 5.3** deduces that the regular polygon has less perimeter than any other *convex* polygonal tile.

**Remark 5.1.** By Gauss-Bonnet, the regular $k$-gon $\mathrm{R}_k$ of area $\mathrm{A}_k = (k-6)\pi/3$ ($k \geq 7$) has interior angles of $2\pi/3$ (section 3). It therefore tiles $\mathbf{H}^2$. It also tiles many closed hyperbolic surfaces (Proposition 5.1). Every such $\mathrm{R}_k$ is thought to be isoperimetric. However, for area not a multiple of $\pi/3$, there is no conjectured isoperimetric tile.

**Proposition 5.1.** *For $k \geq 7$, there exist infinitely many closed hyperbolic surfaces tiled by the regular $k$-gon of area $(k-6)\pi/3$ and angles $2\pi/3$.*

*Proof.* These surfaces are provided by work of Edmonds, Ewing, and Kulkarni [4, Main Thm.] on torsion-free subgroups of Fuchsian groups and tessellations (see also [5, 6]).

Their work yields torsion-free subgroups S of arbitrarily large finite index of the triangle group $(2,3,k)$. This triangle group is the orientation-preserving symmetry group of the hyperbolic triangle of angles $\pi/2, \pi/3$, and $\pi/k$. Each quotient of $\mathbf{H}^2$ by such a subgroup S is a closed hyperbolic surface tiled by these triangles, which can be joined in groups of $2k$ to form a tiling by the regular $k$-gon of area $(k-6)\pi/3$ and hence angles $2\pi/3$ (by Gauss-Bonnet). ■

**Example 5.2.** The Klein Quartic Curve in $\mathbf{C}\mathrm{P}^2$ is the set of complex solutions to the homogeneous equation [13]

$$u^3 v + v^3 w + w^3 u = 0.$$

The curve is a hyperbolic 3-holed torus. It is famously tiled by 24 regular heptagons.

The following results are instrumental in eliminating competing $n$-gons of large $n$.

**Lemma 5.2.** *Consider a tiling of a closed hyperbolic surface by curvilinear polygons* $\mathrm{Q}_i$ *of average area* $\mathrm{A}_k = (k-6)\pi/3$*. Then each polygon has on average at most k vertices of degree at least* 3*, with equality if and only if every vertex has degree two or three.*

*Proof.* A tile with $n$ edges and $v$ vertices of degree at least 3 contributes to the tiling 1 face, $n/2$ edges, and at most $(n-v)/2 + v/3$ vertices, with equality precisely if no vertices have degree greater than 3. Therefore its contribution to the Euler characteristic $\mathrm{F}-\mathrm{E}+\mathrm{V}$ is at most $1 - v/6$. The Gauss-Bonnet theorem says that

$$\int \mathrm{G} = 2\pi(\mathrm{F}-\mathrm{E}+\mathrm{V}).$$

Hence the average contributions per tile satisfy

$$-\mathrm{A}_k = -(k-6)\pi/3 \leq 2\pi(1-\overline{v}/6).$$

Therefore $\overline{v} \leq k$, with equality if and only if no vertices have degree more than 3. ■

**Proposition 5.3.** *Let* Q *be an* $n$*-gon of area* $\mathrm{A}_k = (k-6)\pi/3$ *with* $\ell_1$ *(interior) angles of measure* $\pi$ *and* $\ell_2$ *of measure greater than* $\pi$*. If* Q *tiles* M*, then* $\ell_1 + 2\ell_2 \geq n-k$*. Equality holds for a tiling (and therefore every tiling) if and only if every vertex is of degree two or three, and every concave angle has degree two.*

*Proof.* Take any tiling of M by Q. Each vertex of degree two in the tiling has either two angles of measure $\pi$ or exactly one angle of measure greater than $\pi$. By Lemma 5.2,

$$\ell_1 + 2\ell_2 \geq n-k,$$

with equality precisely when every vertex has degree two or three, and every concave angle has degree 2. ■

**Corollary 5.3.** *The regular $k$-gon $R_k$ has less perimeter than any non-equivalent convex polygonal tile of area $A_k = (k-6)\pi/3$.*

*Proof.* Let Q be a convex $n$-gonal tile of area $A_k$. By Proposition 5.3, Q contains at least $n-k$ angles of measure $\pi$. Hence Q is equivalent to a polygon with at most $k$ sides. Unless Q is equivalent to $R_k$, Q has strictly more perimeter by **corollary 3.3**. ■

## 6 Octagonal Tiles

The next step to proving that the regular 7-gon $R_7$ with 120° angles is isoperimetric is to show that no octagonal tile is better. **corollary 6.1** proves that the regular heptagon of area $\pi/3$ has less perimeter than any octagonal tile of the same area. Since strictly convex tiles of this area do not exist for $n \geq 8$, we consider octagonal tiles which are not strictly convex.

**Proposition 6.1.** *The regular heptagon of area $\pi/3$ has less perimeter than any non-equivalent non-strictly-convex octagon of the same area.*

*Proof.* The proposition is immediate from **corollary 3.4**. ■

**Corollary 6.1.** *Let M be a closed hyperbolic surface that is tiled by the regular heptagon $R_7$ of area $\pi/3$. Then $R_7$ has less perimeter than any non-equivalent octagonal tile of the same area.*

*Proof.* By Proposition 5.3, the octagon contains an angle of measure at least $\pi$. The corollary follows from Proposition 6.1. ■

## 7 Nonagonal Tiles

Proving that the regular heptagon has less perimeter than any 9-gonal tile (Proposition 7.2) is more difficult than the octagonal case because we must consider what happens when the tile has strictly concave angles. **corollary 7.4** first proves that "flattening" degree-two concave angles and their corresponding convex angles reduces perimeter while preserving area.

**Definition 7.1** (Flattening)**.** Consider a polygonal chain $A_1A_2 \ldots A_n$ in $\mathbf{H}^2$. To flatten adjacent vertices $A_2 \ldots A_{n-1}$, replace $A_1A_2, \ldots, A_n$ with the geodesic $A_1A_n$. In a hyperbolic surface, flattening is done in the cover $\mathbf{H}^2$.

**Lemma 7.1.** *Let M be a surface which admits a monohedral tiling by a polygon Q. Suppose that Q has a degree-2 vertex $v$ with measure $m(v)$. Then Q also has a vertex $w$ of measure $2\pi - m(v)$. If Q has no angles of measure $\pi$, then $v$ and $w$ are distinct vertices. Furthermore, the incident edges of $v$ are equal in length to the incident edges of $w$.*

*Proof.* In the tiling, vertex $v$ on Q has measure $m(v)$, and since it is degree-2 it is shared by exactly one other copy of Q in the tiling; on this other tile, $v$ has measure $2\pi - m(v)$. Since the tiling is monohedral, all tiles are congruent, and so there must exist a vertex of measure $2\pi - m(v)$ on Q as well. If Q contains no angles of measure $\pi$, then it is not possible that $m(v) = 2\pi - m(v)$, and so $v$ and $w$ must be distinct vertices. Since tilings are edge-to-edge, it must be the case that the edges incident to $v$ coincide with edges incident to $w$, and so they are equal in length. ■

**Corollary 7.2.** *Let* $B, B'$ *be distinct complementary vertices on a monohedral tile* $Q$. *Let* $A, C$ *be the vertices adjacent to* $B$ *and let* $A', C'$ *be those adjacent to* $B'$. *Then* $ABC$ *is congruent to* $A'B'C'$.

*Proof.* This follows immediately from **lemma 7.1**. ■

**Corollary 7.3.** *Let* $B, B'$ *be distinct but adjacent complementary vertices on a monohedral tile* $Q$. *Let* $A$ *be the other vertex adjacent to* $B$ *and* $C$ *be the other vertex adjacent to* $B'$. *Let* $D$ *be the intersection of the segments* $BB'$ *and* $AC$. *Then* $\triangle ABD$ *is congruent to* $\triangle DB'C$.

*Proof.* This follows immediately from **lemma 7.1**. ■

**Corollary 7.4.** *Flattening distinct complementary vertices* $B, B'$ *of a tile* $Q$ *does not change the area of* $Q$.

*Proof.* Without loss of generality, let $m(B) < \pi$. If B and B′ are adjacent, then flattening them amounts to removing the area of $\triangle ABD$ and adding the area of $\triangle DB'C$ to Q, as shown in **corollary 7.3**. Since these triangles are congruent, the area of Q does not change. If B and B′ are not adjacent, then flattening them amounts to removing the area of $\triangle ABC$ and adding the area of $\triangle A'B'C'$ to Q, as shown in **corollary 7.2**. Since these triangles are congruent, the area of Q does not change. ■

**Proposition 7.2.** *Let* $M$ *be a closed hyperbolic surface. Then the regular heptagon* $R_7$ *of area* $\pi/3$ *has less perimeter than any non-equivalent* $9$*-gonal tile* $Q$ *of* $M$ *of the same area.*

*Proof.* Suppose Q has an angle of measure $\pi$. If there is only one such angle then by Proposition 5.3, Q is equivalent to an octagon with at least one strictly concave angle. By Proposition 6.1, $P(Q) > P(R_7)$. If there are two or more angles of measure $\pi$, Q is equivalent to a polygon with seven or fewer sides, and so by **corollary 3.3**, $P(Q) > P(R_7)$.

On the other hand, suppose that Q does not have an angle of measure $\pi$. By **lemma 5.2**, there exist distinct vertices $B, B'$ on Q such that $m(B) + m(B') = 2\pi$ and the edges incident to each vertex are equal in length. Let A and C be the vertices adjacent to B and let A′ and C′ be those adjacent to B′. Since Q has no angles of measure $\pi$, let $m(B) < \pi$ without loss of generality. Then there are no vertices in the interior of $\triangle ABC$ and B is the only vertex in the interior of $\triangle A'B'C'$, since otherwise the convex hull $H(Q)$ would be a polygon with seven or fewer sides, so $P(Q) > P(H(Q)) \geq P(R_7)$, and we are done. Let $Q'$ be the heptagon formed by flattening both $B$ and $B'$, whether or not they are adjacent. By **corollary 7.4**, the area of $Q$ is equal to that of $Q'$, and the perimeter is reduced. Since $Q'$ is a heptagon, it has at least as much perimeter as $R_7$, so $P(Q) > P(Q') \geq P(R_7)$. ■

## 8 Decagonal Tiles

Proving that the regular heptagon has less area than any 10-gonal tile (Proposition 8.2) is more difficult than the 9-gonal case since there may be multiple concave angles, which means we need to also worry about the adjacency of the angles on the tile. This in turn requires case work to address every possible configuration of angles on the 10-gon.

**Definition 8.1.** Let $B$ be a vertex of a polygon, and consider the adjacent vertices. Let $\triangle B$ be the triangle with those vertices.

**Lemma 8.1** (Angle Nesting)**.** *Let $Q$ be a decagon of area $\pi/3$ with $2$ concave angles $A, A'$ with corresponding convex angles $V, V'$. Then $P(Q) > P(R_7)$ if either*

(1) *there exists a vertex in the interior of $\triangle A$; or*

(2) *there exists a vertex in the interior of $\triangle V$ or $\triangle V'$ which is neither $A$ nor $A'$.*

*Proof.* We show that in either case, the convex hull $H(Q)$ contains at least three vertices in its interior, and so has at most seven sides.

1. Suppose there exists a vertex in the interior of $\triangle A$. If this vertex is $A'$, notice that the line connecting $AA'$ intersects $Q$ at some point $M$, which is neither $A$ nor $A'$, that is inside $\triangle A$. If $M$ is a vertex of $Q$, then $M$, $A$, and $A'$ are in the interior of $H(Q)$; otherwise, one of the two vertices $B$ of the edge on which $M$ lies must be in the interior of $\triangle A$, and so $B$, $A$, and $A'$ are in the interior of $H(Q)$.

   Now, consider when some vertex $B \neq A'$ is inside $\triangle A$. Then $B$, $A$, and $A'$ all are in the interior of $H(Q)$.

2. Suppose there exists a vertex $B$ in the interior of $\triangle V$ or $\triangle V'$ which is neither $A$ nor $A'$. Then $B$, $A$, and $A'$ are all in the interior of $H(Q)$.

Thus $H(Q)$ is a polygon with at most seven sides. If $H(Q) = R_7$, then since $Q$ is not equivalent to $R_7$, $Q$ must contain a strictly concave angle, and so $P(Q) > P(H(Q)) = P(R_7)$. On the other hand, if $H(Q)$ is an irregular heptagon or is a polygon with fewer than seven sides, then we know that $P(Q) \geq P(H(Q)) > P(R_7)$. ■

Therefore we may flatten the pairs of concave and convex angles of a decagonal tile without self-intersections, as otherwise the conditions of **lemma 8.1** hold and the decagon already has more perimeter than $R_7$.

**Proposition 8.2.** *Let* $M$ *be a closed hyperbolic surface. Then the regular heptagon* $R_7$ *of area* $\pi/3$ *has less perimeter than any non-equivalent* $10$*-gonal tile* $Q$ *of* $M$ *of the same area.*

*Proof.* By **corollary 3.4**, it suffices to consider only decagons with two or fewer concave angles. These angles cannot both have measure $\pi$, since otherwise by Proposition 3.1 (Gauss-Bonnet), the remaining eight angles would have an average measure of $17\pi/24 > 2\pi/3$, implying they could not meet in threes and $Q$ would have to contain another concave angle.

Suppose $Q$ contains a single angle $A$ of measure $\pi$. Then there exists an angle $A'$ of measure greater than $\pi$ and a vertex $V'$ that fits into $A'$. Further, by **lemma 5.2**, $Q$ has an average of at least 3 degree-two vertices per tile, so $A$, $A'$, and $V'$ always meet in twos. First suppose $A$ and $V'$ are not adjacent. Let $Q'$ be the polygon formed by flattening $A'$ and $V'$, and then taking the convex hull of the resulting shape. Then $Q'$ is a heptagon of area $A \geq \pi/3$ and less perimeter. If instead $A$ and $V'$ are adjacent, since $A$ must always meet in twos and therefore at $A$ on another copy of $Q$, $A'$ is adjacent to $A$ as well. Flattening all three of $V', A, A'$ forms a heptagon $Q'$ of area $A = \pi/3$ and less perimeter, as in fig. 3. In either case, by Proposition 3.5, $P(Q) < P(Q') \leq P(H(Q')) \leq P(R_7)$.

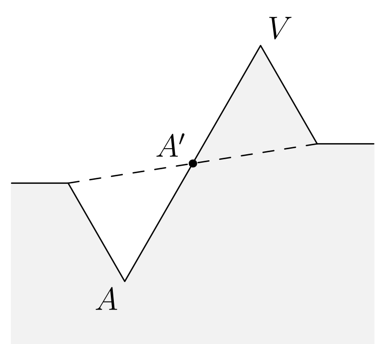


Figure 3: Flattening $AA'V$ reduces perimeter while preserving area.

Finally, we consider the case when $Q$ contains two strictly concave angles $A, A'$, and two distinct corresponding strictly convex angles $V, V'$. If $A'$ is not adjacent to $A$ or $V$, then flattening $A'$ would turn $Q$ into a 9-gon $Q'$ with concave angle $A$ and corresponding angle $V$ that fits into $A$. By Proposition 7.2, $Q'$ has more perimeter than $R_7$, so $P(Q) > P(Q') \geq P(R_7)$. By symmetry, this also covers the case that $A$ is not adjacent to $A'$ or $V'$. If, however, $A'$ *is* adjacent to $A$ or $V$ (or, by symmetry, $A$ is adjacent to $A'$ or $V'$), we enumerate the six possible orientations of the vertices and show that the claim holds for each. Cases (1) and (2) cover when A′ is adjacent to A but not V; cases (3) and (4) cover when A′ is adjacent to V but not A; and cases (5) and (6) cover when A′ is adjacent to both A and V. In the following proof, "−" is used to denote vertices that are not V, A, V′, or A′.

(1) −A′A−: Flatten A′A as in fig. 4 and flatten V. This reduces perimeter and increases area, because the triangle removed at V is congruent to the triangle added at A.

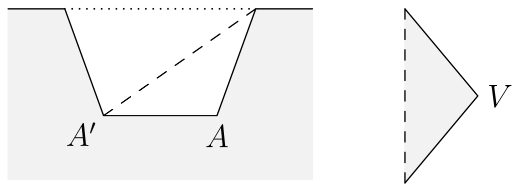


Figure 4: In the −A′A− case, flattening reduces perimeter and increases area, because the triangle removed at V is congruent to a triangular portion of the trapezoidal region added at A′A.

(2) A′AV: Flatten AV as in the dashed line of fig. 5, preserving area and reducing perimeter. Note that A′ remains concave after flattening AV. Taking the convex hull (dotted line) yields a polygon with seven or fewer sides, and so $P(Q) > P(H(Q')) \geq P(R_7)$ Note that V′ may occur anywhere without affecting the argument, including adjacent to A′.

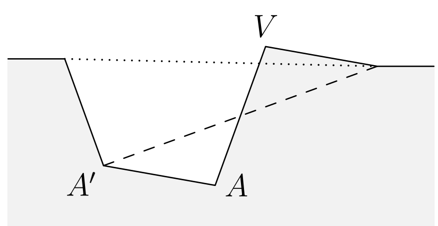


Figure 5: In the A′AV case, the dashed-line flattening is followed by taking the convex hull (dotted line).

(3) V′AA′V: By **lemma 5.2**, the average number of degree-two vertices per tile is at least 3, which means that some copy of Q must have both A and A′ degree two. When V fits into A, A′ cannot fit into itself (recall $A' > \pi$), so A′ must simultaneously fit into another V′. Thus necessarily the other angle adjacent to V is congruent to V′, and so this reduces to the following Case (4), as illustrated in fig. 6.

(4) AA′VV′ and AA′V−V′: Flatten A′ and V′ as in fig. 7, preserving area and reducing perimeter. Note that the angle at A remains concave. Take the convex hull of the

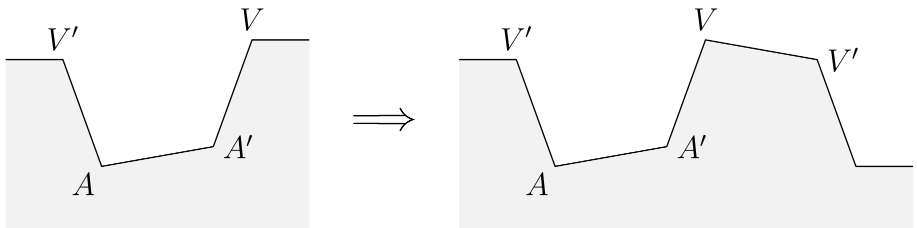


Figure 6: The V′AA′V configuration necessarily implies that there is an angle congruent to V′ adjacent to V, reducing to Case (4).

resulting polygon and the resulting shape has seven or fewer sides with at least as much area and less perimeter than Q. Therefore $P(Q) > P(Q_7)$.

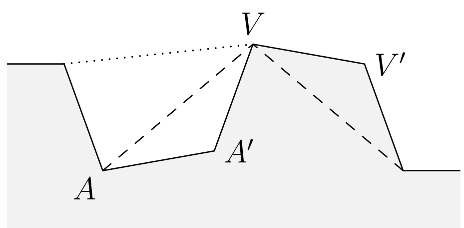


Figure 7: In the AA′VV′ case, the dashed-line flattening is followed by taking the convex hull (dotted line).

(5) VA′—AV′ Note that every concave angle must be part of a degree 2 vertex, so the polygonal curve consisting of the three edges incident to V and A′ must be congruent to the polygonal curve with edges incident to A and V′. Therefore flattening VA′ and AV′ yields a figure with seven or fewer sides with the same area and less perimeter. Therefore $P(Q) > P(R_7)$.

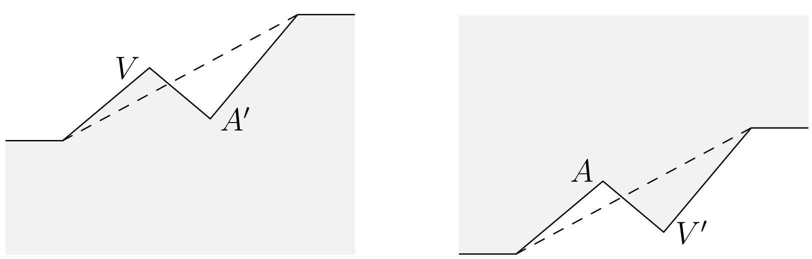


Figure 8: The polygonal chain containing V and A′ is congruent to the one containing A and V′, so we may flatten simultaneously (dashed line) with no net change in area.

(6) A′VAV′ Flatten VA as in fig. 9, which preserves area and reduces perimeter. The measure of angle A′ will decrease, but the measure of angle V′ will increase by the

same amount. This guarantees that $A'$ or $V'$ will have measure at least $\pi$. Taking the convex hull yields a heptagon or less with at least as much area and less perimeter than Q. Thus $P(Q) > P(R_7)$.

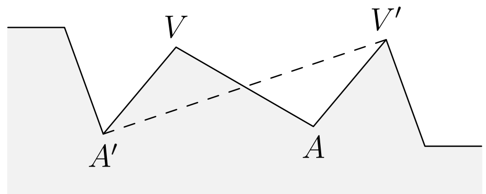


Figure 9: In the $A'VAV'$ case, the dashed-line flattening is followed by taking the convex hull.

Since these six cases enumerate all possible permutations of the relevant angles, the proof is complete. ■

The following theorem is our main result.

**Theorem 8.3.** *Let* $M$ *be a closed hyperbolic surface. Then the regular heptagon* $R_7$ *of area* $\pi/3$ *has less perimeter than any non-equivalent* $n$*-gonal tile of* $M$ *of the same area for* $n \leq 10$.

*Proof.* The theorem follows from corollaries 3.3 and 6.1 and Propositions 7.2 and 8.2. ■

## 9 11-gonal Tiles

Finally our work stalls with partial results on 11-gonal tiles. They are similar to decagonal tiles in that they have at least two concave angles. However, they can have more concave angles, and there are more possible permutations of the concave $A_i$ and the corresponding convex $V_i$. We first employed casework based on the number of $A_i$, and further subdivided based on the number of angles exactly equal to $\pi$. The most difficult cases are when there are three concave angles, one of which might never meet in twos, with multiple copies of some $V_i$, and several subcases of exactly two concave angles. We resolved all but three of around 20 cases for 11-gonal tiles before discovering a general proof [12]. We leave the reader with several examples of the 11-gon casework.

**Lemma 9.1.** *An* $11$*-gonal tile* $Q$ *with exactly two concave angles* $A_1, A_2$ *must have* $A_1, A_2$ *meet in twos for every copy of* $Q$.

*Proof.* By **lemma 5.2**, there must be on average at least $11 - 7 = 4$ degree-two vertices per tile. The result follows. ■

**Corollary 9.2.** *An* $11$*-gonal tile* $Q$ *with exactly one strictly concave angle* $A_1$ *and two angles* $m(A_2) = m(A_3) = \pi$ *must have* $A_1$, $A_2$, $A_3$ *meeting in twos for every copy of* $Q$.

*Proof.* The result follows from **lemma 5.2**. ■

**Corollary 9.3** (Angle Measures)**.** *Note that an* $11$*-gonal tile* $Q$ *cannot have exactly one strictly concave angle and exactly one angle of measure* $\pi$*, as there could not be an average of* $4$ *degree-two vertices per tile. Similarly,* $Q$ *cannot have exactly three concave angles all of which have measure* $\pi$.

**Lemma 9.4** (Four or More Concave Angles)**.** *A non-equivalent* $11$*-gonal tile* $Q$ *of area* $\pi/3$ *with four or more concave angles is worse than* $R_7$.

*Proof.* Suppose $Q$ has four or more concave angles. By **corollary 3.4**, $P(Q) \geq P(R_7)$, with equality if and only if $Q \sim R_7$. ■

**Proposition 9.1** (Three Concave Angles)**.** *A non-equivalent* $11$*-gonal tile* $Q$ *of area* $\pi/3$ *with exactly three concave angles is worse than* $R_7$.

*Proof.* Consider such a $Q$ with concave angles $A_1, A_2, A_3$ and corresponding convex angles $V_1, V_2, V_3$. We first consider the cases where some of the $A_i$ have measure $\pi$. By **corollary 9.3**, we may assume without loss of generality $m(A_1) > \pi$.

1. If $m(A_2) = m(A_3) = \pi$, by **corollary 9.2**, each $A_i$ always meets in twos. Removing $A_2$ and $A_3$ forms a 9-gonal *tile* $Q'$ since both vertices always met in twos and thus were only ever aligned with each other. Since $Q'$ is a 9-gonal tile, it reduces to Section 7.

2. If only $m(A_3) = \pi$, consider its neighbors. If none of $A_i, V_i, i \leq 2$ neighbor $A_3$, remove $A_3$, resulting in a 10-gon which, while not a tile, satisfies the concave and convex angle requirements of a 10-gon, and thus, with a little more work, can be shown to be sub-optimal. ■

The following represents some incomplete results necessary for the 11-gon proof.

**Lemma 9.5.** $A_i$ *neighbors* $A_3$ *if and only if* $V_i$ *neighbors* $A_3$.

*Proof.* Without loss of generality, assume $A_1$ neighbors $A_3$. Assume $V_1$ does not. Since on average $Q$ has four degree-two vertices per tile, $A_1$ must sometimes meet in twos. When $A_3$ meets in twos—and therefore meets $A_3$ on another copy of $Q$—both copies of $A_1$ cannot meet in twos, since the only other neighbor of $A_3$ is not $V_1$. But then $A_3$ adds two degree-two vertices to the overall sum, but subtracts both potential $A_1$ and $V_1$, a total of four. Thus the average is too small, and $V_1$ must neighbor $A_3$.

Now without loss of generality assume $V_1$ neighbors $A_3$. Similar to the above, when $A_3$ meets in twos and meets $A_3$ on another copy of $Q$, both copies of $V_1$ cannot meet in twos, which leaves two other copies of $A_1$ unfilled as well. A separate case covers when there is more than 1 copy of $V_1$; if so, that's advantageous, as we can then use the other $V_1$ instead. Again, this makes the average too small, so $A_1$ also neighbors $A_3$. Flattening $V_1, A_1$ and $A_3$ as in the diagram forms an 8-gon with at least one convex angle ($A_2$). Taking the convex hull to form $Q'$ yields a 7-gon with equal or greater area. Hence $P(R_7) < P(Q') < P(Q)$. ■

More casework would be necessary to fully resolve the case of 11-gons.

## 10 Euclidean Hexagons

A subsequent paper [12] simultaneously proves Conjecture 1.1 in comparison with polygons of any number $n$ of sides, generalizes the result from 7 to all $k \geq 7$, and remarks that the same methods yield a relatively simple proof of a weak version (Proposition 10.3) of Hales's theorem [11] on Euclidean hexagons. Here we provide the details behind the extension to Euclidean hexagons. The following propositions and lemmas 10.1–10.2 provide (generally easier) Euclidean versions of the hyperbolic cases presented in [12, Lemma 4.3, Proposition 5.3, Lemma 5.4, and Lemma 5.5].

**Lemma 10.1.** *Consider a tiling of a flat torus by curvilinear polygons. Then each polygon has on average at most* 6 *vertices of degree at least* 3*, with equality if and only if every vertex has degree two or three.*

*Proof.* A tile with $n$ edges and $v$ vertices of degree at least 3 contributes to the tiling 1 face, $n/2$ edges, and at most $(n-v)/2 + v/3$ vertices, with equality precisely if no vertices have degree greater than 3. Therefore it adds at most $1 - v/6$ to the Euler characteristic $F - E + V$. The Gauss-Bonnet theorem says that

$$\int G = 2\pi(F - E + V).$$

Hence the average contributions per tile satisfy

$$0 \leq 2\pi(1 - \overline{v}/6).$$

Therefore $\overline{v} \leq 6$, with equality if and only if no vertices have degree more than 3. ■

**Proposition 10.1.** *Let* M *be a flat torus tiled by curvilinear polygons* $Q_i$*. Let* $Q_i^*$ *be the convex hull of the vertices of degree three or higher of* $Q_i$*. Then* $\{Q_i^*\}$ *covers* M *and the average number of sides is less than or equal to* 6.

*Proof.* By the Euclidean restatement of [12, Lemma 5.2], straightening edges and flattening all degree-2 vertices yields a covering by immersed polygons, each covered by the corresponding $Q_i^*$. Hence $\{Q_i^*\}$ covers M. By **lemma 10.1** the average number of sides is less than or equal to 6. ■

**Proposition 10.2.** *The area of the regular $n$-gon with perimeter* $P$ *is given by*

$$A(n) = \frac{P^2 \cot\alpha}{4n},$$

*where* $\alpha = \pi/n$. *The function* $A(n)$ *is strictly increasing and strictly concave on* $[2,\infty)$. *We extend* $A(n)$ *continuously to be identically* $0$ *on the interval* $[0,2]$.

*Proof.* Let $R$ be the circumradius of the regular $n$-gon of perimeter $P$. Its area is

$$\frac{n}{2} R^2 \sin(2\alpha).$$

But

$$\sin\alpha = \frac{P}{2Rn},$$

and a simple substitution yields the claimed expression for $A(n)$. Its second derivative with respect to $n$ is

$$\frac{P^2}{4} \cdot \frac{n^2[2\cot\alpha\left(1+\alpha^2\csc^2\alpha\right)) - 4\alpha\csc^2\alpha]}{n^5}.$$

The numerator can be rewritten as

$$\frac{P^2 n^2}{\sin^3\alpha} \cdot \left(2\cos\alpha \cdot (\sin^2\alpha + \alpha^2) - 4\alpha\sin\alpha\right). \qquad (\star)$$

The derivative (with respect to $\alpha$) of the term in parentheses is

$$-2\alpha^2 \sin\alpha - 6\sin^3\alpha,$$

which is negative over $0 < \alpha \leq \pi/2$. Since the term in the parentheses is zero at $\alpha = 0$, it follows that $(\star)$ and hence the second derivative of $A(n)$ are negative for $0 < \alpha \leq \pi/2$.

Finally, strict monotonicity of $A(n)$ follows from strict concavity, since $A(n)$ remains positive for $n > 2$. ∎

**Lemma 10.2.** *Fix* $P > 0$. *For all real* $n \geq 6$,

$$A(n) < 2A\left(\frac{n}{2}\right).$$

*Proof.* The desired inequality simplifies to

$$\cot(\pi/n) < 4\cot(2\pi/n),$$

and for $n > 4$ further rearranges to

$$\frac{2}{3} < \cos^2(\pi/n),$$

which is true for $n \geq 6$. ∎

**Proposition 10.3.** *Consider a curvilinear polygonal tiling of a flat torus with* N *tiles of average area* A *and no more perimeter than the regular hexagon* $R_6$ *of area* A*. Then every tile is equivalent to* $R_6$.

*Proof.* Let P be the perimeter of the regular hexagon of area A. By Proposition 10.1, the collection of convex hulls $Q_i^*$ of the vertices with degree at least 3 on each tile covers M, and of course $P(Q_i^*) \leq P(Q_i) \leq P$ by assumption. Since the $Q_i^*$ cover,

$$\frac{1}{N}\sum \text{Area}(Q_i^*) \geq A. \tag{5}$$

By Proposition 10.1, the number of sides $n_i$ of $Q_i^*$ satisfy

$$\frac{1}{N}\sum n_i \leq 6.$$

The areas can be estimated in terms of $A(n)$ for P as

$$\sum \text{Area}(Q_i^*) \leq \sum A(n_i) \leq N \cdot A\left(\frac{\sum n_i}{N}\right) \leq N \cdot A(6) = N \cdot A. \tag{6}$$

The first inequality follows from the well-known fact that regular (Euclidean) $n$-gons maximize area for given perimeter. The second inequality follows from the concavity of $A(n)$ for $n \geq 2$ (Proposition 10.2) and Jensen's inequality. If any of the $n_i$ are 0 or 1, choose some $n_i \geq 6$, and use **lemma 10.2** first to replace $0 + A(n_i)$ with $2A(n_i/2)$. If you run out of large enough $n_i$, the next inequality holds already. The third inequality follows from the fact that $A(n)$ is strictly increasing (again Proposition 10.2). The final equality holds by the definition of $A(n)$ for P.

By eq. (5), equality must hold in every inequality. By the strict concavity of $A(n)$, equality in the second inequality implies that every $n_i = 6$. Equality in the first inequality implies that every $Q_i^*$ has area A. Since regular hexagons uniquely maximize area, $Q_i^*$ is the regular hexagon $R_6$ of area A. Finally

$$P(Q_i) \geq P(Q_i^*) = P,$$

and equality implies that $Q_i \sim R_6$. ∎

## Acknowledgements

This work is a product of the 2019 Summer Undergraduate Mathematics Research at Yale (SUMRY) and the 2016 Williams College NSF SMALL Geometry Groups, under the guidance of Frank Morgan of Williams College. The authors greatly thank Prof. Morgan for his help and insight over the many weeks we spent researching and writing this paper. We thank the National Science Foundation (NSF), the Williams College

Science Center, the John and Louise Finnerty Fund, and Yale for support. We also thank the Mathematical Association of America, Stony Brook University, Williams College, the Young Mathematicians Conference, and Yale for supporting our trips to speak at MathFest 2016 and the 2019 Young Mathematicians Conference, both in Columbus, Ohio.

## References

[1] Colin Carroll et al. "On generalizing the Honeycomb Theorem to compact hyperbolic manifolds and the sphere." Report of SMALL Geometry Group '06, Williams College. Aug. 2006.

[2] Chris Cox. "The Honeycomb Problem on Hyperbolic Surfaces." Aug. 2005.

[3] Chris Cox. "The Honeycomb Problem on Hyperbolic Surfaces." July 2011.

[4] Allan L. Edmonds, John H. Ewing, and Ravi S. Kulkarni. "Regular tessellations of surfaces and $(p, q, 2)$-triangle groups". In: *Ann. Math.* 116.1 (1982), pp. 113–132. ISSN: 0003486X. URL: http://www.jstor.org/stable/2007049.

[5] Allan L. Edmonds, John H. Ewing, and Ravi S. Kulkarni. "Torsion free subgroups of Fuchsian groups and tessellations of surfaces." In: *Bull. Amer. Math. Soc.* 6.3 (May 1982), pp. 456–458. ISSN: 1088-9485. DOI: 10.1090/S0273-0979-1982-15014-5. URL: http://www.ams.org/journals/bull/1982-06-03/S0273-0979-1982-15014-5/S0273-0979-1982-15014-5.pdf.

[6] Allan L. Edmonds, John H. Ewing, and Ravi S. Kulkarni. "Torsion free subgroups of Fuchsian groups and tessellations of surfaces." In: *Invent. Math.* 69.3 (Oct. 1982), pp. 331–346. ISSN: 1432-1297. DOI: 10.1007/BF01389358. URL: https://doi.org/10.1007/BF01389358.

[7] László Fejes Tóth. *Regular Figures*. New York: The Macmillan Company, 1964.

[8] László Fejes Tóth. "Über das kürzeste Kurvennetz das eine Kugeloberfläche in flächengleiche konvexe Teil zerlegt". In: *Mat. Term.-tud. Ertesitö* 62 (1943), pp. 349–354.

[9] László Fejes Tóth. "What the bees know and what they do not know". In: *Bull. Amer. Math. Soc.* 70.4 (1964), pp. 468–481. URL: https://projecteuclid.org/download/pdf_1/euclid.bams/1183526078.

[10] Chaim Goodman-Strauss. "Regular production systems and triangle tilings." In: *Theoret. Comput. Sci.* 410.16 (2009). Theory and Applications of Tiling, pp. 1534–1549. ISSN: 0304-3975. DOI: 10.1016/j.tcs.2008.12.012. URL: http://www.sciencedirect.com/science/article/pii/S0304397508008803.

[11] Thomas C. Hales. "The Honeycomb Conjecture." In: *Discrete Comput. Geom.* 25.1 (Jan. 2001), pp. 1–22. ISSN: 1432-0444. DOI: 10.1007/s004540010071.

[12] Jack Hirsch et al. "Certain regular polygonal tiles are isoperimetric". In: *Geometriae Dedicata* (Feb. 2021). DOI: 10.1007/s10711-021-00605-2. arXiv: 1910.12966 [Math.MG].

[13] Felix Klein. "Ueber die Transformation siebenter Ordnung der elliptischen Functionen." In: *Math. Ann.* 4 (Nov. 1878), pp. 428–471. URL: https://gdz.sub.uni-goettingen.de/download/pdf/PPN235181684%5C_0014/PPN235181684_0014.pdf. Trans. by Silvio Levy as "On the order-seven transformation of elliptic functions." (1998).

[14] Gregori Aleksandrovich Margulis and Shahar Mozes. "Aperiodic tilings of the hyperbolic plane by convex polygons." In: *Israel J. Math.* 107.1 (Dec. 1998), pp. 319–325. ISSN: 1565-8511. DOI: 10.1007/BF02764015.

[15] Vojislav Šešum. "The Honeycomb Problem on Hyperbolic Surfaces." Undergraduate Thesis. Williams College, May 2006.

**Leonardo Di Giosia**
Rice University
lsd2@rice.edu

**Jahangir Habib**
Williams College
jih1@williams.edu

**Jack Hirsch**
Yale University
jack.hirsch@yale.edu

**Lea Kenigsberg**
Columbia University
lea@math.columbia.edu

**Kevin Li**
Yale University
k.li@yale.edu

**Dylanger Pittman**
Emory University

dylanger.skyler.pittman@emory.edu

**Jackson Petty**
Yale University
jackson.petty@yale.edu

**Christopher Xue**
Yale University
christopher.xue@yale.edu

**Weitao Zhu**
Columbia University
wz2453@math.columbia.edu