Rose-Hulman Undergraduate Mathematics Journal

Volume 22
Issue 2

Article 4

The Optimal Double Bubble for Density r^p

Jack Hirsch
Yale University, jack.hirsch@yale.edu

Kevin Li
Yale University, k.li@yale.edu

Jackson Petty
Yale University, jackson.petty@yale.edu

Christopher Xue
Yale University, christopher.xue@yale.edu

Cover Page Footnote
All authors are supported by Yale University through the SUMRY fellowship.

Rose-Hulman Undergraduate Mathematics Journal
VOLUME 22, ISSUE 2, 2021

The Optimal Double Bubble for Density r^p

By *Jack Hirsch, Kevin Li, Jackson Petty,* and *Christopher Xue*[*]

Abstract. In 2008 [13] proved that the optimal Euclidean double bubble—the least-perimeter way to enclose and separate two given volumes in $\mathbf{R}^n$—is three spherical caps meeting along a sphere at 120 degrees. We consider $\mathbf{R}^n$ *with density* r^p, joining the surge of research on manifolds with density after their appearance in Perelman's 2006 proof of the Poincaré Conjecture. [1] proved that the best single bubble is a sphere *through* the origin. We conjecture that the best double bubble is the Euclidean solution with the singular sphere passing through the origin, for which we have verified equilibrium (first variation or "first derivative" zero). To prove the exterior of the minimizer connected, it would suffice to show that least perimeter is increasing as a function of the prescribed areas. We give the first direct proof of such monotonicity in the Euclidean plane. Such arguments were important in the 2002 *Annals* proof [7] of the double bubble in Euclidean 3-space.

1 Introduction

The isoperimetric problem is one of the oldest in mathematics. It asks for the least-perimeter way to enclose given volume. For a single volume in Euclidean space of any dimension with uniform density, the well-known solution is any sphere. In Euclidean space with density r^p, [1] found that the solution for a single volume is a sphere *through* the origin. For *two* volumes in Euclidean space, [13] showed that the standard double bubble of Figure 3, consisting of three spherical caps meeting along a sphere in threes at 120° angles, provides an isoperimetric cluster. Conjecture 3.2 states that the isoperimetric cluster for two volumes in $\mathbf{R}^n$ with density r^p for $p > 0$ is the same Euclidean standard double bubble, with the singular spherical cap (enclosed between the two outer spherical caps) passing through the origin, as in Figure 1.

Corollary 3.4 verifies equilibrium (first variation or "first derivative" zero) by scaling arguments and by direct computation. As to whether our candidate is the minimizer, it is not even known whether for the minimizer each region and the whole cluster are connected. Focusing on the 2D case, **Proposition 5.6** notes that to prove the exterior is

*All authors are supported by Yale University through the SUMRY fellowship.
Mathematics Subject Classification. MSC 53A10
Keywords. isoperimetric problems, double bubble

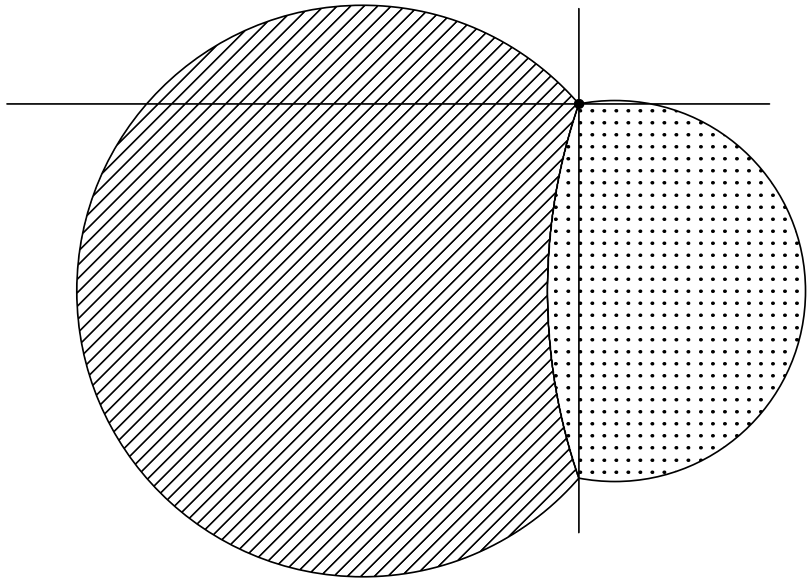


Figure 1: A standard double bubble with vertex at the origin is our conjectured double bubble in the plane with density r^p.

connected, it would suffice to show that the least perimeter $P(A_1, A_2)$ for the two areas is increasing in each variable. **Proposition 5.8** gives the first direct proof in the Euclidean plane of the "obvious" but nontrivial fact that $P(A_1, A_2)$ is an increasing function of the prescribed areas. The original proof of the Euclidean planar double bubble by [4] finessed the question by considering the alternative problem of minimizing perimeter for areas *at least* A_1 and A_2, which is obviously nondecreasing. Later [6] deduced that least perimeter is increasing in higher dimensions from his ingenious proof of concavity. Such arguments were important in the 2002 *Annals* proof [7] of the double bubble in Euclidean space.

For our direct proof that $P(A_1, A_2)$ is increasing in the Euclidean plane (**Proposition 5.8**), we consider the consequences of local minima. In particular, if $P(A_1, A_2)$ is not strictly increasing in A_1 for fixed A_2, there is a local minimum never again attained. Because it is a local minimum, in a corresponding isoperimetric cluster, the first region has zero pressure. Because this minimum is never again attained, the exterior must be connected; otherwise a bounded component could be absorbed into the first region, increasing A_1 and decreasing perimeter. It follows that the dual graph has no cycles. Since one can show that components are surrounded by many other components as in Figure 4, the cluster would have infinitely many components, a contradiction of known regularity.

History

Examination of isoperimetric regions in the plane with density r^p began in 2008 when [2] showed that the isoperimetric solution for a single area in the plane with density r^p is a convex set containing the origin. It was something of a surprise when [3] proved that the solution is a circle through the origin. In 2016 [1] extended this result to higher dimensions. In 2019 [5] studied the 1-dimensional case, showing that the best single bubble is an interval with one endpoint at the origin and that the best double bubble is a pair of adjacent intervals which meet at the origin. As for the triple bubble, the minimizer in the plane with density r^p cannot just be the Euclidean minimizer [15] with central vertex at the origin, because the outer arcs do not have constant generalized curvature.

Acknowledgements

This paper is a product of the 2019 Yale Summer Undergraduate Research in Mathematics (SUMRY) Geometry Group, advised by Frank Morgan. The authors would like to thank Morgan for his advice, guidance, and support throughout the program, as well as Yale University and the Young Mathematicians Conference for providing funding.

2 Definitions

Definition 2.1 (Density Function)**.** Given a smooth Riemannian manifold M, a *density* on M is a positive continuous function that weights each point p in M with a certain mass $f(p)$. Given a region $\Omega \subset \mathrm{M}$ with piecewise smooth boundary, the weighted volume and perimeter of Ω are given by

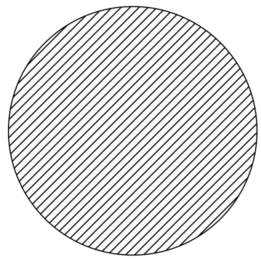
$$\mathrm{V}(\Omega) = \int_\Omega f\, d\mathrm{V}_0 \quad \text{and} \quad \mathrm{A}(\Omega) = \int_{\partial\Omega} f\, d\mathrm{P}_0,$$

where $d\mathrm{V}_0$ and $d\mathrm{P}_0$ denote Euclidean volume and perimeter. We may also refer to the perimeter of Ω as the perimeter of its boundary.

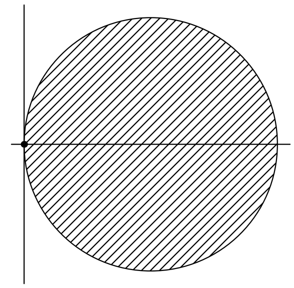
Definition 2.2 (Isoperimetric Region)**.** A region $\Omega \subset \mathrm{M}$ is *isoperimetric* if it has the smallest weighted perimeter of all regions with the same weighted volume. The boundary of an isoperimetric region is also called isoperimetric.

We can generalize the idea of an isoperimetric region by considering two or more volumes.

Definition 2.3 (Isoperimetric Cluster)**.** An isoperimetric cluster is a set of disjoint open regions Ω_i of volume V_i such that the perimeter of the union of the boundaries is minimized.

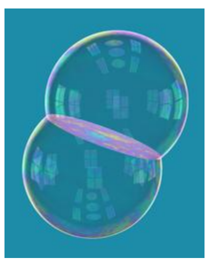


(a) $f = 1$



(b) $f = r^2$

Figure 2: Known single-volume isoperimetric solutions. In the Euclidean plane, it is any circle of the prescribed area; for density r^p, it is a circle *through the origin.*



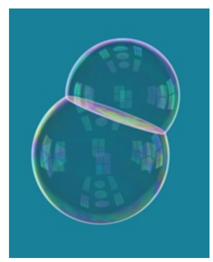


Figure 3: The standard double bubbles for volumes $V_1 = V_2$ and $V_1 > V_2$. John M. Sullivan, `http://torus.math.uiuc.edu/jms/Images/double/`, used with permission.

To provide an example of the concepts we have introduced, consider the isoperimetric solution for a single unit volume in $\mathbf{R}^n$ with constant density 1. The solution is simply a sphere.

For density r^p, the solution in the plane is a circle passing through the origin [3, Thm. 3.16], as shown in Figure 2; in higher dimensions, the solution is a sphere passing through the origin [1, Thm. 3.3].

[7] proved in 2002 that the isoperimetric solution for two volumes in Euclidean space with constant density is the standard double bubble, so called because of how soap bubbles combine in three-dimensional space, as in Figure 3. The standard double bubble illustrates the existence, boundedness, and regularity theorems:

Lemma 2.4. *Consider $\mathbf{R}^n$ with radial non-decreasing density f such that $f(r) \to \infty$ as $r \to \infty$. If a sequence of clusters Ω_i with uniformly bounded perimeter converge to Ω, there is no loss of volume at infinity in the limit.*

Proof. This fact is shown almost identically in the proof of one region by [14, Thm. 2.1]. Consider each region of the sequence of clusters separately to obtain a sequence of regions of fixed volume and uniformly bounded perimeters. The proof of [14, Thm. 2.1] implies that there is no loss in volume at infinity for each region and therefore for the whole cluster. □

Theorem 2.5 (Existence)**.** *Consider* $\mathbf{R}^n$ *endowed with a nondecreasing radial density* f *such that* $f(r) \to \infty$ *as* $r \to \infty$*. Given volumes* $V_1, \ldots, V_n$*, there exists an isoperimetric cluster that separates and encloses the given volumes.*

Proof. The proof is almost identical to the proof of one region by [14, Thm. 2.5], because their argument does not depend on the number of regions. We give the generalization of the proof here to multiple regions for the convenience of the reader. Consider a sequence of clusters enclosing and separating volumes $V_1, \ldots, V_n$ such that their perimeter approaches $I(V_1, \ldots, V_n)$ and is less than $I(V_1, \ldots, V_n) + 1$. By the Compactness Theorem [9, Sect. 9.1], we may assume this sequence converges. By **Lemma 2.4**, there is no loss of volume at infinity, so the limit gives the isoperimetric region. □

Proposition 2.6 (Boundedness)**.** *In* $\mathbf{R}^n$ *with radial non-decreasing* C^1 *density* f*, a perimeter-minimizing cluster is bounded.*

Proof. The proof follows [10, Thm. 5.9], generalizing their proof to clusters. Some analogous details of their proof are omitted for brevity. Note that while their proof is emphasized for $n \geq 3$, it nonetheless applies for $n = 2$. Assume to the contrary that E is an unbounded isoperimetric set with unbounded regions $E_1, \ldots, E_b$ and bounded regions $E_{b+1}, \ldots, E_n$. Each region has bounded perimeter.

Let $B(r)$ and $S(r)$ be the closed ball and sphere of radius r, and let $\mathscr{H}_f^k$ denote the k-dimensional Hausdorff measure in $\mathbf{R}^n$ with density f. Define

$$E_i(r) := E_i \cap B(r) \qquad E_i^r := E_i \cap S(r)$$
$$P_i(r) := \mathscr{H}_f^{n-1}(\partial E_i \setminus B(r)) \qquad V_i(r) := \mathscr{H}_f^n(E_i \setminus B(r)).$$

Since E_1 is unbounded, the proof of [10, Thm. 4.3] shows

$$P(E_1(r)) < P(E_1),$$

which, as $P(E_1(r)) = P(E_1) - P_1(r) + \mathscr{H}_f^{n-1}(E_1^r)$, is equivalent to

$$P_1(r) > \mathscr{H}_f^{n-1}(E_1^r).$$

After a careful application of the isoperimetric inequality on the $(n-1)$-sphere and some manipulations, the above inequality yields for sufficiently large r

$$P_1(r)^{\frac{n}{n-1}} \geq c V_1(r) \tag{1}$$

for some positive constant c.

Note that there must be some unbounded region with a component C which borders the exterior. Without loss of generality, let this be the unbounded region E_1. Pick an R such that B(R) completely contains the bounded components of E and such that Inequality 1 holds for any $r > R$, and such that the part of C which borders the exterior has nonzero measure $\mathscr{H}_f^{n-1}$ in B(R). There exists a constant $\bar{\varepsilon}$ such that, for any $0 < \varepsilon < \bar{\varepsilon}$, it is possible to make small variations to C along the exterior and replace the set E_1 with another set E^ε such that

$$E^\varepsilon \setminus B(R) = E_1 \setminus B(R),\ V(E^\varepsilon) = V(E_1) + \varepsilon,\ P(E^\varepsilon) \le P(E_1) + \varepsilon(H(E_1) + 1),$$

where H is the generalized mean curvature (2.7), well defined because the density f is C^1.

Now, for any r sufficiently large, set $\varepsilon = V_1(r) < \bar{\varepsilon}$ and $\widetilde{E} = E^\varepsilon \cap B(r)$. By construction,

$$V(\widetilde{E}) = V(E_1)$$

and

$$\begin{aligned} P(\widetilde{E}) &= P(E^\varepsilon) - P_1(r) + \mathscr{H}_f^{n-1}(E_1^r) \\ &\le P(E_1) + \varepsilon(H(E_1) + 1) - P_1(r) + \mathscr{H}_f^{n-1}(E_1^r). \end{aligned}$$

Since the volume of $\widetilde{E}$ equals the corresponding volume $V(E_1)$ and the variation was made only to E_1 and only along the exterior, i.e., no shared perimeter changes, it must be that

$$P(E_1) \le P(\widetilde{E}),$$

as E is isoperimetric for its given volumes.

Hence, taking ε arbitrarily small by picking arbitrarily large $r > R$ and using Inequality 1, it follows that

$$\mathscr{H}_f^{n-1}(E_1^r) \ge c^{\frac{n-1}{n}} V_1(r)^{\frac{n-1}{n}},$$

which is equivalent to

$$-\frac{\partial}{\partial r}\left(V_1(r)^{\frac{1}{n}}\right) \ge \frac{c^{\frac{n-1}{n}}}{n},$$

contradicting the fact that $V_1(r) > 0$ for all r.

□

Definition 2.7 (Generalized Curvature)**.** In $\mathbf{R}^2$ with density f, the generalized curvature κ_f of a curve with inward-pointing unit normal N is given by the formula

$$\kappa_f = \kappa_0 - \frac{\partial \log f}{\partial N},$$

where κ_0 is the (unweighted) geodesic curvature. This comes from the first variation formula, so that generalized curvature has the interpretation as minus the perimeter cost dP/dA of moving area across the curve, and constant generalized mean curvature is the equilibrium condition dP = 0 if dA = 0 (see [14, Sect. 3]).

More generally, for a smooth open region Ω in $\mathbf{R}^{n+1}$ with boundary Σ with smooth density $f = e^{\Psi}$, we can define the generalized mean curvature to be

$$\mathrm{H}_f = \mathrm{H}_0 - \langle \nabla \psi, \mathrm{N} \rangle,$$

where N is the inward normal unit vector to Σ, and H_0 is the Euclidean mean curvature (sum of principal curvatures) with respect to N.

Theorem 2.8 (Regularity)**.** *An isoperimetric cluster in* $\mathbf{R}^2$ *with smooth density consists of smooth constant-generalized-curvature curves meeting in threes at* $120°$. *The sum of the curvatures encountered along a generic closed path is* 0.

Proof. An isoperimetric cluster is a so-called ($\mathbf{M}$,Cr^{α}, δ)-minimal set, and therefore consists of curves meeting in threes at 120° (see [9, Sect. 13.10]). The rest is the equilibrium conditions (see [14, Sect. 3]). □

For regularity in higher dimensions, see [9, Sect. 13.10] for a detailed discussion.

Remark 2.9. Consider, in the plane of density r^p, a circle C of radius R centered at (x_0, y_0). At some point $(a, b) \in \mathrm{C}$, the normal vector is $\frac{1}{\mathrm{R}}(a - x_0, b - y_0)$. If $(a, b) \neq (0, 0)$, the generalized curvature is

$$\begin{aligned}
\kappa_{r^p} = \kappa_0 - \frac{\partial \log f}{\partial n} &= \frac{1}{\mathrm{R}} - \frac{p}{2} \frac{\partial \log\left(a^2 + b^2\right)}{\partial n} \\
&= \frac{1}{\mathrm{R}} - \frac{p}{\mathrm{R}} \frac{a(a - x_0) + b(b - y_0)}{a^2 + b^2} \\
&= \frac{1}{\mathrm{R}} - \frac{p}{\mathrm{R}} \left(1 - \frac{a x_0 + b y_0}{a^2 + b^2}\right).
\end{aligned}$$

From

$$(a - x_0)^2 + (b - y_0)^2 = \mathrm{R}^2,$$

we find that

$$a^2 + b^2 + x_0^2 + y_0^2 = \mathrm{R}^2 + 2(a x_0 + b y_0).$$

Therefore $(a x_0 + b y_0)/(a^2 + b^2)$ is constant if and only if $(x_0^2 + y_0^2 - \mathrm{R}^2)/(a^2 + b^2)$ is as well. This happens if and only if either $x_0^2 + y_0^2 = \mathrm{R}^2$ or $a^2 + b^2$ is constant. In other words, C has constant generalized curvature if and only if it either passes through, or is centered at, the origin.

This result extends nicely to $\mathbf{R}^n$: the spheres in $\mathbf{R}^n$ with density r^p with constant generalized curvature are precisely those passing through, or centered at, the origin.

3 Double Bubble in density r^p

We conjecture that the isoperimetric cluster for two regions in $\mathbf{R}^n$ with density r^p has the exact same shape as the Euclidean standard double bubble, but with the singular sphere passing through the origin. Notice that every cap is now part of a sphere through the origin, proved by [1] to be the best *single* bubble.

Proposition 3.1. *For any two given volumes, there is a standard double bubble with singular sphere passing through the origin, unique up to rotation.*

Proof. This proof follows directly from [9, Prop. 14.1], the existence of unique standard bubbles in unit density. □

Conjecture 3.2. Consider $\mathbf{R}^n$ with density r^p for positive p. The isoperimetric solution for two regions in space is the standard double bubble with singular sphere passing through the origin, unique up to rotation.

The proof (**Corollary 3.4**) that our candidate is in equilibrium (first variation zero) will require the following scaling lemma.

Lemma 3.3. *In the space $\mathbf{R}^n$ with density r^p, if a surface is scaled by λ about the origin, then the generalized curvature is scaled by $1/\lambda$.*

Proof. In space with density r^p, perimeter is scaled by λ^{p+n-1}, and volume is scaled by λ^{p+n}. Since generalized curvature has the interpretation of dP/dV, it is scaled by $1/\lambda$. □

Corollary 3.4. *The standard double bubble in $\mathbf{R}^n$ with density r^p for some $p > 0$ and singular sphere passing through the origin is in equilibrium.*

Proof. [13] showed that the standard double bubble in $\mathbf{R}^n$ with unit density is isoperimetric, in particular in equilibrium. Thus, the three spherical caps meet at 120 degrees, have constant Euclidean curvature, and the sum of the Euclidean curvatures encountered along a generic closed path is 0. Observe that the spherical caps also have constant generalized curvature since they all pass through the origin. By **Lemma 3.3**, their generalized curvatures are in proportion to their inverse radii, i.e., Euclidean curvature. It follows that the sum of the generalized curvatures encountered along a generic closed path must be 0. □

Proposition 3.5. *In a bounded isoperimetric cluster in $\mathbf{R}^n$ with non-decreasing radial density, the region farthest from the origin must have positive pressure.*

Proof. Since at the point farthest from the origin the cluster lies in a halfspace, the tangent cone must be a hyperplane and the cluster must be regular by [8], with normal vector pointing toward the origin. Also, at the point farthest from the origin, the unweighted Euclidean curvature is positive. Since the log of the density is radially nondecreasing, the generalized curvature also must be positive, and hence the region must have positive pressure. □

4 Geodesics in plane with density r^p

Geodesics in the plane with density r^p can be completely analyzed by mapping the plane with density to a Euclidean cone with area density.

Proposition 4.1. *The conformal map $w = z^{p+1}/(p+1)$ takes the plane with area and perimeter density r^p to a Euclidean cone with angle $(p+1)\pi$ about the origin, with area density $r^{-p} \sim |w|^{-p/(p+1)}$ (and perimeter density 1).*

Proof. Since the derivative z^p has modulus r^p, the image perimeter density is $r^{-2p}r^p = r^{-p}$ and the image area density is $r^{-p}r^p = 1$. □

Note that in the image, a geodesic is either a straight line or two straight lines meeting at the origin.

Corollary 4.2. *In the plane with density r^p, the unique geodesic from any point to the origin is the straight line. For two points with $\Delta\theta$ at least $\pi/2(p+1)$, the unique geodesic consists of two lines to the origin. For two points with $\Delta\theta$ less than $\pi/2(p+1)$, there is a unique geodesic corresponding to a straight line segment in the Euclidean cone.*

5 Properties of the Isoperimetric Function

Understanding the isoperimetric function, and thus how least perimeter depends on volume, has important consequences for the shape of minimizers. We begin with some preliminary results about scaling. As noted by [3, Sect. 3.6] for the plane and mentioned in our proof of **Lemma 3.3**, the density r^p has nice scaling properties. If a cluster Ω has perimeter P and volume V, then $\lambda\Omega$ has perimeter λ^{p+n-1}P and volume λ^{p+n}V.

Lemma 5.1. *In Euclidean space with density r^p, if a cluster Ω is scaled such that the volume is scaled by λ, then the perimeter is scaled by $\lambda^{\frac{p+n-1}{p+n}}$.*

Proof. When the cluster Ω is scaled to $\lambda^{\frac{1}{p+n}}\Omega$, the volume is scaled by λ and the perimeter is scaled by $\lambda^{\frac{p+n-1}{p+n}}$. □

Definition 5.2 (Isoperimetric Function)**.** The isoperimetric function $I(V_1, V_2)$ has the least perimeter to enclose and separate volumes V_1 and V_2. In our applications, minimizers exist. In general, I would be defined as an infimum.

The isoperimetric function I reflects the nice scaling properties of density r^p.

Proposition 5.3. *In $\mathbf{R}^n$ with density r^p, for any volumes v and w,*

$$I(\lambda v, \lambda w) = \lambda^{\frac{p+n-1}{p+n}} I(v, w).$$

Proof. By **Lemma 5.1**, $\mathrm{I}(\lambda v, \lambda w) \leq \lambda^{\frac{p+n-1}{p+n}} \mathrm{I}(v, w)$. Reapplying the lemma with $1/\lambda$ yields the opposite inequality. □

The following proposition that the isoperimetric profile is continuous is by no means clear for spaces of infinite measure. Indeed, [11] and [12] give examples of (noncompact) two- and three-dimensional Riemannian manifolds with discontinuous isoperimetric profile.

Proposition 5.4. *In* $\mathbf{R}^n$ *with density* r^p*, the isoperimetric profile* $\mathrm{I}(v, w)$ *is continuous.*

Proof. To prove upper semicontinuity, note that small changes in volume can be attained by a small change in perimeter. For lower semicontinuity, consider a sequence of volumes $(v_i, w_i) \to (v, w)$. Let Ω_i be an isoperimetric cluster of volume (v_i, w_i). By the Compactness Theorem [9, Sect. 9.1], we may assume that $\Omega_i \to \Omega$, and by **Lemma 2.4**, volume does not escape to infinity, so Ω encloses and separates volumes v and w. By the lower semicontinuity property [9, Ex. 4.22] $\mathrm{A}(\Omega_i) \leq \liminf_{i\to\infty} \mathrm{I}(v_i, w_i)$. Since $\mathrm{I}(v, w)$ is the perimeter of the isoperimetric cluster, we must have $\mathrm{I}(v, w) \leq \liminf_{i\to\infty} \mathrm{I}(v_i, w_i)$. Therefore I is lower semicontinuous, and hence continuous. □

Properties of the isoperimetric profile imply connectivity properties of an isoperimetric cluster. **Proposition 5.5** gives the trivial implication that if I subadditive, then that the cluster is connected.

Proposition 5.5. *Consider a Riemannian manifold with density in which an isoperimetric cluster exists for all volumes. If the isoperimetric profile is strictly subadditive, then any isoperimetric cluster is connected.*

Proof. Suppose the isoperimetric cluster of volumes (v, w) is not connected. Then the cluster can be separated into two disjoint clusters, one with volumes (v_1, w_1) and another with volumes (v_2, w_2) with $v_1 + v_2 = v$ and $w_1 + w_2 = w$. Then

$$\mathrm{I}(v, w) \geq \mathrm{I}(v_1, w_1) + \mathrm{I}(v_2, w_2),$$

which contradicts strict subadditivity. □

The following proposition proves in a general context that I increasing implies that the exterior is connected.

Proposition 5.6. *Consider* $\mathbf{R}^n$ *with radial density* $f(r)$ *such that* $\liminf_{r\to\infty} f(r) > 0$. *If the isoperimetric profile is non-decreasing, the exterior of an isoperimetric cluster is connected.*

Proof. Since $\liminf f(r)$ does not vanish, an unbounded hypersurface yields infinite weighted perimeter. Therefore the unbounded component of the exterior is connected. If there is a bounded component, simply absorb it into an adjacent region, which decreases the perimeter and increases volume, a contradiction of the assumption that the isoperimetric profile is non-decreasing. □

On the other hand, focusing on the plane, the following proposition shows that if the isoperimetric profile is not increasing, at least one of the regions is not connected.

Proposition 5.7. *In the plane with density r^p, if the isoperimetric profile* I *is not (strictly) increasing in each variable, then there exists an isoperimetric cluster such that the region farthest from the origin has at least two components.*

Proof. Since I is not increasing, there exist v_0 and w_0 such that say $\mathrm{I}(v_0, w_0)$ is a local minimum of $\mathrm{I}_{v_0}(w) = \mathrm{I}(v_0, w)$ and

$$\mathrm{I}(v_0, w_0) \le \mathrm{I}(v_0, w) \qquad \text{for all } w > w_0. \tag{$\star$}$$

Since w_0 is a local minimum, the second region R_2 must have 0 pressure. The image of each component of R_2 under the map of **Proposition 4.1** to the flat cone with only area density is bounded by geodesics and negative curvature curves meeting at 120°, bounding alternately R_1 and the exterior. If it does not pass through the origin, it has at least eight edges. Since regularity does not hold at the origin, where the density is 0, a geodesic could turn at a small angle there, but it still has at least four edges, two of which border R_1. To see that they are different components of R_1, note that the exterior cannot have a bounded component, because such a component could be absorbed into R_2, contradicting ($\star$). Therefore the component is bounded by at least two distinct components of R_1. □

The following proposition proves for the Euclidean plane the "obvious" but nontrivial fact that least perimeter $\mathrm{I}(v, w)$ is an increasing function of the prescribed areas. The original proof of the Euclidean double bubble by [4] finessed the question by considering the alternative problem of minimizing perimeter for areas *at least* v and w, which is obviously nondecreasing. Later [6] deduced I increasing in higher dimensions from his ingenious proof of I concave.

Proposition 5.8. *In the Euclidean plane, the isoperimetric profile* $\mathrm{I}(v, w)$ *is (strictly) increasing in each variable.*

Proof. If not, there exists a v_0 such that $\mathrm{I}_{v_0}(w) = \mathrm{I}(v_0, w)$ is not increasing. Since $\mathrm{I}_{v_0}(w) \to \infty$ as $w \to \infty$ and is continuous, there exists a w_0 such that $\mathrm{I}_{v_0}(w_0)$ is a local minimum and $\mathrm{I}_{v_0}(w_0) < \mathrm{I}_{v_0}(w)$ for all $w > w_0$. Let Ω be an isoperimetric cluster of areas v_0 and w_0, and let R_1 and R_2 denote the regions of areas v_0 and w_0 respectively. The second region

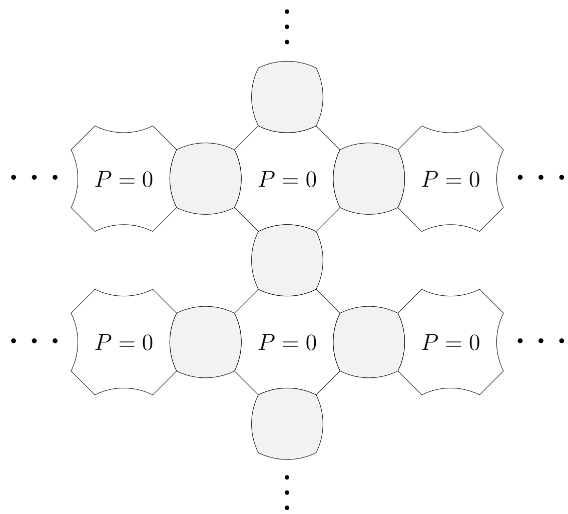


Figure 4: Our direct proof that the isoperimetric function $\mathrm{I}(\mathrm{A}_1, \mathrm{A}_2)$ in the Euclidean plane is increasing shows that otherwise there would be a region of pressure 0 and infinite branching.

R_2 must have zero pressure, since otherwise, it is possible to decrease perimeter while changing area. On the other hand, by **Proposition 3.5** the region farthest from the origin must have positive pressure and hence must be R_1.

The exterior of Ω is connected; otherwise, since the cluster is bounded (**Proposition 2.6**), there would be a bounded component of the exterior, which could be absorbed into R_2, contradicting $\mathrm{I}_{v_0}(w_0) < \mathrm{I}_{v_0}(w)$ for all $w > w_0$. Therefore the dual graph of Ω, with a labeled vertex for each component of R_1 and of R_2, does not have any cycles. Since as in Figure 4 a component of R_2 is bounded by alternating geodesic and strictly concave segments meeting at 120°, it has at least eight edges. Since a component of R_1 is convex with 120° angles, it must have two or four edges (alternately shared with R_2 and the exterior). If it has two edges as in Figure 5, the two adjacent geodesics are collinear. Hence at least two components of R_1 on the boundary of every component of R_2 both have four edges. Since the dual graph has no cycles, starting at a component of R_2, moving to an adjacent component of R_1 with four edges, moving to the other adjacent component of R_2, moving to another adjacent component of R_1, etc., would yield infinitely many components, a contradiction of boundedness and regularity (**Proposition 2.6**, **Theorem 2.8**). □

Remark 5.9. Note that this argument does not extend to $\mathbf{R}^3$, since the dual graph may contain cycles even if the exterior is connected.

Remark 5.10. For an isoperimetric double bubble in $\mathbf{R}^3$ with unit density, [6, Sect. 4.5]

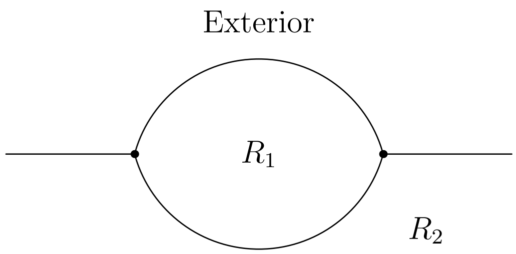


Figure 5: If a component of R_1 has just two edges, then the adjacent two edges bounding R_2 are two collinear geodesics.

proved the much stronger result that there are at most three components. In particular, the larger region has exactly one component, and the smaller region has at most two components.

6 Connectedness

Although *a priori* we do not know that an isoperimetric cluster is connected, we can say something about what multiple components would look like.

Lemma 6.1. *In* $\mathbf{R}^n$ *with radial density, components of an isoperimetric cluster must lie in disjoint open hyperspherical shells.*

Proof. If components are not in disjoint open hyperspherical shells, then one can rotate a component about the origin until it contacts another component, contradicting regularity (**Theorem 2.8**). □

In general, a conformal map takes a surface with density to a surface with different area and perimeter densities. For the right conformal map, however, one of the densities could be made to be 1. With unit area density it is easier to find transformations that preserve area.

Now we focus on the two-dimensional problem. A conformal map takes a surface with density to a surface with different area and perimeter densities. For the right conformal map, however, one of the densities can be made to be 1. With unit area density it is easier to find transformations that preserve area.

Proposition 6.2. *The conformal map*

$$w = \frac{2}{p+2} z^{\frac{p+2}{2}}$$

takes the plane with area and perimeter density r^p *to a Euclidean cone with perimeter density* $r^{p/2} \sim |w|^{p/(p+2)}$ *(and area density* 1*).*

Proof. Since the derivative $z^{p/2}$ has modulus $r^{p/2}$, the image perimeter density is $r^{-p/2}r^p = r^{p/2} \sim |w|^{p/(p+2)}$ and the image area density is $r^{-p}r^p = 1$. □

The following lemma gives a nice map that preserves area.

Lemma 6.3. *Consider an open set* U *in the plane with area density 1 such that* U *is outside some ball* $\mathrm{B}(0,\sqrt{\epsilon})$. *Let* $\varphi_\epsilon(r) := \sqrt{r^2-\epsilon}$. *The polar map*

$$\Phi_\epsilon\colon (r,\theta) \mapsto (\varphi_\epsilon(r),\theta)$$

preserves the area of U.

Proof. Note that $\Phi_\epsilon\colon \mathrm{U} \to \mathbf{R}^2$ is injective. A computation shows that the determinant of the Jacobian $\det(\mathrm{D}\Phi_\epsilon) = 1$, so the area of U is preserved. □

Remark 6.4. Given $\epsilon > 0$, the maps Φ_α for $\alpha \le \epsilon$ are actually the only radially symmetric differentiable maps that preserve area outside $\mathrm{B}(0,\sqrt{\epsilon})$. Indeed, suppose $\Phi(r,\theta) = (\varphi(r),\theta)$ preserves area outside that ball. For every open set $\mathrm{V} \subseteq \mathbb{R}^2 \setminus \mathrm{B}(0,\sqrt{\epsilon})$,

$$\int_{\mathrm{V}} r\,\mathrm{d}r\,\mathrm{d}\theta = \int_{\mathrm{V}} \varphi(r)\det(\mathrm{D}\Phi_\epsilon)\,\mathrm{d}r\,\mathrm{d}\theta = \int_{\mathrm{V}} \varphi(r)\varphi'(r)\,\mathrm{d}r\,\mathrm{d}\theta.$$

Thus, φ must satisfy

$$r = \varphi(r)\varphi'(r) = \frac{1}{2}(\varphi(r)^2)'$$

for almost all $r \ge \sqrt{\epsilon}$. One can extend this equation to all $r \ge \sqrt{\epsilon}$ by continuity and solve to conclude that φ takes the form $\varphi(r) = \sqrt{r^2-\alpha}$ for some $\alpha \le \epsilon$.

Next we consider how the map Φ_ϵ affects the perimeter.

Lemma 6.5. *Consider a smooth curve in the plane with perimeter density* r^k *with* $k > 1$, *outside some ball* $\mathrm{B}(0,\sqrt{\epsilon})$. *The map* Φ_ϵ *strictly decreases the length of the curve.*

Proof. Note that Φ_ϵ clearly decreases the length of infinitesimal tangential elements. Therefore it suffices to consider an infinitesimal radial element at r. The Euclidean length is scaled by

$$\lambda = \frac{\mathrm{d}}{\mathrm{d}r}\sqrt{r^2-\epsilon} = \frac{r}{\sqrt{r^2-\epsilon}} > 1$$

by Φ_ϵ. The density changes from r^k to $(r^2-\epsilon)^{k/2}$, scaled by λ^{-k}. So the weighted length is scaled by λ^{1-k}, which is less than 1 because by hypothesis $k > 1$. □

Now we use the map Φ_ϵ to show that the cluster is connected for certain densities.

Proposition 6.6. *In the plane with density* r^p, $p < -2$, *any isoperimetric cluster (including the interior) must be connected and unbounded.*

Proof. We work in the Euclidean cone of **Proposition 6.2** with only perimeter density and origin corresponding to infinity back in the plane. For small enough ϵ we can apply the map Φ_ϵ of **Lemma 6.3** to a component that does not contain the origin, yielding a cluster with the same area and less perimeter by **Lemma 6.5**. Thus in the cone the cluster must be connected and contain the origin, which implies that back in the plane the cluster must be connected and unbounded. □

Remark 6.7. Consider the plane with density r^p for $p > 0$. Note that under the conformal map given in **Proposition 6.2**, the perimeter density is always in the form $|w|^k$ for some $0 < k < 1$. Therefore this map does not decrease perimeter.

Consider maps in the form $(r^k - \epsilon)^{1/k}$. The determinant of the Jacobian is given by

$$\det(D\Phi_\epsilon) = r^{k-2}(r^k - \epsilon)^{2/k-1}.$$

For $0 < k < 2$ and any $R > 0$, there exists a small enough ϵ such that $[\det(D\Phi_\epsilon)](r) < 1$ for all $r > R$. For $k > 2$ and any $R > 0$, there exists a small enough ϵ such that $[\det(D\Phi_\epsilon)](r) > 1$ for all $r > 1$. Therefore there are no area-increasing maps in this form that decrease perimeter for perimeter density r^p for $0 < p < 1$.

7 Comparisons with Other Candidates

In this section we focus on the plane and compare our standard double bubble with vertex at the origin against three other candidates, offering numerical and theoretical evidence that our standard double bubble is isoperimetric. All candidates are in equilibrium and separate and enclose two regions of equal area 1. Without loss of generality, they are plotted symmetric about the y-axis and shown here for density r^2.

Figure 6 shows our conjectured champion, the standard Euclidean double bubble with one vertex at the origin. Figure 7 shows the next best candidate, a double bubble symmetric about the y-axis, composed of two constant-generalized-curvature arcs and a segment of the y-axis, meeting at 120°. Note that the arcs do not have constant *Euclidean* curvature and hence are not circular. Figure 8 shows the next best candidate, two circles meeting tangentially at the origin. Recall that a circle at the origin is the isoperimetric solution for the single bubble problem. Adding another circle, despite sharing no perimeter, does reasonably well, closely matching the perimeter of the symmetric double bubble for large p. The general 120° equilibrium condition does not apply at the origin, because the density vanishes there; indeed, Section 4 shows that shortest paths can have sharp (but not arbitrarily sharp) corners at the origin. Equilibrium still holds for variations that are smooth diffeomorphisms, because each circle is minimizing. Nevertheless, **Proposition 7.1** below shows that in fact equilibrium fails because perimeter can be reduced to first order by a Lipschitz deformation that pinches the two circles together, the very kind of deformation used in proving that curves meet at 120° angles where the

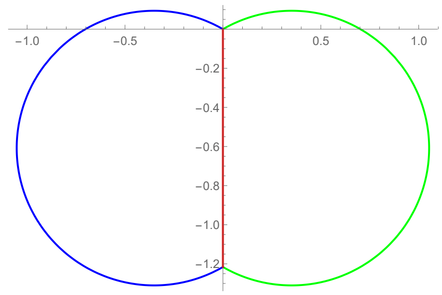


Figure 6: The standard double bubble, our conjectured champion, $p = 2$.

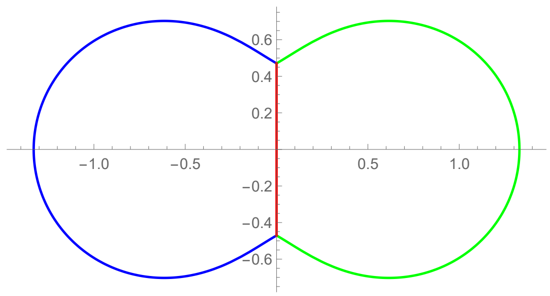


Figure 7: The symmetric double bubble, $p = 2$.

density is positive. Figure 9 shows two concentric circles, evidently worse even than the previous two-circle candidate, because each of its bubbles does worse than a circle at the origin, the isoperimetric single bubble. Nonetheless, circles centered at the origin have constant generalized curvature, so the configuration is in (unstable) equilibrium. Table 1 gives the perimeters of the computed configurations in the plane with densities r^p, for $1 \le p \le 10$.

The following proposition shows that although the candidate of Figure 8 is in equilibrium under smooth diffeomorphisms (because each circle is minimizing), it is not in equilibrium under small Lipschitz deformations about the origin that can pinch pieces

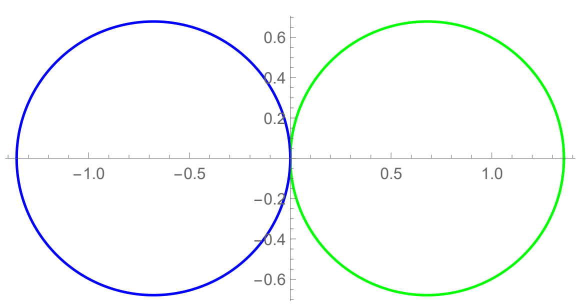


Figure 8: The two circles double bubble, $p = 2$.

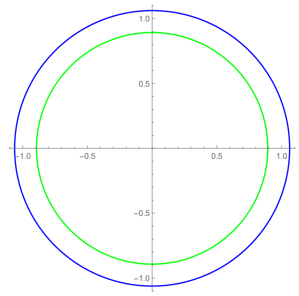


Figure 9: The concentric double bubble, $p = 2$.

p	1	2	3	4	5	6	7	8	9	10
standard	6.490	7.597	8.979	10.493	12.085	13.731	15.416	17.132	18.872	20.632
symmetric	6.720	7.837	9.176	10.650	12.212	13.835	15.502	17.203	18.932	20.683
two circles	6.868	7.858	9.177	10.650	12.212	13.835	15.502	17.203	18.932	20.683
concentric	9.931	12.009	14.346	16.820	19.379	21.998	24.661	27.359	30.085	32.834

Table 1: Perimeters of equilibrium double bubble candidates, rounded to the nearest thousandth. Computations are done numerically in Mathematica.

together.

Proposition 7.1. *A double bubble consisting of two circles tangent to each other at the origin is not in equilibrium under (small) area-preserving Lipschitz deformations.*

Proof. Given small $\epsilon > 0$, there is a $\delta > 0$ such that part of the portion of the smaller circle C_1 in the lower half of an ϵ-ball about the origin can be Lipschitz deformed to a chord to reduce area by any amount less than δ. As in Figure 10, for small $r > 0$ first Lipschitz deform the top half of the arc of the smaller circle C_1 inside the circle C about the origin of radius r onto the other circle and a portion of C. The perimeter saved is greater than the weighted length of a ray, which is $\int r^p \sim r^{p+1}$. The arc of C adds perimeter on the order of $r^2 r^p = r^{p+2}$. Hence for small r, perimeter is reduced. The area added to the first region can now be returned by deforming an initial arc of C_1 below the origin to a chord, with further reduction of perimeter, all inside the ϵ-ball. □

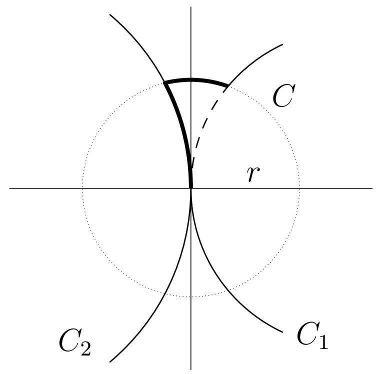


Figure 10: Deforming the dashed portion of C_1 onto the bold portion of C and C_2 reduces perimeter, belying equilibrium.

References

[1] Wyatt Boyer, Bryan Brown, Gregory R. Chambers, Alyssa Loving, and Sarah Tammen. "Isoperimetric regions in R^n with density r^p." In: *Anal. Geom. Metr. Spaces* **4** (1 2016), 236–265. ISSN: 2299-3274. arXiv: 1504.01720.

[2] Colin Carroll, Adam Jacob, Conor Quinn, and Robin Walters. "The isoperimetric problem on planes with density". In: *Bull. Austral. Math. Soc.* **78** (2008), 177–197. DOI: 10.1017/S000497270800052X.

[3] Jonathan Dahlberg, Alexander Dubbs, Edward Newkirk, and Hung Tran. "Isoperimetric regions in the plane with density r^p." In: *New York J. Math.* **16** (May 2010), 31–51. ISSN: 1076-9803.

[4] Joel Foisy, Manuel Alfaro, Jeffrey Brock, Nickelous Hodges, and Jason Zimba. "The standard double soap bubble in $\mathbf{R}^2$ uniquely minimizes perimeter." In: *Pacific J. Math.* **159**.1 (1993), 47–59. ISSN: 0030-8730. URL: https://projecteuclid.org:443/euclid.pjm/1102634378.

[5] Juiyu Huang, Xinkai Qian, Yiheng Pan, Mulei Xu, Lu Yang, and Junfei Zhou. "Isoperimetric problems on the line with density $|x|^p$." Feb. 2019. URL: https://sites.williams.edu/Morgan/files/2019/02/x18Feb19.pdf. See https://sites.williams.edu/Morgan/2019/02/22/china/.

[6] Michael Hutchings. "The structure of area-minimizing double bubbles." In: *J. Geom. Anal.* **7**.2 (June 1997), 285–304. ISSN: 1559-002X. DOI: 10.1007/BF02921724.

[7] Michael Hutchings, Frank Morgan, Manuel Ritoré, and Antonio Ros. "Proof of the double bubble conjecture." In: *Ann. of Math.* **155**.2 (2002), 459–489. ISSN: 0003486X. URL: http://www.jstor.org/stable/3062123.

[8] F. Morgan. "Regularity of isoperimetric hypersurfaces in Riemannian manifolds". In: *Transactions of the American Mathematical Society* **355** (2003), 5041–5052.

[9] Frank Morgan. *Geometric Measure Theory: A Beginner's Guide*. 5th ed. Academic Press, May 2016. ISBN: 9780128044896.

[10] Frank Morgan and Aldo Pratelli. "Existence of isoperimetric regions in $\mathbf{R}^n$ with density". In: *Annals of Global Analysis and Geometry* **43**.4 (Apr. 2013), 331–365. ISSN: 1572-9060. DOI: 10.1007/s10455-012-9348-7. URL: https://doi.org/10.1007/s10455-012-9348-7.

[11] Stefano Nardulli and Pierre Pansu. *A discontinuous isoperimetric profile for a complete Riemannian manifold*. June 2015. arXiv: 1506.04892.

[12] Panos Papasoglu and Eric Swenson. "A surface with discontinuous isoperimetric profile and expander manifolds". In: (July 2019). arXiv: 1509.02307.

[13] Ben W. Reichardt. "Proof of the Double Bubble Conjecture in $\mathbf{R}^n$". In: *Journal of Geometric Analysis* **18**.1 (Nov. 2007), 172. ISSN: 1559-002X. DOI: 10.1007/s12220-007-9002-y. URL: https://doi.org/10.1007/s12220-007-9002-y.

[14] César Rosales, Antonio Cañete, Vincent Bayle, and Frank Morgan. "On the isoperimetric problem in Euclidean space with density." In: *Calc. Var. Partial Differential Equations* **31**.1 (Jan. 2008), 27–46. ISSN: 1432-0835. DOI: 10.1007/s00526-007-0104-y.

[15] Wacharin Wichiramala. "Proof of the planar triple bubble conjecture". In: *J. Reine Angew. Math.* **2004** (567), 1–49. ISSN: 1435-5345. DOI: 10.1515/crll.2004.011.

Jack Hirsch
Yale University
jack.hirsch@yale.edu

Kevin Li
Yale University
k.li@yale.edu

Jackson Petty
Yale University
jackson.petty@yale.edu

Christopher Xue
Yale University
christopher.xue@yale.edu