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Peter A. Cermak
Christendom College, peter.cermak@christendom.edu

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A Single Criterion for Polynomial Symmetry

By *Peter Cermak*

Abstract. We develop an understanding of the relationship between the symmetry of polynomial graphs and the calculus that underlies this symmetry. We arrive at a method to determine whether a single-variable polynomial with real coefficients has a symmetric graph. We then encode this method into a closed formula that is a necessary and sufficient condition for the polynomial to have symmetry.

1 Introduction

It is a fact that is known to high school algebra students that the parabola has reflectional symmetry about the vertical axis given by its vertex. It is not as well known that all cubic polynomials have rotational symmetry about their point of inflection, though this is not surprising, nor is it difficult to prove. However, higher degree polynomials do not always have symmetry, but may when certain conditions are met. The trivial case is when all the odd degree coefficients are zero, or all the even degree coefficients are zero, but this is not the only case for a higher degree polynomial to have symmetry. *The object of this paper is to present a single criterion which is both necessary and sufficient for polynomials to be symmetric (having reflectional symmetry if the degree is even, and rotational symmetry if it is odd).* To do this, we must develop a recursive test that we can apply to any polynomial. This requires the use of basic integral calculus to increase the degree of symmetric polynomials. At its root, this paper is pointing out a phenomenon that is primarily geometric, but has its true cause in calculus.

For the remainder of this paper, we will adhere to the following definitions of the primary kinds of symmetry we are concerned with.

Definition 1.1. A function f is said to be *even* at some real number a if for all real x, we have that $f(a-x) = f(a+x)$ (i.e. reflectional symmetry about the $x = a$ axis).

Definition 1.2. Similarly, a function f is said to be *odd* at some real number a if for all real x, we have that $f(a+x) = -f(a-x)$ (i.e. rotational symmetry about the point $(a,0)$).

The other kind of symmetry we are interested in is a more general one.

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Definition 1.3. Rotational symmetry about a point $(a, f(a))$ is when for all real x, we have $f(a+x) - f(a) = -f(a-x) + f(a)$ (note that this gives us the definition of odd symmetry when $f(a) = 0$).

2 The Quartic Case, and a More General Result

We first must examine the general quartic polynomial, since we know that its derivative, being cubic, is symmetric about its point of inflection. It is not surprising, given the geometric idea that the integral is the area under a curve, that a quartic polynomial is symmetric if and only if the point of inflection of its derivative is on the x-axis. In terms of the defenitions given above, a quartic polynomial has even symmetry if and only if its derivative has odd symmetry. Before continuing, we can see a more general theorem which will be useful in our pursuit of this question, and extends beyond the realm of polynomials.

This result comes directly from the Fundamental Theorem of Calculus:

Theorem 2.1. *Let f be a differentiable function. We have*

a) f is even at a if and only if its derivative is odd at a.

b) f has rotational symmetry about $(a, f(a))$ if and only if its derivative is even at a.

Proof. The forward implication of both parts is a simple application of the chain rule to the statements given in the definition:

Definition of 'even'	Definition of 'rotational symmetry'
$f(a+x) = f(a-x)$	$g(a+x) - g(a) = -g(a-x) + g(a)$
$f'(a+x) = -f'(a-x)$	$g'(a+x) = g'(a-x)$

So f' satisfies the definition of 'odd,' while g' satisfies the definition of 'even.' Now, for the reverse implication of a), suppose that $f'(x)$ is odd at a. From the definition of 'odd' given above, we see that the following two Riemann sums are equal.

$$\sum_{i=1}^{n}\left[f'\left(a+\frac{ci}{n}\right)\frac{c}{n}\right] = \sum_{i=1}^{n}\left[-f'\left(a-\frac{ci}{n}\right)\frac{c}{n}\right]$$

Taking the limit of these expressions as n approaches infinity gives us the following (which should be obvious, given the geometric interpretation of what is being said):

$$\int_{a}^{a+c} f'(x)\,dx = -\int_{a-c}^{a} f'(x)\,dx.$$

And by the Fundamental Theorem of Calculus, $f(a+c) - f(a) = -(f(a) - f(a-c))$, and so $f(a+c) = f(a-c)$; therefore, f is even at a.

The reverse implication of b) follows similarly: If f' is even at a, then

$$\int_a^{a+c} f'(x)\,dx = \int_{a-c}^{a} f'(x)\,dx,$$

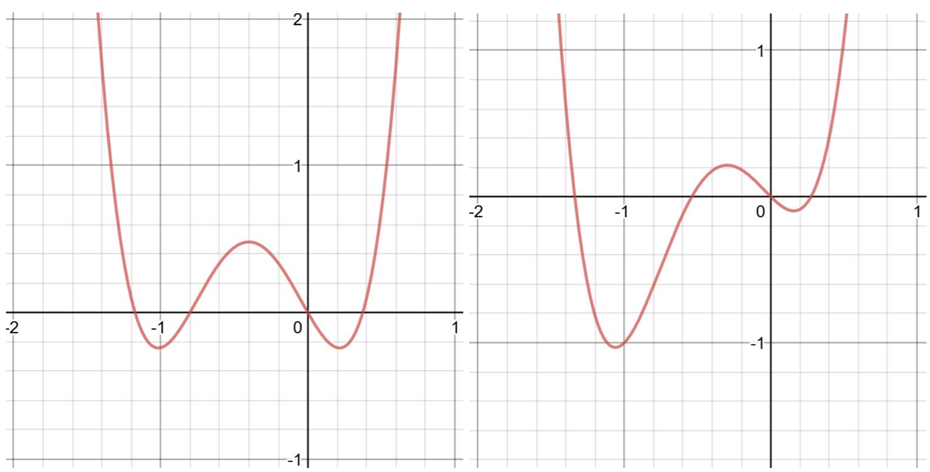
and so $f(a+c)-f(a) = -f(a-c)+f(a)$, meaning f has rotational symmetry at a. □

Now, to return to polynomials, we need to answer the following question. If a polynomial is symmetric, about what point/axis is it symmetric? Let $f(x) = a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n$ be a polynomial with real coefficients. The authors of [1] show that if $f(x)$ has symmetry, it must be at $x = -\frac{a_1}{na_0}$. The reason for this is the following. Since $f^{(n-1)}(x) = n!a_0x + (n-1)!a_1$, we may repeatedly apply **theorem 2.1** to conclude that the only possible point of symmetry for f is the x-intercept of $f^{(n-1)}(x)$, that is the point of odd symmetry for the linear expression $n!a_0x + (n-1)!a_1$. So, if f is symmetric at some point a, this means $f^{(n-1)}(a) = 0$, so we have that $a = -\frac{a_1}{na_0}$.

This observation allows one to create a test for determining whether a quartic polynomial is symmetric based on its coefficients. Namely, if f is a quartic polynomial, $f'(x) = 4a_0x^3 + 3a_1x^2 + 2a_2x + a_3$ is a cubic polynomial, and as such, it has rotational symmetry about the point $\left(-\frac{a_1}{4a_0}, f'\left(-\frac{a_1}{4a_0}\right)\right)$. In order for **theorem 2.1** to apply, and so for us to conclude f is symmetric, we need $f'\left(\frac{a_1}{4a_0}\right) = 0$, which happens when

$$a_3 = -\frac{a_1^3 - 4a_0a_1a_2}{8a_0^2}$$

The following graphs depict *a*) a quartic function that satisfies the above equation, and *b*) one that does not.



(a) $y = 5x^4 + 8x^3 + x^2 - 1.76x$ (b) $y = 5x^4 + 8x^3 + x^2 - x$

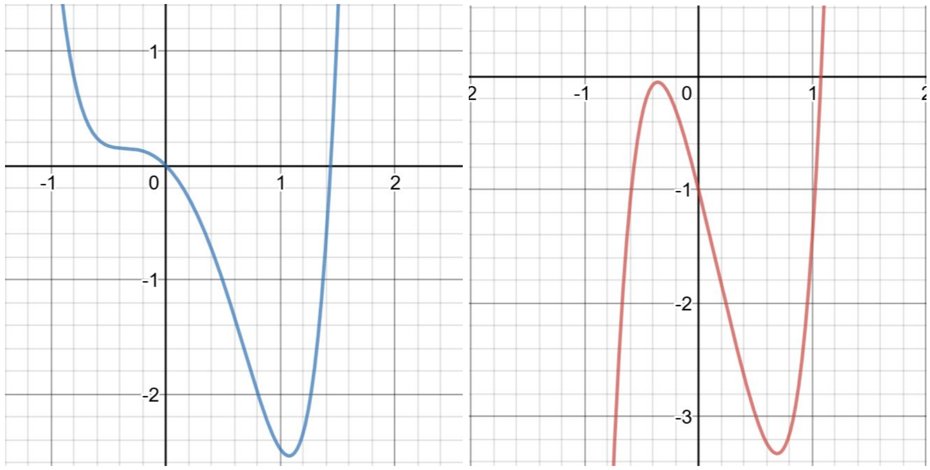
3 A General Process from Applying Theorem 1

We are nearing the solution to our general question, when do polynomial graphs have symmetry? We can use **theorem 2.1** and our work with the quartic case to work our way up to higher order polynomials as follows. A quintic polynomial $f(x) = a_0x^5 + a_1x^4 + a_2x^3 + a_3x^2 + a_4x + a_5$ has rotational symmetry if and only if its quartic derivative has even symmetry, and we already have the test for this. Translating this test over to the general quintic function, we arrive at the following criterion for a_3.

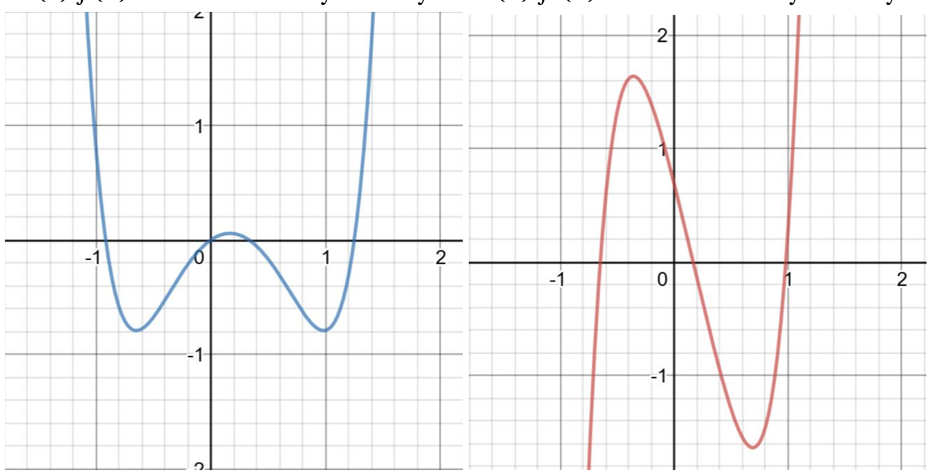
$$a_3 = -\frac{4a_1^3 - 15a_0a_1a_2}{25a_0^2}$$

Sixth-degree polynomials become more interesting. In this case, it is not enough for the coefficient a_3 to have the proper value, but we must also have the proper value for a_5. The cubic coefficient, a_3, will be dictated by the coefficients a_0, a_1, and a_2 as in the previous cases, then the linear coefficient, a_5 will result from these same coefficients in addition to the value calculated for a_3 and the given a_4. The cubic coefficient, a_3, will determine whether the quintic derivative has rotational symmetry at $\left(-\frac{a_1}{6a_0}, f'\left(-\frac{a_1}{6a_0}\right)\right)$, and the linear coefficient a_5 will determine whether that point is on the x-axis, meaning this roational symmetry is in fact odd symmetry. Both of these are necessary for the symmetry of the sixth degree polynomial, due to **theorem 2.1**.

The following examples illustrate a) a sixth-degree polynomial function $f(x)$ with the necessary value for a_3 but not for a_5, b) $f'(x)$ which has rotational symmetry but not odd symmetry, c) a sixth-degree polynomial function $g(x)$ with the necessary values for both a_3 and a_5, and d) $g'(x)$, which is identical to $f'(x)$ but with a vertical shift.



(a) $f(x)$ does not have symmetry (b) $f'(x)$ has rotational symmetry



(c) $g(x)$ has even symmetry (d) $g'(x)$ has odd symmetry

Intuitively, the process of checking for symmetry runs as follows. Given a polynomial of arbitrary degree, we differentiate repeatedly until a cubic is achieved; then we solve for the appropriate constant to place the point of inflection on the x-axis, and then we integrate twice and repeat. In fact, what is really going on is more akin to the construction of a polynomial that has symmetry, given arbitrary coefficients $a_0, a_1, a_2, a_4, a_6, \ldots$ by choosing the corresponding values for $a_3, a_5, a_7, \ldots$. Every time an odd degree is reached, we check the integration constant to make sure the rotational symmetry is in fact *odd* symmetry.

4 A Criterion for Polynomial Symmetry

We are able to encode this process of repeatedly applying **theorem 2.1** into a closed formula. In order to see whether a polynomial of degree n has symmetry at $-\frac{a_1}{na_0}$, we need to check that when differentiated to the point of being a cubic, quintic, etc., that the constant term is such that the derivative in question has a zero at the point of symmetry. Since taking the derivative $(n-i)$ times gives a polynomial of degree i, this is equivalent to checking whether $f^{(n-i)}\left(-\frac{a_1}{na_0}\right) = 0$ for each odd i such that $3 \le i < n$. And so we obtain the following criterion, which answers the main question of this paper.

Theorem 4.1. *Let $f(x) = a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n$ be a polynomial with real coefficients. Then, f has symmetry if and only if for all odd $3 \le i < n$ we have*

$$a_i = -\frac{1}{(n-i)!}\sum_{j=0}^{i-1}\frac{(n-j)!}{(i-j)!}a_j\left(-\frac{a_1}{na_0}\right)^{i-j}$$

Proof. Let $f(x) = a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n$ be a general polynomial of a real variable with degree n. Taking the derivative $(n-i)$ times gives us

$$f^{(n-i)}(x) = \sum_{j=0}^{i}\frac{(n-j)!}{(i-j)!}a_jx^{i-j}.$$

Note that this is a polynomial of degree i, so $f^{(n-3)}(x)$ has a_3 as its constant term, $f^{(n-5)}(x)$ has a_5 as its constant, etc. Now, by **theorem 2.1**, it is necessary and sufficient that for any such polynomial to have symmetry at $x = -\frac{a_1}{na_0}$, which has been shown to be the only possible place for symmetry to occur, every derivative of f of odd degree must have a zero at $x = -\frac{a_1}{na_0}$. So, setting the above expression for $f^{(n-i)}(x)$ equal to zero for $x = -\frac{a_1}{na_0}$ and for all odd i such that $3 \le i < n$ results in the following:

$$\begin{aligned} 0 &= f^{(n-i)}\left(-\frac{a_1}{na_0}\right) \\ &= \sum_{j=0}^{i}\frac{(n-j)!}{(i-j)!}a_j\left(-\frac{a_1}{na_0}\right)^{i-j} \\ &= \left[\sum_{j=0}^{i-1}\frac{(n-j)!}{(i-j)!}a_j\left(-\frac{a_1}{na_0}\right)^{i-j}\right] + (n-i)!a_i \end{aligned}$$

Solving for a_i gives us the following formula.

$$a_i = -\frac{1}{(n-i)!}\sum_{j=0}^{i-1}\frac{(n-j)!}{(i-j)!}a_j\left(-\frac{a_1}{na_0}\right)^{i-j}$$

This is the criterion given in the statement of **theorem 4.1**. □

5 Question for Further Study: Symmetric Approximations

Now, having an idea of what one would need to change to make a given asymmetric polynomial symmetric, we can ask: How do we find the symmetric polynomial that best approximates the given asymmetric one? Our conjecture is that the polynomial produced by the above method is in fact the best symmetric approximation for any polynomial with those given values of $a_0, a_1, a_2, a_4, a_6, \ldots$. One simple way to measure the error in the approximation would be to use the absolute value of the polynomial given by the difference of the two functions (which will always diverge unless the functions are vertical translations of each other). To minimize the rate of this divergence, the above method makes this error function of three degrees lesser than the original functions (i.e. in the case of quartic polynomials, this error function would be linear, etc.), and because of Theorem 1, it is the only method that can so reduce the order. Any other way of constructing a symmetric approximation cannot retain the given values for a_0, a_1, and a_2, and so this error function will have a higher order. Clearly there is more to be said about such questions that cannot be said here. Nevertheless, we hope that the criterion which this study has produced can assist in answering related questions such as this.

References

[1] Geoff Goehle and Mitsuo Kobayashi, Polynomial Graphs and Symmetry, *Coll. Math. J.*, **44:1**, (2013) 37-42, DOI: 10.4169/college.math.j.44.1.037

Peter Cermak
Christendom College
peter.cermak@christendomcollege.edu