Rose-Hulman Undergraduate Mathematics Journal

Volume 21
Issue 1

Article 10

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Recommended Citation

Riddlesden, Connor (2020) "Generalised Fibonacci Sequences under Modular Arithmetic," *Rose-Hulman Undergraduate Mathematics Journal*: Vol. 21 : Iss. 1 , Article 10.
Available at: https://scholar.rose-hulman.edu/rhumj/vol21/iss1/10

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Cover Page Footnote

I would like to thank Ha Tran and Amy Feaver for the opportunity to conduct my own research and for their guidance throughout the writing of this paper. I would also like to thank Concordia University of Edmonton for the opportunity to study there on my placement year, Finally, I would like to thank the Mathematics department at Coventry University for their continued support.

Rose-Hulman Undergraduate Mathematics Journal

VOLUME 21, ISSUE 1, 2020

Generalised Fibonacci Sequences under Modular Arithmetic

By *Connor Riddlesden*

Abstract. In this paper, we find patterns and count the number of distinct generalised Fibonacci sequences under modular arithmetic. We will start with the repetition of the normal Fibonacci sequence modulo an integer $m \geq 2$ and make connections to its dependency on the prime factorisation of m. We will then extend the complexity of the problem into generalised Fibonacci sequences with different starting values. Finally we will present some interesting observations that are still open problems.

1 Introduction

Pingala first presents Mount Meru (more commonly known as Pascal's triangle) in his works on patterns in Sanskrit poetry circa 450BC [2]. The Fibonacci sequence was first demonstrated in Pingala's Mount Meru in which the shallow diagonals of the Mount Meru sum to the Fibonacci sequence (**figure** 1). This same sequence appeared many

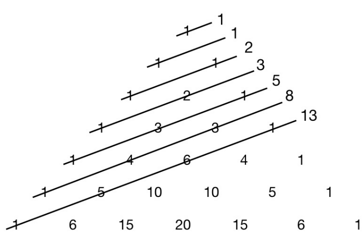

Figure 1: The Fibonacci Sequence from Mount Meru

times throughout history and was made popular by Leonardo de Pisa (more commonly known as Fibonacci) in 1202 in his book Liber Abaci to model the population growth of rabbits. He invoked this mathematical sequence to describe the number of pairs of rabbits if a pair was introduced out of thin air and then were allowed to reproduce

Mathematics Subject Classification. 11B39

Keywords. cycles, Fibonacci sequences, modular arithmetic, recurrence relations

unbounded month after month [1]. As a result, Fibonacci's name became attached to this infinite sequence. The sequence begins with

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots,$$

where each new term is obtained by adding together the previous two terms. Thus, the Fibonacci sequence is the sequence $\{F_n\}_{n=0}^{\infty}$ with $F_0 = 0, F_1 = 1$, and all other terms are given by the recurrence relation $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$.

A generalised Fibonacci sequence $\{G_k\}_{k=0}^{\infty}$ is defined by:

$$G_n = \begin{cases} a & \text{if } n = 0; \\ b & \text{if } n = 1; \\ G_{n-1} + G_{n-2} & \text{if } n \geq 2; \end{cases}$$

with $a, b \in \mathbb{Z}$. Therefore the Fibonacci sequence is the specific case of generalised Fibonacci sequences in which $a = 0$ and $b = 1$.

1.1 Fibonacci Sequences Modulo An Integer

In this subsection, we state some basic definitions and theorems on generalised Fibonacci sequences modulo an integer, and set up the notation used throughout the paper.

Clearly generalised Fibonacci sequences will grow unbounded. However, in this paper, we consider what happens to a generalised Fibonacci sequence when we take each term and reduce modulo an integer m. We may guess a priori that at least in some cases the sequence may end up repeating ad infinitum. We will actually discover that each of these sequences, modulo any m will fall into a repeating pattern. Before we discuss this in detail, we will define some useful terminology. Also, unless otherwise stated, we will use m to denote an integer greater than 1 that will serve as the modulus for our sequence.

Definition 1.1. Let $m \in \mathbb{Z}^+$ be any modulus and $\{G_i\}_{i=0}^{\infty}$ be any generalized Fibonacci sequence. We call the sequence in which these terms have been reduced modulo m, i.e. $\{G_i \pmod{m}\}_{i=0}^{\infty}$, a *modular Fibonacci sequence*.

Definition 1.2. The *period* is the length $t \in \mathbb{N}$ of the repeating pattern in a sequence. In other words, if $\{G_n\}_{n=0}^{\infty}$ is a generalised Fibonacci sequence with period t, then t is the smallest integer such that

$$G_n \equiv G_{n+t} \pmod{m}, \text{ for all } n \geq 0.$$

In the above definition, we do claim that the repeating pattern starts at G_0. This is of course not true for all eventually periodic sequences, but it is true in this case as we will see in **lemma 1.3**:

Lemma 1.3 (Renault [3])**.** *The first pair to repeat in a modular Fibonacci sequence will be the pair of integers that begin the sequence.*

Definition 1.4. In a periodic sequence $\{G_i \pmod{m}\}_{i=0}^{\infty}$ of period t, a *cycle* is a finite sequence of the form $(G_n, G_{n+1}, \dots, G_{n+t-1}) \pmod{m}$ for some $n \geq 0$.

Two cycles which derive from the same periodic sequence $\{G_i \pmod{m}\}_{i=0}^{\infty}$ are said to be rotations of one another. Two cycles are distinct if they are not rotations of one another.

For example, the Fibonacci sequence modulo 2 is

$$0, 1, 1, 2, 3, 5, 8, \dots \equiv 0, 1, 1, 0, 1, 1, 0 \dots \pmod{2}.$$

Hence the period of the Fibonacci sequence modulo 2 is $t = 3$ and a cycle is $0, 1, 1$. Therefore the cycles 0,1,1 and 1,1,0 and 1,0,1 are all derived from the same periodic sequence, so are rotations of one another.

Definition 1.5. Given a cycle with period t of the form $(G_n, G_{n+1}, \dots, G_{n+t-1}) \pmod{m}$ for some $n \geq 0$, then the *consecutive rotation* of this cycle will be of the form $(G_{n+1}, \dots, G_{n+t-1}, G_n) \pmod{m}$.

1.2 Main Result

The main result of this paper will be presented in the last section of this paper. In order to reach this, we must develop notation and provide some properties of generalised Fibonacci sequences. We will also introduce conjectures for sequences both modulo prime numbers and powers of prime numbers. These conjectures are based on a large amount of computational evidence which will be discussed in more detail later in the paper. The main result will allow us to extrapolate the information of both the length of the cycles and number of cycles of each length of basic cycles (those generated modulo p, where p is prime) in order to find the length of the cycles and number of cycles of each length for any m based on the prime factorisation of m.

For example, if we knew the behaviour of the cycles modulo the primes 2 and 3, and the behaviour of prime powers, then we can use the main result to find the behaviour of the generalised Fibonacci sequences modulo 24.

2 Properties of cycles

In this section, we will present and prove some basic results on cycles of modular Fibonacci sequences. We also state results from Renault's paper [3] and provide examples to clarify them. This will lead to further results in the latter part of the paper.

Definition 2.1. The complete *set of distinct cycles for generalised Fibonacci sequences modulo m* is denoted by $S(m)$.

As we observed in the example of the Fibonacci sequence modulo 2, the cycle of length 3 can be expressed in three different ways (the cycle in that example was $1,1,0$ or $1,0,1$ or $0,1,1$). In general, a cycle of length n will have n different expressions for the same cycle. To avoid this confusion, we will standardise how we express cycles: given any cycle of a generalised Fibonacci sequence mod m, we will always express the cycle using the smallest integer first. If we revisit the example just mentioned, we would choose to express this cycle as $0,1,1$. Sometimes, the smallest integer can appear more than once in a cycle, if that's the case, we will choose to start with the pair consisting of the smallest integer followed by its smallest antecessor. Note that there will never be a tie since each pair only generates one cycle.

Definition 2.2. Denote the *ith cycle modulo m* as the cycle $S_i(m)$ for $i \geq 0$ in the set $S(m)$, where the cycles are first ordered by length with increasing order and then by the smallest starting integer and smallest antecessor of the cycles.

For example, consider the case where the modulus $m = 3$. We will work out the description of $S(3)$ here. Since the integers modulo 3 consist only of the three integers 0, 1 and 2, we have $3^2 = 9$ ways to begin a generalised Fibonacci sequence mod 3. These can be defined by the initial elements G_0 and G_1 given by the following ordered pairs (G_0, G_1):

(0,0)	(0,1)	(0,2)
(1,0)	(1,1)	(1,2)
(2,0)	(2,1)	(2,2)

The sequence beginning with $(0,0)$ will consist only of zeros and have a cycle of length 1. This is clearly the shortest cycle length and has the smallest starting numbers, so $S_0(3) = 0$ since it is the cycle consisting only of 0.

Working out the other sequences mod 3, we find that the sequences starting with all of the other pairs listed above have the cycle

$$0,1,1,2,0,2,2,1.$$

This cycle is written in the desired form since it starts with the smallest pair, $(0,1)$. Thus $S_1(3) = 0,1,1,2,0,2,2,1$ and the set $S(3)$ equals

$$\{S_0(3), S_1(3)\}.$$

Given this example, you might guess that the only cycle of length 1 is going to be the trivial cycle, consisting just of zero. This was proven by Renault:

Lemma 2.3 (Renault [3])**.** *Considering the set of all generalised Fibonacci sequences under all possible moduli, there is exactly one cycle of length 1. This is the trivial cycle consisting of 0.*

Additionally, Renault discovered a number of interesting properties about the lengths of cycles of generalised Fibonacci sequences mod m. The next two theorems describe relationships between moduli, where at least one modulus is a divisor of another. These theorems are a precursor to our **theorem 4.12**, which is our main result and also connects cycles modulo m to cycles modulo a factor of m.

Theorem 2.4 (Renault [3, Theorem 3.2])**.** *Let* $\mathrm{G} := \{\mathrm{G}_n\}_{n=0}^{\infty}$ *be a generalised Fibonacci sequence. Let* m_1 *and* m_2 *be two moduli and let* r *and* t *denote the periods of this sequence mod* m_1 *and* m_2*, respectively. Then if* $m_1 \mid m_2$ *we have that* $r \mid t$*.*

Theorem 2.5 (Renault [3, Theorem 3.3])**.** *Let* m *have the prime factorisation* $m = p_1^{e_1} p_2^{e_2} \dots p_k^{e_k}$ *for* $k \in \mathbb{N}$*. Let* t *denote the period of any fixed generalised Fibonacci sequence mod* m*, and let* $r_1, \dots, r_k$ *denote the periods of this sequence mod* $p_1^{e_1}, \dots, p_k^{e_k}$ *respectively. Then* $t = \mathrm{lcm}(r_1, \dots, r_k)$*.*

We have already seen in previous examples that the Fibonacci sequence, which begins with $0, 1, 1, 2, 3, 5\dots$ has a period $r_1 = 3$ modulo 2 and a period $r_2 = 8$ modulo 3. Therefore, if we considered the Fibonacci sequence modulo 6, it would have a period

$$t = \mathrm{lcm}(r_1, r_2) = \mathrm{lcm}(3, 8) = 24.$$

In fact, we can see that this cycle of length 24 is

$$0, 1, 1, 2, 3, 5, 2, 1, 3, 4, 1, 5, 0, 5, 5, 4, 3, 1, 4, 5, 3, 2, 5, 1.$$

Lemma 2.6. *Let* m *be a positive integer. For any integers* a *and* b *in* $\{0, \dots, m-1\}$*, the pair* (a, b) *is a subsequence of some generalised Fibonacci sequence modulo* m*.*

Proof. When generating a generalised Fibonacci sequence modulo m, either (a, b) is already included in $\mathrm{S}(m)$ or (a, b) will start a new sequence that will be added to the set. □

Theorem 2.7. *Let* m *be a positive integer. Assume* $\mathrm{S}(m)$ *contains* q *distinct cycles and* l_i *is the length of the* i*th cycle for* $0 \le i \le q-1$*. Then*

$$m^2 = \sum_{i=0}^{q-1} l_i.$$

Proof. From **lemma 2.6** we know that every ordered pair is included in a cycle. Additionally, every ordered pair is only included in exactly one cycle, since it can only generate one unique cycle because each term G_n is defined to be the sum of the previous two terms $n \geq 2$. Therefore, we have m^2 distinct ordered pairs in $S(m)$. Now, a cycle $S_i(m)$ with period l_i is given by

$$\left(G_0, G_1, \ldots, G_{l_i-2}, G_{l_i-1}\right) \pmod{m}.$$

Therefore it has pairs

$$\{(G_0, G_1), \ldots, \left(G_{l_i-2}, G_{l_i-1}\right), \left(G_{l_i-1}, G_0\right)\}$$

which means for every cycle with length l_i there are l_i pairs from that cycle. Hence if $S(m)$ has q distinct cycles then

$$m^2 = \sum_{i=0}^{q-1} l_i.$$

□

Corollary 2.8. *If n_i is the number of distinct cycles in $S(m)$ for which the ith distinct cycle length is l_i for $0 \leq i \leq k-1$ and k is the number of distinct lengths of the cycles in $S(m)$, then*

$$m^2 = \sum_{i=0}^{k-1} n_i \times l_i.$$

For example if we look at generalised Fibonacci sequences modulo 4 then we have cycles

$$(0), (0,2,2), (0,1,1,2,3,1), (0,3,3,2,1,3).$$

Using **theorem 2.7** we have

$$\sum_{i=0}^{3} l_i = 1+3+6+6 = 16 = 4^2.$$

Using **corollary 2.8** we have

$$\sum_{i=0}^{2} n_i \times l_i = 1+3+2\times 6 = 16 = 4^2.$$

As m increases, **corollary 2.8** can be applied more easily than using **theorem 2.7**. Since we will store generalised Fibonacci sequences modulo m in the form defined in **definition 3.1** instead of in the form of the complete set of cycles modulo m, because this will be far less computationally expensive. Thus the data will already be in the form required for **corollary 2.8**.

3 Lengths of Generalised Fibonacci cycles

Definition 3.1 (Lengths of generalised cycles modulo m)**.** All the distinct cycles modulo m and their associated lengths are denoted by

$$\mathrm{G}(m) = \{1 \times |1|,\ n_1 \times |l_1|,\ n_2 \times |l_2|, \ldots\}$$

where n_i is the number of distinct cycles in which the cycle length is l_i and $1 \times |1|$ is the unique cycle of length 1 given by **lemma 2.3**.

Conjecture 3.1. We conjecture that there are only two cases for periods of generalised sequences modulo m, where m is prime. This conjecture is supported by computational experiment for all primes $m \leq 1000$ and the two cases are:

i. $\mathrm{G}(m) = \{1 \times |1|,\ n_1 \times |l_1|\}$,

ii. $\mathrm{G}(m) = \{1 \times |1|,\ n_1 \times |l_1|,\ n_2 \times |l_2|\}$.

Conjecture 3.2 (Powers of Primes)**.** We also conjecture that there are only three cases for periods of generalised sequences modulo m^k, where m is prime. This conjecture is supported by experiment for all powers of primes $m \leq 1000$ and the three cases are:

i. If $m = 5$, we get $\mathrm{G}(5) = \{1 \times |1|,\ 1 \times |4|,\ 1 \times |20|\}$, then

$$\mathrm{G}\left(5^k\right) = \{1 \times |1|,\ 1 \times |4|,\ \left(5^{k-1}\right) \times |4 \times 5^k|,\ \left(5^j + 5^{j-1}\right) \times |4 \times 5^j|;\ 1 \leq j \leq k-1\}.$$

ii. If $\mathrm{G}(m) = \{1 \times |1|,\ n_1 \times |l_1|\}$ where $m \neq 5$, then

$$\mathrm{G}\left(m^k\right) = \{1 \times |1|,\ \left(n_1 \times m^j\right) \times |l_1 \times m^j|;\ 0 \leq j \leq k-1\}.$$

iii. If $\mathrm{G}(m) = \{1 \times |1|,\ n_1 \times |l_1|,\ n_2 \times |l_2|\}$ where $m \neq 5$, for $2 \leq k$ and $\alpha = \dfrac{l_1 \times n_1}{l_2} \in \mathbb{Z}$, then

$$\mathrm{G}\left(m^k\right) = \left\{ \begin{array}{c} 1 \times |1|,\ n_1 \times |l_1|,\ n_2 \times |l_2|,\ \left([n_2 \times m^j] + \alpha \sum_{i=1}^{j} m^{i-1}[m-1]\right) \times |l_2 \times m^j|, \\ n_1 \times |l_1 \times m^j|;\ 1 \leq j \leq k-1 \end{array} \right\}.$$

4 Using the Prime Factorisation of the Modulus

Theorem 4.12 will be the main result for this paper. The rest of this section will be committed to proving it and giving supporting examples. This theorem allows us to build one set of cycle lengths for a modulus m, from two sets of cycle lengths which represent the factors of m. Therefore we introduce the language and definitions to work with two cycles simultaneously, so we do this by defining pairs of blocks, as in the definition below.

Definition 4.1 (Pairs of blocks)**.** Let S_1 and S_2 be two cycles with lengths r and t respectively, with $r \le t$. Then we define two *blocks* of length, $\mathrm{lcm}(r,t)$. The first block, $B_1[S_1, S_2]$ consists of $\frac{\mathrm{lcm}(r,t)}{r}$ copies of S_1 and the second block, $B_2[S_1, S_2]$ consists of $\frac{\mathrm{lcm}(r,t)}{t}$ copies of S_2 as shown below

$$B_1 : \boxed{\alpha_1, \alpha_2, \ldots\ldots, \alpha_r}\,\boxed{\alpha_1, \alpha_2, \ldots\ldots, \alpha_r}\ \ldots\ldots\ldots\ldots\ldots\ \boxed{\alpha_1, \alpha_2, \ldots\ldots, \alpha_r},$$
$$B_2 : \boxed{\beta_1, \beta_2, \ldots\ldots\ldots, \beta_t}\,\boxed{\beta_1, \beta_2, \ldots\ldots\ldots, \beta_t}\ \ldots\ldots\ldots\ \boxed{\beta_1, \beta_2, \ldots\ldots\ldots, \beta_t}.$$

Definition 4.2 (Correspondence of Elements in Blocks)**.** Let B_1 and B_2 be a pair of blocks of cycles with lengths r and t, respectively. Without loss of generality assume that $r \le t$. Fix any value i, $1 \le i \le r$. Then we say that α_i *corresponds* with all elements β_j in B_2 that occupy the same respective positions of α_i in B_1. In particular, α_i corresponds with

$$\left\{\beta_j : j \equiv rn + i \pmod{t} \text{ with } 0 \le n \le \frac{\mathrm{lcm}(r,t)}{r} - 1\right\}.$$

For example, let S_1 be $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ and S_2 be $(\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6)$. Then we have the pair of blocks

$$B_1 : \boxed{\alpha_1, \alpha_2, \alpha_3, \alpha_4}\,\boxed{\alpha_1, \alpha_2, \alpha_3, \alpha_4}\,\boxed{\alpha_1, \alpha_2, \alpha_3, \alpha_4},$$
$$B_2 : \boxed{\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6}\,\boxed{\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6}.$$

Let us choose $i = 2$ then α_2 corresponds to the set of β_j's given by the set $\{\beta_2, \beta_6, \beta_4\}$.

Note that we can also study the correspondence of pairs of elements in two blocks. Let us fix any value i, $1 \le i \le r$. If α_i corresponds to the elements β_j then by construction α_{i+1} will correspond to the elements β_{j+1}, where $\alpha_{r+1} = \alpha_1$ and $\beta_{t+1} = \beta_1$. Therefore we will say that the pair (α_i, α_{i+1}) corresponds to the pairs $\left(\beta_j, \beta_{j+1}\right)$.

The following lemma uses this concept of the pair of blocks and correspondence of pairs. This will be important when we construct sequences from two smaller moduli to form a sequence with a larger modulus.

Lemma 4.3. *Let* S_1 *and* S_2 *be two cycles of length* r *and* t *respectively. Without loss of generality, these can be chosen so that* $r \le t$*. Let* B_1 *and* B_2 *be a pair of blocks. Fix a value* i*,* $1 \le i \le r$*. Then the pair* (α_i, α_{i+1}) *corresponds with* $\dfrac{t}{\gcd(r,t)}$ *distinct pairs* $\left(\beta_j, \beta_{j+1}\right)$*.*

Proof. Fixing any i, we know that the pair (α_i, α_{i+1}) appears $\dfrac{\mathrm{lcm}(r,t)}{r}$ times in a pair of blocks, by the definition of pairs of blocks, so it corresponds to $\dfrac{\mathrm{lcm}(r,t)}{r}$ pairs $\left(\beta_j, \beta_{j+1}\right)$.

Thus, we only need to prove that for each instance of the pair (α_i, α_{i+1}) it will correspond with a distinct pair (β_j, β_{j+1}) in order to establish the lemma.

Now assume that for some j, $1 \le j \le t$, there are two instances of the pair (α_i, α_{i+1}) in the pair of blocks that correspond with the pair (β_j, β_{j+1}) for this fixed j. As each pair (α_i, α_{i+1}) repeats every r terms, and each pair (β_j, β_{j+1}) repeats every t terms, then in order to have the two pairs correspond a second time, that would have to occur $\operatorname{lcm}(r, t)$ terms after their first correspondence. Therefore the length of the blocks would need to be greater than $\operatorname{lcm}(r, t)$, leading to a contradiction. □

Corollary 4.4. *Assume that the pair of blocks is given by*

$$B_1 : \boxed{x_1, x_2, x_3, \ldots\ldots x_l},$$
$$B_2 : \boxed{y_1, y_2, y_3, \ldots\ldots\ y_l},$$

then the quadruples $\{(x_i, x_{i+1}, y_i, y_{i+1})\}_{\{i=1,\cdots,l\}}$ *are distinct. Thus we have* $l = \operatorname{lcm}(r, t)$ *distinct quadruples.*

Lemma 4.5. *Using the notation defined in* ***definition 4.2*** *and subsequently for the correspondence of pairs, if 2 pairs* $(\alpha_i, \alpha_{i+1}), (\alpha_k, \alpha_{k+1})$ *in* S_1 *are corresponding to the same pair* (β_j, β_{j+1}) *in* S_2 *then* $i \equiv k \pmod{\gcd(r, t)}$.

Proof. By **definition 4.2**, when we study the correspondence of the first terms of each pair we have that

$$\begin{cases} j \equiv rn + i \pmod{t} \\ j \equiv rm + k \pmod{t} \quad \text{for some } n, m \in \mathbb{Z} \end{cases},$$

which implies that $r(n-m) + (i-k) \equiv 0 \pmod{t}$, therefore $i - k \equiv 0 \pmod{\gcd(r, t)}$. □

Corollary 4.6. *Let* (α_i, α_{i+1}) *denote the ith pair of* S_1 *and* (β_j, β_{j+1}) *denote the jth pair of* S_2*, where* S_1 *and* S_2 *are defined in* ***lemma 4.3*** *. Then, using* ***lemma 4.3*** *and* ***lemma 4.5*** *we can consecutively rotate* S_1 *the* $\gcd(r, t)$ *number of times in order to ensure that each pair* (α_i, α_{i+1}) *in* S_1 *corresponds to every pair* (β_j, β_{j+1}) *in* S_2.

Let us denote r_i and t_j as the respective lengths of $S_i(m_1)$ and $S_j(m_2)$. Also denote the number at the ath position in $B_1[S_i(m_1), S_j(m_2)]$ as $\alpha_{i,j,a}$ and the number at the bth position in $B_2[S_i(m_1), S_j(m_2)]$ as $\beta_{i,j,b}$. Note the definition of a pair of blocks states that they are also periodic so we don't need to restrict the maximum value of a and b. We will define $\oplus$ as the piecewise addition between the two blocks B_1 and B_2, for each of the $\gcd(r_i, t_j)$ consecutive rotations of $S_i(m_1)$. Finally, let

$$\left(m_2 B_1[S_i(m_1), S_j(m_2)] \oplus m_1 B_2[S_i(m_1), S_j(m_2)]\right) \pmod{m_1 m_2}$$

be the formula which we will use to form the new cycles modulo $m_1 m_2$.

Lemma 4.7. $\left(m_2 \mathrm{B}_1[\mathrm{S}_i(m_1), \mathrm{S}_j(m_2)] \oplus m_1 \mathrm{B}_2[\mathrm{S}_i(m_1), \mathrm{S}_j(m_2)]\right) \pmod{m_1 m_2}$ *is periodic and has length* $\operatorname{lcm}(r_i, t_j)$*, where* r_i *and* t_j *are the respective lengths of* $\mathrm{S}_i(m_1)$ *and* $\mathrm{S}_j(m_2)$*.*

Proof. The definition of a pair of blocks states that they are both periodic with period $\operatorname{lcm}(r_i, t_j) = l$, so the period of the new cycle, $p \le \operatorname{lcm}(r_i, t_j)$. For a generalised Fibonacci cycle to be periodic a new ordered pair in cycle must agree with the starting ordered pair. Now let us assume that $1 \le p < \operatorname{lcm}(r_i, t_j)$, so for the cycle to be periodic we have

$$m_2 \alpha_{i,j,p+1} + m_1 \beta_{i,j,p+1} \equiv m_2 \alpha_{i,j,1} + m_1 \beta_{i,j,1} \pmod{m_1 m_2},$$

$$m_2 \alpha_{i,j,p+2} + m_1 \beta_{i,j,p+2} \equiv m_2 \alpha_{i,j,2} + m_1 \beta_{i,j,2} \pmod{m_1 m_2}.$$

Using the definition of modular arithmetic, we obtain

$$\begin{aligned} m_2(\alpha_{i,j,p+1} - \alpha_{i,j,1}) + m_1(\beta_{i,j,p+1} - \beta_{i,j,1}) &= k_1 m_1 m_2, && (1)\\ m_2(\alpha_{i,j,p+2} - \alpha_{i,j,2}) + m_1(\beta_{i,j,p+2} - \beta_{i,j,2}) &= k_2 m_1 m_2, && (2)\\ m_2(\alpha_{i,j,p+1} - \alpha_{i,j,1}) &= m_1(k_1 m_2 - \beta_{i,j,p+1} + \beta_{i,j,1}),\\ m_2(\alpha_{i,j,p+2} - \alpha_{i,j,2}) &= m_1(k_2 m_2 - \beta_{i,j,p+2} + \beta_{i,j,2}), \end{aligned}$$

which should hold for some $k_1, k_2 \in \mathbb{Z}$.

Using the definition of modular arithmetic on both equations, we get

$$m_2(\alpha_{i,j,p+1} - \alpha_{i,j,1}) \equiv 0 \pmod{m_1},$$

$$m_2(\alpha_{i,j,p+2} - \alpha_{i,j,2}) \equiv 0 \pmod{m_1}.$$

Now since m_1 and m_2 are coprime, $\left(\alpha_{i,j,p+1} - \alpha_{i,j,1}\right) \equiv 0 \pmod{m_1}$ and $\left(\alpha_{i,j,p+2} - \alpha_{i,j,2}\right) \equiv 0 \pmod{m_1}$. However, by definition the values of $\alpha_{i,j,a}$ found in B_1 are elements of $\mathrm{S}_i(m_1)$, which are all modulo m_1. Therefore, $0 \le \alpha_{i,j,1}, \alpha_{i,j,p+1}, \alpha_{i,j,2}, \alpha_{i,j,p+2} \le m_1 - 1$, which consequently results in the bounds $-(m_1 - 1) \le \alpha_{i,j,p+1} - \alpha_{i,j,1} \le (m_1 - 1)$ and $-(m_1 - 1) \le \alpha_{i,j,p+2} - \alpha_{i,j,2} \le m_1 - 1$. Therefore $\alpha_{i,j,p+1} = \alpha_{i,j,1}$ and $\alpha_{i,j,p+2} = \alpha_{i,j,2}$.

Similarly, if we rearrange **equations** 1 and 2 and using a similar principle as before one obtains $\beta_{i,j,p+1} = \beta_{i,j,1}$ and $\beta_{i,j,p+2} = \beta_{i,j,2}$. The assumption implies $\alpha_{i,j,p+1} = \alpha_{i,j,1}$, $\alpha_{i,j,p+2} = \alpha_{i,j,2}$, $\beta_{i,j,p+1} = \beta_{i,j,1}$ and $\beta_{i,j,p+2} = \beta_{i,j,2}$ for $1 \le p < \operatorname{lcm}(r_i, t_j)$.

Therefore we have a contradiction since **corollary 4.4** states that there cannot be two quadruples in the pairs of blocks which are the same. So

$$\left(m_2 \mathrm{B}_1[\mathrm{S}_i(m_1), \mathrm{S}_j(m_2)] \oplus m_1 \mathrm{B}_2[\mathrm{S}_i(m_1), \mathrm{S}_j(m_2)]\right) \pmod{m_1 m_2}$$

is periodic and has length $\operatorname{lcm}(r_i, t_j)$. □

Lemma 4.8. *Consecutively rotating* B_1 *the* $\gcd\left(r_i, t_j\right)$ *number of times gives a unique cycle for*

$$\left(m_2\,\mathrm{B}_1[\mathrm{S}_i(m_1),\mathrm{S}_j(m_2)] \oplus m_1\,\mathrm{B}_2[\mathrm{S}_i(m_1),\mathrm{S}_j(m_2)]\right) \pmod{m_1 m_2}$$

for each rotation.

Proof. Using **corollary 4.6** we know that we need to consecutively rotate the cycle $\mathrm{S}_i(m_1)$ the $\gcd\left(r_i, t_j\right)$ number of times in order to look at every combination of the addition of the cycles. Now let us assume that there are two distinct cycles for different values after consecutively rotating p times where $1 \le p \le \gcd\left(r_i, t_j\right) - 1$. Therefore for the cycle to be periodic we have

$$\left(m_2\,\alpha_{i,j,p+1} + m_1\,\beta_{i,j,1}\right) \equiv \left(m_2\,\alpha_{i,j,1} + m_1\,\beta_{i,j,1}\right) \pmod{m_1 m_2},$$

$$\left(m_2\,\alpha_{i,j,p+2} + m_1\,\beta_{i,j,2}\right) \equiv \left(m_2\,\alpha_{i,j,2} + m_1\,\beta_{i,j,2}\right) \pmod{m_1 m_2}.$$

By construction, we have $\beta_{i,j,1} = \beta_{i,j,1}$ and $\beta_{i,j,2} = \beta_{i,j,2}$. Now, we can use a similar argument to the proof for **lemma 4.7** to show that the assumption implies $\alpha_{i,j,p+1} = \alpha_{i,j,1}$ and $\alpha_{i,j,p+2} = \alpha_{i,j,2}$ for $1 \le p \le \gcd\left(r_i, t_j\right) - 1$.

Thus we have a contradiction since $1 \le p \le \gcd\left(r_i, t_j\right) - 1 < r_i$ and by construction the first time the pair $\left(\alpha_{i,j,1}, \alpha_{i,j,2}\right)$ will be repeated is after r_i terms. Therefore consecutively rotating B_1 the $\gcd\left(r_i, t_j\right)$ number of times gives a unique solution for

$$\left(m_2\,\mathrm{B}_1[\mathrm{S}_i(m_1),\mathrm{S}_j(m_2)] \oplus m_1\,\mathrm{B}_2[\mathrm{S}_i(m_1),\mathrm{S}_j(m_2)]\right) \pmod{m_1 m_2}$$

for each rotation. □

Lemma 4.9. *Every distinct pair* (i, j) *and number of consecutive rotations give a unique cycle for*

$$\left(m_2\,\mathrm{B}_1[\mathrm{S}_i(m_1),\mathrm{S}_j(m_2)] \oplus m_1\,\mathrm{B}_2[\mathrm{S}_i(m_1),\mathrm{S}_j(m_2)]\right) \pmod{m_1 m_2}.$$

Proof. Now let us assume that we have $(i, j) \ne (k, l)$ hence two distinct pairs of blocks and the addition of these pairs blocks will give two distinct cycles. We will let $0 \le p \le \gcd\left(r_i, t_j\right) - 1$ and $0 \le q \le \gcd\left(r_k, t_l\right) - 1$ because this will take into account all the possible consecutive rotations. For the cycle to be periodic we have

$$m_2\,\alpha_{i,j,p+1} + m_1\,\beta_{i,j,1} \equiv m_2\,\alpha_{k,l,q+1} + m_1\,\beta_{k,l,1} \pmod{m_1 m_2},$$

$$m_2\,\alpha_{i,j,p+2} + m_1\,\beta_{i,j,2} \equiv m_2\,\alpha_{k,l,q+2} + m_1\,\beta_{k,l,2} \pmod{m_1 m_2}.$$

Using the definition of modular arithmetic, we get

$$m_2(\alpha_{i,j,p+1} - \alpha_{k,l,q+1}) + m_1(\beta_{i,j,1} - \beta_{k,l,1}) = k_1 m_1 m_2,$$
$$m_2(\alpha_{i,j,p+2} - \alpha_{k,l,q+2}) + m_1(\beta_{i,j,2} - \beta_{k,l,2}) = k_2 m_1 m_2,$$

which should hold for some $k_1, k_2 \in \mathbb{Z}$.

Now, we can use a similar argument to the proof for **lemma 4.7** to show that the assumption implies $\alpha_{i,j,p+1} = \alpha_{k,l,q+1}$, $\alpha_{i,j,p+2} = \alpha_{k,l,q+2}$, $\beta_{i,j,1} = \beta_{k,l,1}$ and $\beta_{i,j,2} = \beta_{k,l,2}$. Now using **lemma 2.6**, since $\left(\alpha_{i,j,p+1}, \alpha_{i,j,p+2}\right) = \left(\alpha_{k,l,q+1}, \alpha_{k,l,q+2}\right)$, then both pairs must be in the same cycle $S_i(m_1)$ so $i = k$. Similarly, $\left(\beta_{i,j,1}, \beta_{i,j,2}\right) = \left(\beta_{k,l,1}, \beta_{k,l,2}\right)$ so once again both pairs must be in the same cycle $S_j(m_2)$ by **lemma 2.6** hence $j = l$.

Thus we have a contradiction since $(i, j) = (k, l)$. Therefore every distinct pair (i, j) and number of consecutive rotations give a unique cycle for

$$\left(m_2 B_1[S_i(m_1), S_j(m_2)] \oplus m_1 B_2[S_i(m_1), S_j(m_2)]\right) \pmod{m_1 m_2}.$$

□

Lemma 4.10. *The set*

$$\left\{\left(m_2 B_1[S_i(m_1), S_j(m_2)] \oplus m_1 B_2[S_i(m_1), S_j(m_2)]\right) \pmod{m_1 m_2};\, 0 \le i \le y-1,\, 0 \le j \le z-1\right\}$$

generates every generalised Fibonacci cycle modulo m.

Proof. First note that from **theorem 2.7** we know that

$$m_1^2 = \sum_{i=0}^{y-1} r_i \text{ and } m_2^2 = \sum_{i=0}^{z-1} t_j.$$

We have already proved that after the addition of the cycles that we get period $\operatorname{lcm}\left(r_i, t_j\right)$ and that each distinct i, j have $\gcd\left(r_i, t_j\right)$ distinct cycles because of the consecutive rotations of $S_i(m_1)$. Therefore we can workout the number of distinct ordered pairs included in these cycles by

$$\sum_{i=0}^{y-1} \sum_{j=0}^{z-1} \operatorname{lcm}\left(r_i, t_j\right) \gcd\left(r_i, t_j\right) = \sum_{i=0}^{y-1} \sum_{j=0}^{z-1} r_i\, t_j = \sum_{i=0}^{y-1} r_i \sum_{j=0}^{z-1} t_j$$
$$= m_1^2\, m_2^2 = (m_1\, m_2)^2 = m^2.$$

Since every ordered pair is distinct and has been included then we can say

$$\left\{\left(m_2 B_1[S_i(m_1), S_j(m_2)] \oplus m_1 B_2[S_i(m_1), S_j(m_2)]\right) \pmod{m_1 m_2};\, 0 \le i \le y-1,\, 0 \le j \le z-1\right\}$$

generates every generalised Fibonacci cycles modulo m. □

Proposition 4.11. *Let* $2 \le m_1, m_2 \in \mathbb{Z}$ *where* $m = m_1 \times m_2$ *and* $\gcd(m_1, m_2) = 1$*. If there are* y *distinct generalised Fibonacci cycles modulo* m_1 *and* z *distinct generalised Fibonacci cycles modulo* m_2 *and we are given* $S(m_1)$ *and* $S(m_2)$*, then*

$$S(m) = \left\{\left(m_2 B_1[S_i(m_1), S_j(m_2)] \oplus m_1 B_2[S_i(m_1), S_j(m_2)]\right)(\text{mod } m_1 m_2); 0 \le i \le y-1, 0 \le j \le z-1\right\}.$$

Proof. We have proved in **lemma 4.7** that every cycle created is periodic and the length is $\text{lcm}\left(r_i, t_j\right)$. We have then proved in **lemma 4.8** and **lemma 4.9** that every cycle generated will be distinct. Finally, we proved in **lemma 4.10** that every cycle has been generated. Therefore the complete set of generalised Fibonacci cycles modulo m is given by

$$S(m) = \left\{\left(m_2 B_1[S_i(m_1), S_j(m_2)] \oplus m_1 B_2[S_i(m_1), S_j(m_2)]\right)(\text{mod } m_1 m_2); 0 \le i \le y-1, 0 \le j \le z-1\right\}.$$

□

Theorem 4.12 (Main Result)**.** *Let* $2 \le m_1, m_2 \in \mathbb{Z}$ *where* $m = m_1 \times m_2$ *and* $\gcd(m_1, m_2) = 1$*. If*

$$G(m_1) = \{n_{1,0} \times |l_{1,0}|,\ n_{1,1} \times |l_{1,1}|, \ldots,\ n_{1,r-1} \times |l_{1,r-1}|\},$$

$$G(m_2) = \{n_{2,0} \times |l_{2,0}|,\ n_{2,1} \times |l_{2,1}|, \ldots,\ n_{2,t-1} \times |l_{2,t-1}|\},$$

then we can use $G(m_1)$ *and* $G(m_2)$ *to find* $G(m)$ *by the formula*

$$G(m) = G(m_1 \times m_2) = \{[n_{1,i} \times n_{2,j} \times \gcd\left(l_{1,i}, l_{2,j}\right)] \times |\text{lcm}\left(l_{1,i}, l_{2,j}\right)|; 0 \le i \le r-1, 0 \le j \le t-1\}.$$

The following proof demonstrates that **proposition 4.11** implies that the main Theorem holds.

Proof. We have proved above in **lemma 4.7** that the cycles are periodic with period $\text{lcm}\left(l_{1,i}, l_{2,j}\right)$. We have proved in **lemma 4.8** that for each value of i, j we get $\gcd\left(l_{1,i}, l_{2,j}\right)$ distinct cycles due to the consecutive rotations. We now only look at the distinct lengths then we take into account the multiple combinations of the different cycles hence we get $n_{1,i} \times n_{2,j}$. Therefore we have proved that

$$G(m) = G(m_1 \times m_2) = \{[n_{1,i} \times n_{2,j} \times \gcd\left(l_{1,i}, l_{2,j}\right)] \times |\text{lcm}\left(l_{1,i}, l_{2,j}\right)|; 0 \le i \le r-1, 0 \le j \le t-1\}.$$

□

Example 4.13. Find $G(15)$ using **theorem 4.12**, $G(3)$ and $G(5)$.
First $G(3) = \{1 \times |1|, 1 \times |8|\}$ with $S(3) = \{(0), (0,1,1,2,0,2,2,1)\}$ and $G(5) = \{1 \times |1|, 1 \times |4|, 1 \times |20|\}$ with $S(5) = \{(0), (1,3,4,2), (0,1,1,2,3,0,3,3,1,4,0,4,4,3,2,0,2,2,4,1)\}$. Then using **theorem 4.12**:

	$1 \times \|1\|$	$1 \times \|4\|$	$1 \times \|20\|$
$1 \times \|1\|$	$1 \times \|1\|$	$1 \times \|4\|$	$1 \times \|20\|$
$1 \times \|8\|$	$1 \times \|8\|$	$4 \times \|8\|$	$4 \times \|40\|$

.

Therefore $G(15) = \{1 \times |1|, 1 \times |4|, 5 \times |8|, 1 \times |20|, 4 \times |40|\}$.

Now to create $S(15)$ we calculate $[5S_i(3) + 3S_j(5)] \pmod{15}$, using all the pairs of blocks where $0 \le i \le 1, 0 \le j \le 2$:

1. Firstly we have $S_0(3) = (0)$ and $S_0(5) = (0)$, so $\gcd(|S_0(3)|, |S_0(5)|) = 1$. Therefore we only have to calculate $[5S_0(3) + 3S_0(5)] \pmod{15}$ once. So we get $[5S_0(3) + 3S_0(5)] \pmod{15} = (0) + (0) = (0)$.

2. We have $S_0(3) = (0)$ and $S_1(5) = (1,3,4,2)$, so $\gcd(|S_0(3)|, |S_1(5)|) = 1$. Therefore we only have to calculate $[5S_0(3) + 3S_1(5)] \pmod{15}$ once. So we get

$$[5S_0(3) + 3S_1(5)] \pmod{15} = (0) + (3,9,12,6) = (3,9,12,6).$$

3. We have $S_0(3) = (0)$ and $S_2(5) = (0,1,1,2,3,0,3,3,1,4,0,4,4,3,2,0,2,2,4,1)$, so $\gcd(|S_0(3)|, |S_2(5)|) = 1$. Therefore we only have to calculate $[5S_0(3) + 3S_2(5)] \pmod{15}$ once. So we get

$$\begin{aligned}[5S_0(3) + 3S_2(5)] \pmod{15} &= (0) + (0,3,3,6,9,0,9,9,3,12,0,12,12,9,6,0,6,6,12,3)\\ &= (0,3,3,6,9,0,9,9,3,12,0,12,12,9,6,0,6,6,12,3).\end{aligned}$$

4. We have $S_1(3) = (0,1,1,2,0,2,2,1)$ and $S_0(5) = (0)$, so $\gcd(|S_1(3)|, |S_0(5)|) = 1$. Therefore we only have to calculate $[5S_1(3) + 3S_0(5)] \pmod{15}$ once. So we get

$$\begin{aligned}[5S_1(3) + 3S_0(5)] \pmod{15} &= (0,5,5,10,0,10,10,5) + (0)\\ &= (0,5,5,10,0,10,10,5).\end{aligned}$$

5. We have $S_1(3) = (0,1,1,2,0,2,2,1)$ and $S_1(5) = (1,3,4,2)$, so $\gcd(|S_1(3)|, |S_1(5)|) = 4$ .Therefore we have to calculate $[5S_1(3) + 3S_1(5)] \pmod{15}$ 4 times, by consecutively rotating $S_1(3)$ by one position 4 times. So we get

$$\begin{aligned}&[5S_1(3) + 3S_1(5)] \pmod{15}\\ \text{i.}\ &= (0,5,5,10,0,10,10,5) + (3,9,12,6)\\ &= (3,14,2,1,3,4,7,11).\\ \text{ii.}\ &= (5,5,10,0,10,10,5,0) + (3,9,12,6)\\ &= (8,14,7,6,13,4,2,6).\\ \text{iii.}\ &= (5,10,0,10,10,5,0,5) + (3,9,12,6)\\ &= (8,4,12,1,13,14,12,11).\\ \text{iv.}\ &= (10,0,10,10,5,0,5,5) + (3,9,12,6)\\ &= (13,9,7,1,8,9,2,11).\end{aligned}$$

6. We have $S_1(3) = (0,1,1,2,0,2,2,1)$ and $S_2(5) = (0,1,1,2,3,0,3,3,1,4,0,4,4,3,2,0,2,2,4,1)$, so $\gcd(|S_1(3)|,|S_2(5)|) = 4$.Therefore we have to calculate $[5S_1(3)+3S_2(5)] \pmod{15}$ 4 times, by consecutively rotating $S_1(3)$ by one position 4 times. So we get

$$\begin{aligned}
&[5S_1(3)+3S_2(5)] \pmod{15}\\
\textbf{i.}\ &= (0,5,5,10,0,10,10,5)+(0,3,3,6,9,0,9,9,3,12,0,12,12,9,6,0,6,6,12,3)\\
&= (0,8,8,1,9,10,4,14,3,2,5,7,12,4,1,5,6,11,2,13,0,13,13,11,9,5,14,4,3,\\
&\quad 7,10,2,12,14,11,10,6,1,7,8).\\
\textbf{ii.}\ &= (5,5,10,0,10,10,5,0)+(0,3,3,6,9,0,9,9,3,12,0,12,12,9,6,0,6,6,12,3)\\
&= (5,8,13,6,4,10,14,9,8,2,10,12,7,4,11,0,11,11,7,3,10,13,8,6,14,5,4,9,13,\\
&\quad 7,5,12,2,14,1,0,1,1,2,3).\\
\textbf{iii.}\ &= (5,10,0,10,10,5,0,5)+(0,3,3,6,9,0,9,9,3,12,0,12,12,9,6,0,6,6,12,3)\\
&= (5,13,3,1,4,5,9,14,8,7,0,7,7,14,6,5,11,1,12,13,10,8,3,11,14,10,9,4,13,\\
&\quad 2,0,2,2,4,6,10,1,11,12,8).\\
\textbf{iv.}\ &= (10,0,10,10,5,0,5,5)+(0,3,3,6,9,0,9,9,3,12,0,12,12,9,6,0,6,6,12,3)\\
&= (10,3,13,1,14,0,14,14,13,12,10,7,2,9,11,5,1,6,7,13,5,3,8,11,4,0,4,4,8,\\
&\quad 12,5,2,7,9,1,10,11,6,2,8).
\end{aligned}$$

$$S(15) = \left\{\begin{array}{l}
(0),(3,9,12,6),(0,5,5,10,0,10,10,5),(3,14,2,1,3,4,7,11),(8,14,7,6,13,4,2,6),\\
(8,4,12,1,13,14,12,11),(13,9,7,1,8,9,2,11),\\
(0,3,3,6,9,0,9,9,3,12,0,12,12,9,6,0,6,6,12,3),\\
(0,8,8,1,9,10,4,14,3,2,5,7,12,4,1,5,6,11,2,13,0,13,13,11,9,5,14,4,3,7,10,\\
\ 2,12,14,11,10,6,1,7,8),\\
(5,8,13,6,4,10,14,9,8,2,10,12,7,4,11,0,11,11,7,3,10,13,8,6,14,5,4,9,13,\\
\ 7,5,12,2,14,1,0,1,1,2,3),\\
(5,13,3,1,4,5,9,14,8,7,0,7,7,14,6,5,11,1,12,13,10,8,3,11,14,10,9,4,13,\\
\ 2,0,2,2,4,6,10,1,11,12,8),\\
(10,3,13,1,14,0,14,14,13,12,10,7,2,9,11,5,1,6,7,13,5,3,8,11,4,0,4,4,8,\\
\ 12,5,2,7,9,1,10,11,6,2,8)
\end{array}\right\}.$$

Therefore we have formed all generalised Fibonacci cycles modulo 15. This also gives $G(15) = \{1\times|1|, 1\times|4|, 5\times|8|, 1\times|20|, 4\times|40|\}$.

5 Conclusion

This research provides a method to calculate the cycles of length, l, and the number of distinct cycles of length l modulo m using the prime factorisation of m. There is also now a method to calculate the cycle lengths and the number of distinct cycles modulo p^k where p is a prime number. We have also shown that every tuple is included in one cycle contained inside $\mathrm{S}(m)$. For further research, the areas for further exploration are:

1. The proofs of the conjectures used to generate the period of generalised Fibonacci cycles modulo m, where $m = p^k$ and p is prime.

2. The formulas of the period of generalised Fibonacci cycles modulo p, where p is prime.

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Appendix

G(m) for 2 ≤ m ≤ 50, 990 ≤ m ≤ 1000.

<table>
<tr><th>m</th><th>G(m)</th></tr>
<tr><td>2</td><td>{1 × |1|, 1 × |3|}</td></tr>
<tr><td>3</td><td>{1 × |1|, 1 × |8|}</td></tr>
<tr><td>4</td><td>{1 × |1|, 1 × |3|, 2 × |6|}</td></tr>
<tr><td>5</td><td>{1 × |1|, 1 × |4|, 1 × |20|}</td></tr>
<tr><td>6</td><td>{1 × |1|, 1 × |3|, 1 × |8|, 1 × |24|}</td></tr>
<tr><td>7</td><td>{1 × |1|, 3 × |16|}</td></tr>
<tr><td>8</td><td>{1 × |1|, 1 × |3|, 2 × |6|, 4 × |12|}</td></tr>
<tr><td>9</td><td>{1 × |1|, 1 × |8|, 3 × |24|}</td></tr>
<tr><td>10</td><td>{1 × |1|, 1 × |3|, 1 × |4|, 1 × |12|, 1 × |20|, 1 × |60|}</td></tr>
<tr><td>11</td><td>{1 × |1|, 2 × |5|, 11 × |10|}</td></tr>
<tr><td>12</td><td>{1 × |1|, 1 × |3|, 2 × |6|, 1 × |8|, 5 × |24|}</td></tr>
<tr><td>13</td><td>{1 × |1|, 6 × |28|}</td></tr>
<tr><td>14</td><td>{1 × |1|, 1 × |3|, 3 × |16|, 3 × |48|}</td></tr>
<tr><td>15</td><td>{1 × |1|, 1 × |4|, 5 × |8|, 1 × |20|, 4 × |40|}</td></tr>
<tr><td>16</td><td>{1 × |1|, 1 × |3|, 2 × |6|, 4 × |12|, , 8 × |24|}</td></tr>
<tr><td>17</td><td>{1 × |1|, 8 × |36|}</td></tr>
<tr><td>18</td><td>{1 × |1|, 1 × |3|, 1 × |8|, 13 × |24|}</td></tr>
<tr><td>19</td><td>{1 × |1|, 2 × |9|, 19 × |18|}</td></tr>
<tr><td>20</td><td>{1 × |1|, 1 × |3|, 1 × |4|, 2 × |6|, 5 × |12|, 1 × |20|, 5 × |60|}</td></tr>
<tr><td>21</td><td>{1 × |1|, 1 × |8|, 27 × |16|}</td></tr>
<tr><td>22</td><td>{1 × |1|, 1 × |3|, 2 × |5|, 11 × |10|, 2 × |15|, 11 × |30|}</td></tr>
<tr><td>23</td><td>{1 × |1|, 11 × |48|}</td></tr>
<tr><td>24</td><td>{1 × |1|, 1 × |3|, 2 × |6|, 1 × |8|, 4 × |12|, 21 × |24|}</td></tr>
<tr><td>25</td><td>{1 × |1|, 1 × |4|, 6 × |20|, 5 × |100|}</td></tr>
<tr><td>26</td><td>{1 × |1|, 1 × |3|, 6 × |28|, 6 × |84|}</td></tr>
<tr><td>27</td><td>{1 × |1|, 1 × |8|, 3 × |24|, 9 × |72|}</td></tr>
<tr><td>28</td><td>{1 × |1|, 1 × |3|, 2 × |6|, 3 × |16|, 15 × |48|}</td></tr>
<tr><td>29</td><td>{1 × |1|, 4 × |7|, 58 × |14|}</td></tr>
<tr><td>30</td><td>{1 × |1|, 1 × |3|, 1 × |4|, 5 × |8|, 1 × |12|, 1 × |20|, 5 × |24|, 4 × |40|, 1 × |60|, 4 × |120|}</td></tr>
<tr><td>31</td><td>{1 × |1|, 2 × |15|, 31 × |30|}</td></tr>
<tr><td>32</td><td>{1 × |1|, 1 × |3|, 2 × |6|, 4 × |12|, , 8 × |24|, 16 × |48|}</td></tr>
<tr><td>33</td><td>{1 × |1|, 2 × |5|, 1 × |8|, 11 × |10|, 24 × |40|}</td></tr>
<tr><td>34</td><td>{1 × |1|, 1 × |3|, 32 × |36|}</td></tr>
<tr><td>35</td><td>{1 × |1|, 1 × |4|, 15 × |16|, 1 × |20|, 12 × |80|}</td></tr>
<tr><td>36</td><td>{1 × |1|, 1 × |3|, 2 × |6|, 1 × |8|, 53 × |24|}</td></tr>
<tr><td>37</td><td>{1 × |1|, 18 × |76|}</td></tr>
<tr><td>38</td><td>{1 × |1|, 1 × |3|, 8 × |9|, 76 × |18|}</td></tr>
<tr><td>39</td><td>{1 × |1|, 1 × |8|, 6 × |28|, 24 × |56|}</td></tr>
<tr><td>40</td><td>{1 × |1|, 1 × |3|, 1 × |4|, 2 × |6|, 25 × |12|, 1 × |20|, 21 × |60|}</td></tr>
<tr><td>41</td><td>{1 × |1|, 42 × |40|}</td></tr>
<tr><td>42</td><td>{1 × |1|, 1 × |3|, 1 × |8|, 27 × |16|, 1 × |24|, 27 × |48|}</td></tr>
<tr><td>43</td><td>{1 × |1|, 21 × |88|}</td></tr>
<tr><td>44</td><td>{1 × |1|, 1 × |3|, 2 × |5|, 2 × |6|, 11 × |10|, 2 × |15|, 59 × |30|}</td></tr>
<tr><td>45</td><td>{1 × |1|, 1 × |4|, 5 × |8|, 1 × |20|, 15 × |24|, 4 × |40|, 12 × |120|}</td></tr>
<tr><td>46</td><td>{1 × |1|, 1 × |3|, 44 × |48|}</td></tr>
<tr><td>47</td><td>{1 × |1|, 69 × |32|}</td></tr>
<tr><td>48</td><td>{1 × |1|, 1 × |3|, 2 × |6|, 1 × |8|, 4 × |12|, 93 × |24|}</td></tr>
<tr><td>49</td><td>{1 × |1|, 3 × |16|, 21 × |112|}</td></tr>
<tr><td>50</td><td>{1 × |1|, 1 × |3|, 1 × |4|, 1 × |12|, 6 × |20|, 6 × |60|, 5 × |100|, 5 × |300|}</td></tr>
<tr><td>990</td><td>{ 1 × |1|, 1 × |3|, 1 × |4|, 2 × |5|, 5 × |8|, 11 × |10|, 1 × |12|, 2 × |15|,
145 × |20|, 65 × |24|, 11 × |30|, 604 × |40|, 145 × |60|, 7852 × |120| }</td></tr>
<tr><td>991</td><td>{1 × |1|, 10 × |99|, 4955 × |198|}</td></tr>
<tr><td>992</td><td>{1 × |1|, 1 × |3|, 2 × |6|, 4 × |12|, 8 × |15|, 8 × |24|, 508 × |30|, 16 × |48|, 768 × |60|, 1536 × |120|, 3072 × |240|}</td></tr>
<tr><td>993</td><td>{1 × |1|, 1 × |8|, 6 × |55|, 993 × |110|, 1992 × |440|}</td></tr>
<tr><td>994</td><td>{1 × |1|, 1 × |3|, 3 × |16|, 2 × |35|, 3 × |48|, 71 × |70|, 2 × |105|, 71 × |210|, 432 × |560|, 432 × |1680|}</td></tr>
<tr><td>995</td><td>{1 × |1|, 1 × |4|, 18 × |11|, 1 × |20|, 1791 × |22|, 3600 × |44|, 3600 × |220|}</td></tr>
<tr><td>996</td><td>{1 × |1|, 1 × |3|, 2 × |6|, 1 × |8|, 5 × |24|, 5904 × |168|}</td></tr>
<tr><td>997</td><td>{1 × |1|, 498 × |1996|}</td></tr>
<tr><td>998</td><td>{1 × |1|, 1 × |3|, 8 × |249|, 1996 × |498|}</td></tr>
<tr><td>999</td><td>{1 × |1|, 1 × |8|, 3 × |24|, 9 × |72|, 18 × |76|, 72 × |152|, 216 × |456|, 648 × |1368|}</td></tr>
<tr><td>1000</td><td>{1 × |1|, 1 × |3|, 1 × |4|, 2 × |6|, 25 × |12|, 6 × |20|, 126 × |60|, 30 × |100|, 630 × |300|, 25 × |500|, 525 × |1500|}</td></tr>
</table>

$G(m)$ for $m = 3^n$, $m = 5^n$, $m = 11^n$; $1 \leq n \leq 4$.

m	G(m)
3	$\{1\times\lvert 1\rvert, 1\times\lvert 8\rvert\}$
$3^2 = 9$	$\{1\times\lvert 1\rvert, 1\times\lvert 8\rvert, 3\times\lvert 24\rvert\}$
$3^3 = 27$	$\{1\times\lvert 1\rvert, 1\times\lvert 8\rvert, 3\times\lvert 24\rvert, 9\times\lvert 72\rvert\}$
$3^4 = 81$	$\{1\times\lvert 1\rvert, 1\times\lvert 8\rvert, 3\times\lvert 24\rvert, 9\times\lvert 72\rvert, 27\times\lvert 216\rvert\}$
5	$\{1\times\lvert 1\rvert, 1\times\lvert 4\rvert, 1\times\lvert 20\rvert\}$
$5^2 = 25$	$\{1\times\lvert 1\rvert, 1\times\lvert 4\rvert, 6\times\lvert 20\rvert, 5\times\lvert 100\rvert\}$
$5^3 = 125$	$\{1\times\lvert 1\rvert, 1\times\lvert 4\rvert, 6\times\lvert 20\rvert, 30\times\lvert 100\rvert, 25\times\lvert 500\rvert\}$
$5^4 = 625$	$\{1\times\lvert 1\rvert, 1\times\lvert 4\rvert, 6\times\lvert 20\rvert, 30\times\lvert 100\rvert, 150\times\lvert 500\rvert, 125\times\lvert 2500\rvert\}$
11	$\{1\times\lvert 1\rvert, 2\times\lvert 5\rvert, 11\times\lvert 10\rvert\}$
$11^2 = 121$	$\{1\times\lvert 1\rvert, 2\times\lvert 5\rvert, 11\times\lvert 10\rvert, 2\times\lvert 55\rvert, 131\times\lvert 110\rvert\}$
$11^3 = 1331$	$\{1\times\lvert 1\rvert, 2\times\lvert 5\rvert, 11\times\lvert 10\rvert, 2\times\lvert 55\rvert, 131\times\lvert 110\rvert, 2\times\lvert 605\rvert, 1451\times\lvert 1210\rvert\}$
$11^4 = 14641$	$\{1\times\lvert 1\rvert, 2\times\lvert 5\rvert, 11\times\lvert 10\rvert, 2\times\lvert 55\rvert, 131\times\lvert 110\rvert, 2\times\lvert 605\rvert, 1451\times\lvert 1210\rvert, 2\times\lvert 6655\rvert, 15971\times\lvert 13310\rvert\}$

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