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Cover Page Footnote

This research was conducted as part of the SURIEM REU program at Lyman Briggs College of Michigan State University, under the supervision of Dr. Robert Bell. We gratefully acknowledge support from the National Security Agency (NSA Award No. H98230-18-1-0042), the National Science Foundation (NSF Award No. 1559776), and Michigan State University. We thank our mentors Dr. Katherine Raoux and David Storey for their guidance throughout this project. In addition, we would like to thank Dr. Peter Feller for his advice. Finally, we thank Eric Zhu for coding support.

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Gordian Adjacency for Positive Braid Knots

By *Tolson H. Bell*, *David C. Luo*, *Luke Seaton*, and *Samuel P. Serra*

Abstract. A knot K_1 is said to be *Gordian adjacent* to a knot K_2 if K_1 is an intermediate knot on an unknotting sequence of K_2. We extend previous results on Gordian adjacency by showing sufficient conditions for Gordian adjacency between classes of positive braid knots through manipulations of braid words. In addition, we explore unknotting sequences of positive braid knots and give a proof that there are only finitely many positive braid knots for a given unknotting number.

1 Introduction

A *mathematical knot* is a non-intersecting embedding of a closed curve in three- dimensional space that is considered unique only up to *isotopy*, a continuous deformation of this embedding without passing the curve through itself. We can visualize a knot by projecting the embedding onto a two-dimensional plane to form a *knot diagram*. A knot diagram contains *crossings*, where parts of the curve intersect in the projection. When we do pass one part of the curve through another, we have made a *crossing change*.

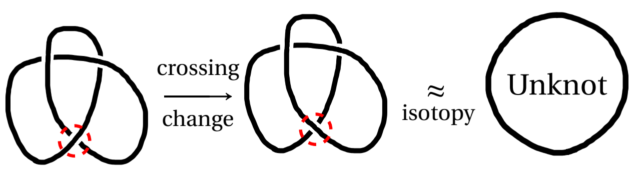

Figure 1: Crossing Change and Isotopy of the Left-handed Trefoil Knot

From these definitions, we obtain the concepts of *unknotting numbers* and *Gordian distance* between knots [1].

Definition 1.1 . *The Gordian distance* $d_g(K_1, K_2)$ *between two knots* K_1 *and* K_2 *is the minimal number of crossing changes needed to change* K_2 *into* K_1.

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Definition 1.2 . *The unknotting number* $u(\mathrm{K})$ *of a knot* K *is given by* $d_g(\mathrm{O},\mathrm{K})$*, where* O *is the unknot.*

Figure 1 shows that the unknotting number of the trefoil knot is no more than one, as it demonstrates how the trefoil can be made into the unknot with only one crossing change.

An important class of knots we study in great detail is torus knots. A *torus knot* is a closed curve on the surface of an unknotted torus that does not intersect itself anywhere [1]. We denote a torus knot by $\mathrm{T}(p,q)$, where p and q denote the number of longitudinal and meridional twists around the torus respectively such that p is coprime to q. Note that $\mathrm{T}(p,q)$ is isotopic to $\mathrm{T}(q,p)$, that is, $\mathrm{T}(p,q) = \mathrm{T}(q,p)$ [1].

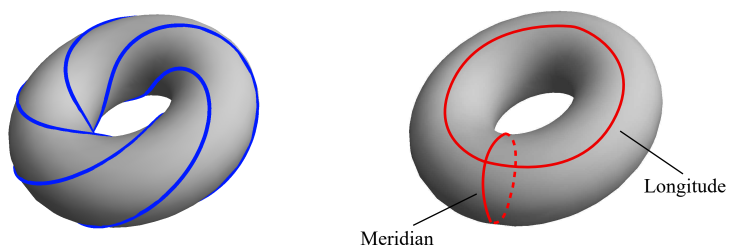

Figure 2: T(2,5) and the Axes of the Torus

In 2014, Peter Feller introduced the concept of *Gordian adjacency* between knots [6], an idea closely related to the description of *unknotting sequences.*

Definition 1.3 (Feller Definition 1)**.** *A knot* K_1 *is said to be Gordian adjacent to a knot* K_2*,* $\mathrm{K}_1 \leq_g \mathrm{K}_2$*, if* $d_g(\mathrm{K}_1,\mathrm{K}_2) = u(\mathrm{K}_2) - u(\mathrm{K}_1)$.

Definition 1.4 . *An unknotting sequence of a knot* K *is a series of knots beginning with* K *and ending with the unknot* O *such that for any two consecutive knots* K_2 *followed by* K_1 *in the series,* $u(\mathrm{K}_2) - u(\mathrm{K}_1) = 1$ *and* $d_g(\mathrm{K}_1,\mathrm{K}_2) = 1$.

Note that in our definition of an unknotting sequence, it is optimal in the sense that it is the fewest number of knots needed to get from a particular knot to the unknot.

In general, the Gordian distance is difficult to compute. However, if K_1 and K_2 are Gordian adjacent, then their Gordian distance is the difference of their unknotting numbers. An equivalent definition for Gordian adjacency between knots K_1 and K_2 is that $\mathrm{K}_1 \leq_g \mathrm{K}_2$ if K_1 appears in an unknotting sequence of K_2 [6].

Feller was able to show that under certain circumstances, Gordian adjacencies are guaranteed between certain classes of torus knots [6].

Theorem 1.1 (Feller Theorem 2)**.** *If* (n, m) *and* (a, b) *are pairs of coprime positive integers with* $n \le a$ *and* $m \le b$*, then* $\mathrm{T}(n, m) \le_g \mathrm{T}(a, b)$.

Theorem 1.2 (Feller Theorem 3)**.** *If* n *and* m *are positive integers with* n *odd and* m *not a multiple of 3, then* $\mathrm{T}(2, n) \le_g \mathrm{T}(3, m)$ *if and only if* $n \le \frac{4}{3}m + \frac{1}{3}$.

Our main goal in this paper is to build off Feller's results by finding Gordian adjacencies among other classes of positive braid knots. In Section 3, we describe new techniques which help prove our results by representing positive braid knots by their braid words. In Section 4, we present our results on Gordian adjacency for certain classes of positive braid knots. Listed below are the major theorems we obtain.

Theorem 4.2. *If* n *and* k *are positive integers, then* $\mathrm{T}(n, n^2k + 1) \le_g \mathrm{T}(n+1, (n^2 - 1)k + 1)$.

Theorem 4.3. *If* n *and* k *are positive integers, then* $\mathrm{T}(n, n^2k + n + 1) \le_g \mathrm{T}(n+1, (n^2 - 1)k + n)$.

Theorem 4.4. *If* a *and* b *are positive integers, then* $\mathrm{T}(3, a) \le_g \mathrm{T}(4, b)$ *if* $a \le \frac{9b+5}{8}$.

Theorem 4.5. *If* a *and* b *are positive integers, then* $\mathrm{T}(2, a) \le_g \mathrm{T}(4, b)$ *if* $a \le \frac{3b+3}{2}$.

Theorem 4.6. *Let* β *in* B_n *be a positive braid where* $\beta = \beta' w$ *and* $\hat{\beta}$ *and* $\hat{\beta}'$ *are knots. If* $\hat{w}$ *is a link with* n *components, then* $\hat{\beta}' \le_g \hat{\beta}$.

Using our new techniques, we present a proof in Section 5 showing that every positive braid knot can be unknotted in a way such that every intermediate knot is a positive braid knot. We also prove that every positive braid knot has only a finite number of these positive unknotting sequences.

Theorem 5.1. *Every positive braid knot has an unknotting sequence that consists of only positive braid knots.*

Theorem 5.2. *For every positive integer* m*, there exist a finite number of positive braid knots with unknotting number* m.

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2 Preliminaries: Braids and Torus Knots

Throughout this paper, we refer to knots by their braid words. In this section, we present the braid group and describe how to represent positive braid knots as braids.

A braid β is defined as a set of n strands which begin on a horizontal bar and end on a lower horizontal bar. Each strand may only intersect any horizontal plane once, which allows the strands to cross each other in the specific ways described later [1].

The *closure* of a braid, $\hat{\beta}$, is formed by attaching the top and bottom bar such that the beginning of each strand connects to the end of a strand, forming a *link*, one or more knots [9]. Every knot can be represented as the closure of some braid [2].

To represent braids algebraically, we refer to their generators σ_i's which represent the $i+1^{st}$ strand from the left crossing over the i^{th} strand (read from top to bottom). If the $i+1^{st}$ strand from the left instead crosses under the i^{th} strand, we denote this by σ_i^{-1}.

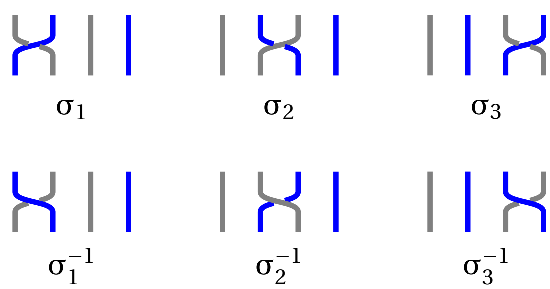


Figure 3: The Generators of B_4 and Their Inverses

We call σ_i^{-1} the inverse of σ_i as it is isotopic to the identity element when concatenated with σ_i. Hence we have that $\sigma_1, \ldots, \sigma_{n-1}$ are the generators of the braid group B_n, which form a group under the operation of concatenation. Note that changing a generator to its inverse corresponds to a crossing change in the knot which the braid represents. We have two relations on the braid group B_n, also known as braid isotopies:

- for any $1 \leq i \leq n-2$, $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$,
- for any $1 \leq i \leq n-3$ and $i+2 \leq j \leq n-1$, $\sigma_j\sigma_i = \sigma_i\sigma_j$.

Both of these braid relations produce isotopic braid closures [3]. Given a knot K, the *braid index* $brd(K)$ of K is the smallest positive integer n such that there exists a braid β in B_n where $\hat{\beta} = K$.

We refer to a positive generator of B_n as a *positive crossing*. A *positive braid* is a braid in which all crossings are positive (we have no inverse generators). A *positive braid knot* is a knot which is isotopic to the closure of some positive braid. While unknotting numbers can be hard to compute in general, we have that for positive braid knots, the unknotting number of $\hat{\beta}$ which is an integer quantity, is given by

$$u(\hat{\beta}) = \frac{\ell - n + 1}{2}$$

where ℓ denotes the length of the braid word (the number of generators it contains) and n the number of strands on which the braid is expressed. This was proven as a lower bound in 1984 [4] and as an upper bound in 2004 [7].

In our study of torus knots, we usually represent torus knots by using braids that they are the closure of. The torus knot $T(p,q)$ can be represented by the braid word $\sigma_{p-1}\cdots\sigma_1$ repeated q times, which we denote as $(\sigma_{p-1}\cdots\sigma_1)^q$ [1].

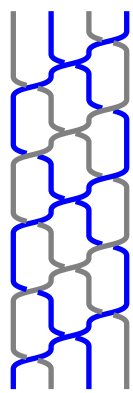


Figure 4: T(4,5) as a Braid in B_4

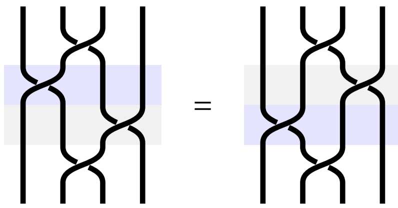
By flipping this braid representation over a vertical axis (viewing it from behind), we see that $(\sigma_1\cdots\sigma_{p-1})^q$ is also a valid braid representation of $T(p,q)$. Since torus knots are a particular class of positive braid knots, the braid representation of torus knots tells us that

$$u\big(T(p,q)\big) = \frac{(p-1)(q-1)}{2}.$$

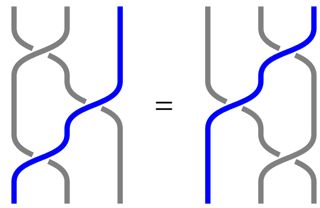
3 Rules for Constructing Gordian Adjacencies

Once we have represented two knots as positive braids, we wish to manipulate these braids in order to uncover Gordian adjacencies between the original knots. We have reduced these manipulations to the application of five rules. Figure 5 shows a graphical representation of these five rules, which we summarize in the following paragraph.

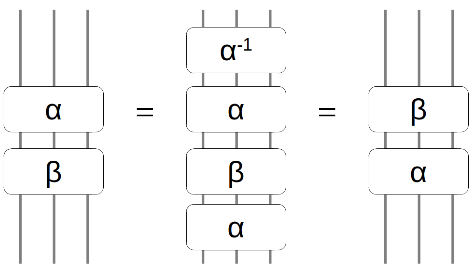
The Distant Generators Rule and Neighboring Generators Rule are relations on the braid group and therefore do not change the braid closure. The Conjugation Rule also



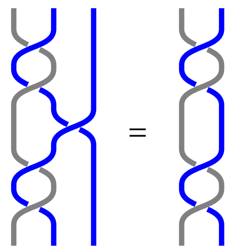
Distant Generators Rule: If $|i-j|>1$, then $\sigma_i\sigma_j=\sigma_j\sigma_i$.



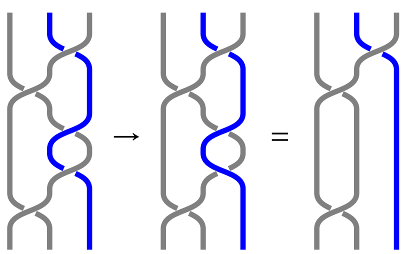
Neighboring Generators Rule: For any $1\le i\le n-2$, $\sigma_i\sigma_{i+1}\sigma_i=\sigma_{i+1}\sigma_i\sigma_{i+1}$.



Conjugation Rule: The closure of the braid $\alpha\beta$ is isotopic to the closure of $\beta\alpha$ via a conjugation by α.



Markov Destabilization Rule: If the braid word contains σ_n and $i<n$ for every other σ_i in the braid word, then σ_n can be deleted.



Crossing Change Rule: If the braid word contains $\sigma_i\sigma_i$, then these two letters can be deleted.

Figure 5: Examples and Descriptions of the Five Rules

does not change the braid closure. These three rules are bidirectional, while the latter two rules are not. The Markov Destabilization Rule similarly does not change the closure of the braid by Markov's Theorem [8]. The Crossing Change Rule is the only rule that changes the closure of a braid up to knot isotopy. It arises from performing a crossing change which switches a generator to its inverse and canceling out the resulting pair.

Throughout this paper, we use an equals sign to denote that the closure of our new braid word is isotopic to the closure of the braid word preceding it and an arrow to denote when we have made one or more crossing changes via the Crossing Change Rule. We note that none of these five rules change the number of components in the closure of our braid word.

Lemma 3.1 . *If we start with a positive braid word* β *whose braid closure is a knot and, by the successive application of the five rules, change* β *into the empty braid word, then the sequence of knots corresponding to the closures of all braid words in the process (up to isotopy) forms an unknotting sequence of* $\hat{\beta}$.

Proof. Since none of the five rules change the number of components in our braid closure, this sequence ends with the unknot, the closure of the empty braid word on one strand. The only time we change our braid closure beyond isotopy is when we use the Crossing Change Rule which creates exactly one crossing change. As this is a sequence that starts from a knot and ends with the unknot, to show that this is an unknotting sequence it is enough to prove that every crossing change decreases the unknotting number by one. Recall that the unknotting number of a positive braid knot is given by $\frac{\ell-n+1}{2}$ where n is the number of strands on which the braid knot is expressed and ℓ is the length of the braid word [7]. An application of the Crossing Change Rule does not change n but decreases ℓ by two. Therefore, it also decreases the unknotting number by one. □

Lemma 3.2 . *Let* β' *be a subword of a positive braid word* β *where* $i \leq n$ *for all* σ_i *in* β'. *Using the five rules,* β *can be made into a braid word that replaces* β' *with a subword containing no more than one* σ_n.

Proof. We proceed by induction on ℓ', the length of β'. Our base case is $\ell' = 0$, when clearly β' is already in the required form for any n. Note that our inductive hypothesis requires that the replacement for β' does not have any generators with index higher than n and is not longer than β' was.

Now, given some β' with $\ell' > 0$, let β'' be the subword that includes all of β' except for the first generator of β'. Because $\ell'' < \ell'$, we can use the inductive hypothesis with the same value of n to replace β'' with a subword that contains at most one σ_n. Now, if the first generator of β' is not σ_n or if the replacement for β'' has no σ_n, we are done, as β' is already in the required form.

Otherwise, we have exactly two σ_n. Let γ be the subword between the two σ_n, not including either σ_n. As the length of γ is less than the original length of β', we can use the inductive hypothesis on γ with $n-1$. Now, the replacement for γ has either one or zero σ_{n-1}, and no generator with index higher than $n-1$.

If the replacement for γ has no σ_{n-1}, we can then move the second σ_n right next to the first using the Distant Generators Rule. Then, we can delete the two σ_n via the Crossing Change Rule. This leaves β' with no σ_n, so we are done.

If the replacement for γ has one σ_{n-1}, we can move the two σ_n to either side of that σ_{n-1} using the Distant Generators Rule. We then use the Neighboring Generators Rule to make $\sigma_n\sigma_{n-1}\sigma_n$ into $\sigma_{n-1}\sigma_n\sigma_{n-1}$. This leaves β' with exactly one σ_n, so we are done. □

The preceding algorithm does not use the Markov Destabilization Rule or the Conjugation Rule, meaning it can be performed locally on any subword of a braid word and still follow the five rules on the whole braid word.

Lemma 3.3 . *Every finite positive braid word can be made into the empty braid word using the five rules.*

Proof. Let σ_n be the letter with the highest subscript in the braid word. By Lemma 3.2, we can make our braid word into a braid word with no more than one σ_n. If this word has a σ_n, delete it using the Markov Destabilization Rule. Then do this on the word again, noting that our highest subscript is now lower than it was before. With each repetition, n decreases by at least one, so eventually the braid word will be the empty word. Because Lemma 3.2 terminates in a finite number of steps, we will always get to the empty word in a finite number of steps. □

Theorem 3.1 . *Let β and β' be positive braid words whose braid closures are knots. If β can be made into β' using the five rules, then $\hat{\beta}' \leq_g \hat{\beta}$.*

Proof. By Lemma 3.3, we know that we can turn β' into the empty braid word using the five rules. Because the five rules do not change the number of components in the closure, that empty braid word corresponds to the unknot. Therefore, if we can go from β to β' using these rules, we can then combine that sequence with the sequence from β' to the unknot to be able to go from β to the unknot using only the five rules. Using Lemma 3.1, we have that this sequence is an unknotting sequence of $\hat{\beta}$. Therefore, $\hat{\beta}'$ is an intermediate knot on an unknotting sequence of $\hat{\beta}$, implying $\hat{\beta}' \leq_g \hat{\beta}$. □

Theorem 3.1 tells us that if we get from one positive braid word to another using these five rules, then we do not need to check how many crossing changes we have used, as the braid closure of the ending braid word is guaranteed to be Gordian adjacent to the braid closure of the starting braid word.

4 Sufficient Conditions for Gordian Adjacency

In this section, we present our main results on Gordian adjacency dealing with certain classes of positive braid knots. We do this constructively using the five rules introduced in the previous section. To shorten the sequence of knots generated by Lemma 3.2, we formulate multi-step moves that are combinations of our rules.

Lemma 4.1 . *If n is a positive integer and β is a positive braid on $n+1$ strands with no σ_n in its braid word, then $\beta\sigma_n\cdots\sigma_1\sigma_1\cdots\sigma_n = \sigma_n\cdots\sigma_1\sigma_1\cdots\sigma_n\beta$ as braids (not allowing conjugation or Markov destabilization).*

Proof. Let $1 \le i \le n-1$. We will show that $\sigma_n\cdots\sigma_1\sigma_1\cdots\sigma_n$ commutes with σ_i. Since the braid word of β consists of a series of σ_i of this form, this implies the lemma.

$\sigma_n\cdots\sigma_1\sigma_1\cdots\sigma_n\sigma_i$
$= \sigma_n\cdots\sigma_1\sigma_1\cdots\sigma_i\sigma_{i+1}\sigma_i\cdots\sigma_n$ by repeating the Distant Generators Rule
$= \sigma_n\cdots\sigma_1\sigma_1\cdots\sigma_{i+1}\sigma_i\sigma_{i+1}\cdots\sigma_n$ by the Neighboring Generators Rule
$= \sigma_n\cdots\sigma_{i+1}\sigma_i\sigma_{i+1}\cdots\sigma_1\sigma_1\cdots\sigma_i\sigma_{i+1}\cdots\sigma_n$ by repeating the Distant Generators Rule
$= \sigma_n\cdots\sigma_i\sigma_{i+1}\sigma_i\cdots\sigma_1\sigma_1\cdots\sigma_n$ by the Neighboring Generators Rule
$= \sigma_i\sigma_n\cdots\sigma_{i+1}\sigma_i\cdots\sigma_1\sigma_1\cdots\sigma_n$ by repeating the Distant Generators Rule
$= \sigma_i\sigma_n\cdots\sigma_1\sigma_1\cdots\sigma_n$.

Since the Distant Generators Rule and the Neighboring Generators Rule are braid isotopies, we have that σ_i commutes with $\sigma_n\cdots\sigma_1\sigma_1\cdots\sigma_n$ only through braid isotopies. □

We can also think about the proof of Lemma 4.1 pictorially. The word $\sigma_n\cdots\sigma_1\sigma_1\cdots\sigma_n$ represents the furthest strand to the right on a braid on $n+1$ strands being wrapped once around the other n strands. Any positive braid on the other n strands has no interaction with the rightmost strand, and so can pass through the rightmost strand. Figure 4 shows this process for $n = 5$.

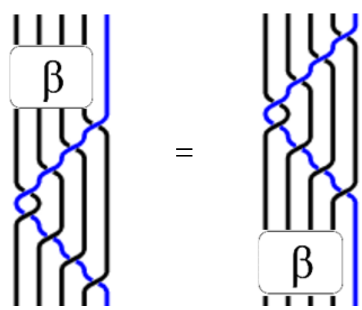


Figure 6: Lemma 4.1

The following standard definition will be of great use in proofs of our major results.

Definition 4.2 . *A full twist on n strands, Δ_n^2, is the braid $(\sigma_{n-1}\cdots\sigma_1)^n$. Δ_n^2 is in the center of the braid group B_n, that is, Δ_n^2 commutes with σ_i for any $1 \le i \le n-1$ [9].*

Lemma 4.3 . *If n is a positive integer, then $\Delta_n^2 = \sigma_{n-1}\sigma_{n-2}\cdots\sigma_1\sigma_1\cdots\sigma_{n-2}\sigma_{n-1}(\Delta_{n-1}^2) = (\Delta_{n-1}^2)\sigma_{n-1}\sigma_{n-2}\cdots\sigma_1\sigma_1\cdots\sigma_{n-2}\sigma_{n-1}$ as braids (not allowing conjugation or Markov destabilization).*

Proof. By definition, $\Delta_n^2 = (\sigma_{n-1}\cdots\sigma_1)(\sigma_{n-1}\cdots\sigma_1)^{n-1}$. Because $T(n, n-1) = T(n-1, n)$ [1] and $(\sigma_1\cdots\sigma_{n-2})^n$ is a valid braid word for $T(n-1, n)$ as explained in Section 2, the closure of $(\sigma_{n-1}\cdots\sigma_1)^{n-1}$ is isotopic to that of $(\sigma_1\cdots\sigma_{n-2})^n$. As $(\sigma_1\cdots\sigma_{n-2})\sigma_{n-1}(\sigma_1\cdots\sigma_{n-2})^{n-1}$ can be made into $(\sigma_1\cdots\sigma_{n-2})^n$ by the Markov Destabilization Rule, these two braids have isotopic closures, and thus the closure of $(\sigma_1\cdots\sigma_{n-2})\sigma_{n-1}(\sigma_1\cdots\sigma_{n-2})^{n-1}$ is also isotopic that of $(\sigma_{n-1}\cdots\sigma_1)^{n-1}$. Etnyre and Van Horn-Morris proved that any positive braids whose closure represent the same link are related by positive Markov moves, braid isotopy, and conjugation [5]. Then as $(\sigma_{n-1}\cdots\sigma_1)^{n-1}$ and $(\sigma_1\cdots\sigma_{n-2})\sigma_{n-1}(\sigma_1\cdots\sigma_{n-2})^{n-1}$ are positive braids on the same number of strands with isotopic braid closures, these two braids must be related by the Conjugation Rule and isotopy. In fact, the repetitive nature of the braid words means that the Conjugation Rule is not necessary, as can be seen in Figure 7. Since this relation requires only braid isotopies, it can be done on any section of the overall braid word. This implies

$$\begin{aligned}\Delta_n^2 &= (\sigma_{n-1}\cdots\sigma_1)^n\\ &= (\sigma_{n-1}\cdots\sigma_1)(\sigma_{n-1}\cdots\sigma_1)^{n-1}\\ &= (\sigma_{n-1}\cdots\sigma_1)(\sigma_1\cdots\sigma_{n-2})\sigma_{n-1}(\sigma_1\cdots\sigma_{n-2})^{n-1}\\ &= \sigma_{n-1}\sigma_{n-2}\cdots\sigma_1\sigma_1\cdots\sigma_{n-2}\sigma_{n-1}(\Delta_{n-1}^2).\end{aligned}$$

Lemma 4.1 tells us that these two sections commute. □

Lemma 4.3 can be thought of pictorially as pulling the strand that starts furthest to the right as far up the braid as possible. Figure 7 shows this process for $n = 4$.

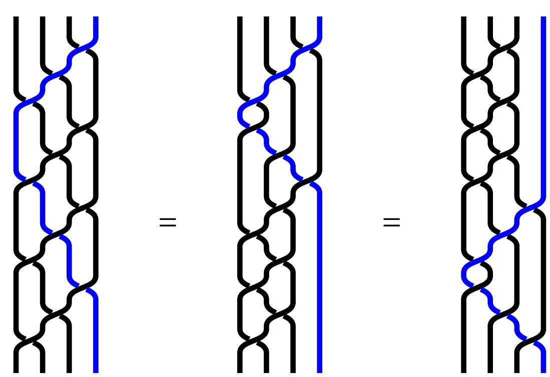


Figure 7: Lemma 4.3

Bear in mind that Lemma 4.3 can be repeated multiple times to obtain

$$\begin{aligned}\Delta_n^2 &= (\sigma_{n-1}\cdots\sigma_1\sigma_1\cdots\sigma_{n-1})(\Delta_{n-1}^2)\\ &= (\sigma_{n-1}\cdots\sigma_1\sigma_1\cdots\sigma_{n-1})(\sigma_{n-2}\cdots\sigma_1\sigma_1\cdots\sigma_{n-2})(\Delta_{n-2}^2)\\ &= (\sigma_{n-1}\cdots\sigma_1\sigma_1\cdots\sigma_{n-1})(\sigma_{n-2}\cdots\sigma_1\sigma_1\cdots\sigma_{n-2})\cdots(\sigma_2\sigma_1\sigma_1\sigma_2)(\sigma_1\sigma_1).\end{aligned}$$

By Lemma 4.1, each of the groups in parentheses above can all commute with one another. Furthermore, these lemmas can be done on any subsequence of our braid word as they only use braid isotopy and follow the five rules on our overall braid word.

With these relations in mind, we now construct Gordian adjacencies between specific classes of torus knots. In each step of the following proofs, we do exactly one of the following:

- rewrite the braid word in a different form for better clarity,
- commute sections of our braid word using some combination of Lemma 4.1 and the Distant Generators Rule,
- use one of Lemma 4.3, Markov Destabilization Rule, or Neighboring Generators Rule on some section or sections of our braid word,
- make crossing changes via the Crossing Change Rule.

Lemma 4.4 . *If n and k are positive integers with $n \geq 2$, then* $(\sigma_{n-1}\cdots\sigma_1)^{nk+1} = (\sigma_{n-1}\cdots\sigma_1\sigma_1\cdots\sigma_{n-1})^k\sigma_{n-1}\cdots(\sigma_2\sigma_1\sigma_1\sigma_2)^k\sigma_2(\sigma_1\sigma_1)^k\sigma_1$ *using braid isotopies and the Conjugation Rule.*

Proof. We proceed by induction on n. When $n = 2$, $(\sigma_1)^{2k+1} = (\sigma_1\sigma_1)^k\sigma_1$. Now assume that this theorem holds true when $n = a$ for some $a \geq 2$. When $n = a+1$, we find

$$\begin{aligned}&(\sigma_a\cdots\sigma_1)^{(a+1)k+1}\\ &= (\Delta_{a+1}^2)^k\sigma_a\cdots\sigma_1\\ &= (\Delta_a^2\sigma_a\cdots\sigma_1\sigma_1\cdots\sigma_a)^k\sigma_a\cdots\sigma_1 \text{ by Lemma 4.3}\\ &= (\Delta_a^2)^k(\sigma_a\cdots\sigma_1\sigma_1\cdots\sigma_a)^k\sigma_a\cdots\sigma_1 \text{ by Lemma 4.1}\\ &= (\sigma_a\cdots\sigma_1\sigma_1\cdots\sigma_a)^k\sigma_a\cdots\sigma_1(\Delta_a^2)^k \text{ by the Conjugation Rule}\\ &= (\sigma_a\cdots\sigma_1\sigma_1\cdots\sigma_a)^k\sigma_a(\Delta_a^2)^k\sigma_{a-1}\cdots\sigma_1 \text{ as } \Delta_a^2 \text{ commutes with } \sigma_i \text{ for all } i \leq a-1\\ &= (\sigma_a\cdots\sigma_1\sigma_1\cdots\sigma_a)^k\sigma_a(\sigma_{a-1}\cdots\sigma_1)^{ak+1}\\ &= (\sigma_a\cdots\sigma_1\sigma_1\cdots\sigma_a)^k\sigma_a(\sigma_{a-1}\cdots\sigma_1\sigma_1\cdots\sigma_{a-1})^k\sigma_{a-1}\cdots(\sigma_2\sigma_1\sigma_1\sigma_2)^k\sigma_2(\sigma_1\sigma_1)^k\sigma_1.\end{aligned}$$

Hence we have that the last step follows from our inductive hypothesis. □

Note that the conversion in the preceding lemma is bidirectional, as all braid isotopies and the Conjugation Rule are reversible. Later, we will use the reverse direction of Lemma 4.4. Since Lemmas 4.1, 4.3, and 4.4 are combinations of our five rules, they follow Theorem 3.1, implying each of these processes results in Gordian adjacencies.

We now focus on constructing Gordian adjacencies between torus knots whose indices differ by one. When going from $\mathrm{T}(a,b)$ to $\mathrm{T}(a-1,c)$ using the five rules, the first question that arises is how many crossing changes are required to eliminate all σ_{a-1} from our braid word. The theorem below shows that this process can be done in $\left\lfloor \frac{b}{a} \right\rfloor$ steps.

Proposition 4.1 . *For all positive integers a and b such that a is coprime to b, there exists* β *in* B_{a-1} *such that* $\hat{\beta} \leq_g \mathrm{T}(a,b)$ *and* $d_g(\mathrm{T}(a,b),\hat{\beta}) = \left\lfloor \frac{b}{a} \right\rfloor$.

Proof. $\mathrm{T}(a,b)$ can be represented by the braid word $(\Delta_a^2)^{\left\lfloor \frac{b}{a} \right\rfloor}(\sigma_{a-1}\cdots\sigma_1)^{(b \pmod a)}$. Taking the second section of this word, as $\mathrm{T}(a, b \pmod a) = \mathrm{T}(b \pmod a, a)$ and $b \pmod a < a$, the braid word of $\mathrm{T}(a, b \pmod a)$ can be changed into the braid word of $\mathrm{T}(b \pmod a, a)$ using only braid isotopies and Markov destabilizations [5]. Note that we can still use the Conjugation Rule because Δ_a^2 commutes with all letters in our braid word. So there is a path using our rules from $\mathrm{T}(a, b \pmod a)$ to $\mathrm{T}(b \pmod a, a)$. On this path, we can pause before the Markov Destabilization Rule is used for the first time. At this point, the subword has only one copy of σ_{a-1}. Moreover, we have only used our rules on our overall braid word as we have only used braid isotopies.

By Lemma 4.3,

$$\begin{aligned}(\Delta_a^2)^{\left\lfloor \frac{b}{a} \right\rfloor} &= (\sigma_{a-1}\cdots\sigma_1\sigma_1\cdots\sigma_{a-1}\Delta_{a-1}^2)^{\left\lfloor \frac{b}{a} \right\rfloor}\\ &= (\sigma_{a-1}\cdots\sigma_1\sigma_1\cdots\sigma_{a-1})^{\left\lfloor \frac{b}{a} \right\rfloor}(\Delta_{a-1}^2)^{\left\lfloor \frac{b}{a} \right\rfloor}.\end{aligned}$$

We can then commute $(\sigma_{a-1}\cdots\sigma_1\sigma_1\cdots\sigma_{a-1})^{\left\lfloor \frac{b}{a} \right\rfloor}$ so that it is next to the final σ_{a-1} made by the process above. We know this is possible because for every other σ_i in that section of the braid word, $i < a-1$, so σ_i commutes with $(\sigma_{a-1}\cdots\sigma_1\sigma_1\cdots\sigma_{a-1})$ by Lemma 4.1. This results in the section of a braid word $(\sigma_{a-1}\cdots\sigma_1\sigma_1\cdots\sigma_{a-1})^{\left\lfloor \frac{b}{a} \right\rfloor}\sigma_{a-1}$ which is the only section of the braid word that contains σ_{a-1}. We use the Crossing Change Rule to delete all consecutive pairs of σ_{a-1}. Since there are $\left\lfloor \frac{b}{a} \right\rfloor$ of these pairs, we need to use $\left\lfloor \frac{b}{a} \right\rfloor$ crossing changes. This leaves only one σ_{a-1}, which we can delete using the Markov Destabilization Rule. We are left with β. Gordian adjacency follows as we have only used the five rules. □

For the following proofs of our theorems and lemmas, we abbreviate “crossing changes" as “CC*s*" for brevity.

Theorem 4.2 . *If n and k are positive integers, then* $\mathrm{T}(n, n^2k+1) \leq_g \mathrm{T}(n+1,(n^2-1)k+1)$.

Proof. We first remove all σ_n from our braid word using the process described in Proposition 4.1.

$$\begin{aligned}
&\mathrm{T}(n+1,(n^2-1)k+1)\\
&=(\sigma_n\cdots\sigma_1)^{(n+1)(n-1)k+1}\\
&=(\Delta_{n+1}^2)^{(n-1)k}\sigma_n\cdots\sigma_1\\
&=(\sigma_n\cdots\sigma_1\sigma_1\cdots\sigma_n\Delta_n^2)^{(n-1)k}\sigma_n\cdots\sigma_1 \text{ by Lemma 4.3}\\
&=(\Delta_n^2)^{(n-1)k}(\sigma_n\cdots\sigma_1\sigma_1\cdots\sigma_n)^{(n-1)k}\sigma_n\cdots\sigma_1\\
&\to(\Delta_n^2)^{(n-1)k}\sigma_n(\sigma_{n-1}\cdots\sigma_1\sigma_1\cdots\sigma_{n-1})^{(n-1)k}\sigma_{n-1}\cdots\sigma_1 (\text{applying } (n-1)k \text{ CC}s)\\
&=(\Delta_n^2)^{(n-1)k}((\sigma_{n-1}\cdots\sigma_1\sigma_1\cdots\sigma_{n-1})^k)^{n-1}\sigma_{n-1}\cdots\sigma_1.
\end{aligned}$$

Next, we remove σ_{n-1} from all but one of the $n-1$ copies of $(\sigma_{n-1}\cdots\sigma_1\sigma_1\cdots\sigma_{n-1})^k$, then remove σ_{n-2} from all of the copies from which σ_{n-1} was removed except one, and so on:

$(\Delta_n^2)^{(n-1)k}((\sigma_{n-1}\cdots\sigma_1\sigma_1\cdots\sigma_{n-1})^k)^{n-1}\sigma_{n-1}\cdots\sigma_1$
applying $(n-2)k$ CCs we find
$\to(\Delta_n^2)^{(n-1)k}(\sigma_{n-1}\cdots\sigma_1\sigma_1\cdots\sigma_{n-1})^k\sigma_{n-1}((\sigma_{n-2}\cdots\sigma_1\sigma_1\cdots\sigma_{n-2})^k)^{n-2}\sigma_{n-2}\cdots\sigma_1$

$$\vdots$$

applying $3k$ CCs,
$\to(\Delta_n^2)^{(n-1)k}(\sigma_{n-1}\cdots\sigma_1\sigma_1\cdots\sigma_{n-1})^k\sigma_{n-1}\cdots(\sigma_3\sigma_2\sigma_1\sigma_1\sigma_2\sigma_3)^{3k}\sigma_3\sigma_2\sigma_1$
applying $2k$ CCs,
$\to(\Delta_n^2)^{(n-1)k}(\sigma_{n-1}\cdots\sigma_1\sigma_1\cdots\sigma_{n-1})^k\sigma_{n-1}\cdots(\sigma_3\sigma_2\sigma_1\sigma_1\sigma_2\sigma_3)^k\sigma_3(\sigma_2\sigma_1\sigma_1\sigma_2)^{2k}\sigma_2\sigma_1$
finally, applying k CCs,
$\to(\Delta_n^2)^{(n-1)k}(\sigma_{n-1}\cdots\sigma_1\sigma_1\cdots\sigma_{n-1})^k\sigma_{n-1}\cdots(\sigma_3\sigma_2\sigma_1\sigma_1\sigma_2\sigma_3)^k\sigma_3(\sigma_2\sigma_1\sigma_1\sigma_2)^k\sigma_2(\sigma_1\sigma_1)^k\sigma_1$

Lemma 4.4 tells us that our braid word, excluding the initial $(\Delta_n^2)^{(n-1)k}$, can be converted to $(\sigma_{n-1}\cdots\sigma_1)^{nk+1}$ using braid isotopy and the Conjugation Rule. Bear in mind that we can still use the Conjugation Rule on this subword here because Δ_n^2 commutes with everything. This implies

$$\begin{aligned}
&(\Delta_n^2)^{(n-1)k}(\sigma_{n-1}\cdots\sigma_1\sigma_1\cdots\sigma_{n-1})^k\sigma_{n-1}\cdots(\sigma_3\sigma_2\sigma_1\sigma_1\sigma_2\sigma_3)^k\sigma_3(\sigma_2\sigma_1\sigma_1\sigma_2)^k\sigma_2(\sigma_1\sigma_1)^k\sigma_1\\
&=(\Delta_n^2)^{(n-1)k}(\sigma_{n-1}\cdots\sigma_1)^{nk+1} \text{ using Lemma 4.4}\\
&=(\sigma_{n-1}\cdots\sigma_1)^{n^2k+1}\\
&=\mathrm{T}(n,n^2k+1).
\end{aligned}$$

Note that $u(\mathrm{T}(n+1,(n^2-1)k+1)-u(\mathrm{T}(n,n^2k+1)=\frac{(n)((n^2-1)k)}{2}-\frac{(n-1)(n^2k)}{2}=\frac{n^2k-nk}{2}=\frac{(n)(n-1)}{2}k$. The number of crossing changes we have made is $(n-1)k+(n-2)k+\cdots+2k+k=\frac{(n)(n-1)}{2}k$, which is exactly the difference in their unknotting numbers. Therefore, the torus knot $\mathrm{T}(n,n^2k+1)\leq_g \mathrm{T}(n+1,(n^2-1)k+1)$. □

Theorem 4.3 . *If n and k are positive integers, then* $\mathrm{T}(n,n^2k+n+1)\leq_g \mathrm{T}(n+1,(n^2-1)k+n)$.

Proof. We first note that by isotopy, the subword $(\sigma_n\cdots\sigma_1)^n$ can be changed into $\sigma_n(\sigma_{n-1}\cdots\sigma_1)^{n+1}$ using identical logic as in the proof of Lemma 4.3.

$$\begin{aligned}
&\mathrm{T}(n+1,(n^2-1)k+n)\\
&=(\sigma_n\cdots\sigma_1)^{(n+1)(n-1)k+n}\\
&=((\sigma_n\cdots\sigma_1)^{(n+1)})^{(n-1)k}(\sigma_n\cdots\sigma_1)^n\\
&=(\Delta_{n+1}^2)^{(n-1)k}\sigma_n(\sigma_{n-1}\cdots\sigma_1)^{n+1}\\
&=(\Delta_{n+1}^2)^{(n-1)k}\sigma_n\sigma_{n-1}\cdots\sigma_1(\sigma_{n-1}\cdots\sigma_1)^n\\
&=(\Delta_{n+1}^2)^{(n-1)k}\sigma_n\cdots\sigma_1\Delta_n^2\\
&=(\sigma_n\cdots\sigma_1\sigma_1\cdots\sigma_n\Delta_n^2)^{(n-1)k}\sigma_n\cdots\sigma_1\Delta_n^2\\
&=(\Delta_n^2)^{(n-1)k}(\sigma_n\cdots\sigma_1\sigma_1\cdots\sigma_n)^{(n-1)k}\sigma_n\cdots\sigma_1\Delta_n^2\\
&=(\Delta_n^2)^{(n-1)k+1}(\sigma_n\cdots\sigma_1\sigma_1\cdots\sigma_n)^{(n-1)k}\sigma_n\cdots\sigma_1.
\end{aligned}$$

From here, we proceed in the same way as in the proof of Theorem 4.2. Since we have one extra Δ_n^2, we end with $\mathrm{T}(n,n^2k+n+1)$ as the second number is increased by n from the previous case. □

When proving adjacencies between torus knots of the form $\mathrm{T}(n,a)$ and $\mathrm{T}(n+1,b)$, Theorems 4.2 and 4.3 cover two cases: when b is congruent to either 1 or n modulo n^2-1 respectively. In general, the number of cases is equal to the amount of numbers between 1 and n^2-1 inclusive that are coprime with $n+1$, which equals $(n-1)(\phi(n+1))$, where ϕ is the Euler Totient function. For $n=2$, this value is 2, so all cases are considered. In fact, the combinations of Theorems 4.2 and 4.3 give an alternate proof of one direction of Feller's Theorem 3 (Theorem 1.2), as it comes as a corollary that $\mathrm{T}(2,a)\leq_g (3,b)$ if $a\leq\frac{4b+1}{3}$. Next, we consider the case when $n=3$, implying that b must be congruent to either 1, 3, 5, or 7 modulo $8=3^2-1$. These cases are dealt with in the four lemmas below, the first two of which follow from Theorems 4.1 and 4.2 respectively.

Lemma 4.5 . *If k is a positive integer, then* $\mathrm{T}(3,9k+1)\leq_g \mathrm{T}(4,8k+1)$.

Lemma 4.6 . *If k is a positive integer, then* $\mathrm{T}(3,9k+4)\leq_g \mathrm{T}(4,8k+3)$.

Lemma 4.7 . *If k is a positive integer, then* $\mathrm{T}(3,9k+5)\leq_g \mathrm{T}(4,8k+5)$.

Proof. First, we eliminate all σ_3 from our braid word by a similar process to Proposition 4.1. We have that

$$\begin{aligned}
&T(4,8k+5)\\
&=(\Delta_4^2)^{2k+1}\sigma_3\sigma_2\sigma_1\\
&=(\Delta_3^2\sigma_3\sigma_2\sigma_1\sigma_1\sigma_2\sigma_3)^{2k+1}\sigma_3\sigma_2\sigma_1\\
&=(\Delta_3^2)^{2k+1}(\sigma_3\sigma_2\sigma_1\sigma_1\sigma_2\sigma_3)^{2k+1}\sigma_3\sigma_2\sigma_1\\
&\to(\Delta_3^2)^{2k+1}\sigma_3(\sigma_2\sigma_1\sigma_1\sigma_2)^{2k+1}\sigma_2\sigma_1 \text{ (applying } 2k+1 \text{ CC}s)\\
&=(\Delta_3^2)^{2k+1}(\sigma_2\sigma_1\sigma_1\sigma_2)^{2k+1}\sigma_2\sigma_1
\end{aligned}$$

Then, we turn our braid on three strands into the required torus knot using similar ideas to Theorem 4.2, with some changes to account for the extra copy of $\sigma_2\sigma_1\sigma_1\sigma_2$.

$$\begin{aligned}
&=(\Delta_3^2)^{2k+1}(\sigma_2\sigma_1\sigma_1\sigma_2)^{k}\sigma_2\sigma_1\sigma_1\sigma_2(\sigma_2\sigma_1\sigma_1\sigma_2)^{k}\sigma_2\sigma_1\\
&\to(\Delta_3^2)^{2k+1}(\sigma_2\sigma_1\sigma_1\sigma_2)^{k}\sigma_2\sigma_1\sigma_1(\sigma_1\sigma_1)^{k}\sigma_1 \text{ (applying } k+1 \text{ CC}s)\\
&=(\Delta_3^2)^{2k+1}(\sigma_2\sigma_1\sigma_1\sigma_2)^{k}(\sigma_1\sigma_1)^{k}\sigma_1\sigma_2\sigma_1\sigma_1 \text{ commuting } \sigma_1\text{'s around the end.}\\
&=(\Delta_3^2)^{2k+1}(\sigma_2\sigma_1\sigma_1\sigma_2\sigma_1\sigma_1)^{k}\sigma_1\sigma_2\sigma_1\sigma_1\\
&=(\Delta_3^2)^{2k+1}(\sigma_2\sigma_1\sigma_2\sigma_1\sigma_2\sigma_1)^{k}\sigma_2\sigma_1\sigma_2\sigma_1\\
&=(\sigma_2\sigma_1)^{3(2k+1)+3(k)+2}\\
&=(\sigma_2\sigma_1)^{9k+5}\\
&=T(3,9k+5).
\end{aligned}$$

Note that $u(T(4,8k+5))-u(T(3,9k+5))=(12k+6)-(9k+4)=3k+2$, which is exactly the number of crossing changes we have used. Therefore, the torus knot $T(3,9k+1)\le_g T(4,8k+1)$. □

Lemma 4.8 . *If k is a positive integer, then* $T(3,9k+8)\le_g T(4,8k+7)$.

Proof. As in the previous lemma, we start by removing all σ_3 from our braid word.

$$\begin{aligned}
&T(4,8k+7)\\
&=((\sigma_3\sigma_2\sigma_1)^4)^{2k+1}\sigma_3\sigma_2\sigma_1\sigma_3\sigma_2\sigma_1\sigma_3\sigma_2\sigma_1\\
&=(\Delta_4^2)^{2k+1}\sigma_1\sigma_2\sigma_3\sigma_2\sigma_1\sigma_2\sigma_1\sigma_2\sigma_1\\
&=(\Delta_3^2\sigma_3\sigma_2\sigma_1\sigma_1\sigma_2\sigma_3)^{2k+1}\sigma_1\sigma_2\sigma_3\Delta_3^2\\
&=(\Delta_3^2)^{2k+2}\sigma_1\sigma_2(\sigma_3\sigma_2\sigma_1\sigma_1\sigma_2\sigma_3)^{2k+1}\sigma_3\\
&\to(\Delta_3^2)^{2k+2}\sigma_1\sigma_2\sigma_3(\sigma_2\sigma_1\sigma_1\sigma_2)^{2k+1} \text{ (applying } 2k+1 \text{ CC}s)\\
&=(\Delta_3^2)^{2k+2}\sigma_1\sigma_2(\sigma_2\sigma_1\sigma_1\sigma_2)^{2k+1}
\end{aligned}$$

Then, we once more convert our three-strand braid to the required form, using a few more steps to deal with the technicalities from the added generators.

$$= (\Delta_3^2)^{2k+2}\sigma_1\sigma_2(\sigma_2\sigma_1\sigma_1\sigma_2)^k\sigma_2\sigma_1\sigma_1\sigma_2(\sigma_2\sigma_1\sigma_1\sigma_2)^k$$
$$\to (\Delta_3^2)^{2k+2}\sigma_1(\sigma_1\sigma_1)^k\sigma_1\sigma_1\sigma_2(\sigma_2\sigma_1\sigma_1\sigma_2)^k \text{ (applying } k+1 \text{ CCs)}$$
$$= (\Delta_3^2)^{2k+2}(\sigma_1\sigma_1)^k\sigma_1\sigma_1\sigma_1\sigma_2(\sigma_2\sigma_1\sigma_1\sigma_2)^k$$
$$= (\Delta_3^2)^{2k+2}\sigma_1\sigma_1\sigma_1\sigma_2(\sigma_2\sigma_1\sigma_1\sigma_2)^k(\sigma_1\sigma_1)^k \text{ commuting } \sigma_1\text{'s around the end.}$$
$$= (\Delta_3^2)^{2k+2}\sigma_1\sigma_1\sigma_1\sigma_2(\sigma_2\sigma_1\sigma_1\sigma_2\sigma_1\sigma_1)^k$$
$$= (\Delta_3^2)^{2k+2}\sigma_1\sigma_1\sigma_1\sigma_2(\sigma_2\sigma_1\sigma_2\sigma_1\sigma_2\sigma_1)^k$$
$$= ((\sigma_2\sigma_1)^3)^{2k+2}\sigma_1\sigma_2\sigma_1\sigma_1((\sigma_2\sigma_1)^3)^k \text{ commuting } \sigma_1\text{'s around the end.}$$
$$= ((\sigma_2\sigma_1)^3)^{2k+2}\sigma_2\sigma_1\sigma_2\sigma_1((\sigma_2\sigma_1)^3)^k$$
$$= (\sigma_2\sigma_1)^{3(2k+2)+2+3(k)}$$
$$= (\sigma_2\sigma_1)^{9k+8}$$
$$= \mathrm{T}(3, 9k+8).$$

Note that $u(\mathrm{T}(4,8k+7)) - u(\mathrm{T}(3,9k+8)) = (12k+9)-(9k+7) = 3k+2$, which is exactly the number of crossing changes we have used. Therefore, the torus knot $\mathrm{T}(3,9k+8) \leq_g \mathrm{T}(4,8k+7)$. □

Theorem 4.4 . *Let a and b be positive integers such that a is coprime to 3 and b is coprime to 4. If* $a \leq \frac{9b+5}{8}$*, then* $\mathrm{T}(3,a) \leq_g \mathrm{T}(4,b)$.

Proof. This proof utilizes the four lemmas above, which cover the four possible cases for a torus knot of index four. We now go through each lemma and replace the number of twists in our 4-strand torus knot with b. For Lemma 4.5, $b = 8k+1$, so

$$\mathrm{T}(3,9k+1) = \mathrm{T}\left(3, 9\left(\frac{b-1}{8}\right)+1\right) = \mathrm{T}\left(3, \frac{9b}{8} - \frac{1}{8}\right) = \mathrm{T}\left(3, \left\lfloor \frac{9b+5}{8} \right\rfloor\right) \leq_g \mathrm{T}(4,b).$$

For Lemma 4.6, $b = 8k+3$, so

$$\mathrm{T}(3,9k+4) = \mathrm{T}\left(3, 9\left(\frac{b-3}{8}\right)+4\right) = \mathrm{T}\left(3, \frac{9b}{8} + \frac{5}{8}\right) = \mathrm{T}\left(3, \left\lfloor \frac{9b+5}{8} \right\rfloor\right) \leq_g \mathrm{T}(4,b).$$

For Lemma 4.7, $b = 8k+5$, so

$$\mathrm{T}(3,9k+5) = \mathrm{T}\left(3, 9\left(\frac{b-5}{8}\right)+5\right) = \mathrm{T}\left(3, \frac{9b}{8} - \frac{5}{8}\right) = \mathrm{T}\left(3, \left\lfloor \frac{9b+5}{8} \right\rfloor - 1\right) \leq_g \mathrm{T}(4,b).$$

This seems to break the trend, but note that $\left\lfloor \frac{9b+5}{8} \right\rfloor = \frac{9b+3}{8}$ in this case, which is divisible by 3 and is therefore not a torus knot. This implies we need to show the index one lower to prove the desired result. For Lemma 4.8, $b = 8k+7$, so

$$T(3,9k+8) = T\left(3, 9\left(\frac{b-7}{8}\right)+8\right) = T\left(3, \frac{9b}{8}+\frac{1}{8}\right) = T\left(3, \left\lfloor \frac{9b+5}{8} \right\rfloor\right) \leq_g T(4,b).$$

These results show that the torus knot $T\left(3, \left\lfloor \frac{9b+5}{8} \right\rfloor\right) \leq_g T(4,b)$. By Feller's Theorem 2 (Theorem 1.1), this means that the torus knot $T(3,a) \leq_g T(4,b)$ if $a \leq \frac{9b+5}{8}$. □

We can also use a similar process to find Gordian adjacencies between torus knots whose indices differ by more than one. Below, we explore Gordian adjacencies between torus knots of index two and index four. Note that for $T(2,b)$ to be a torus knot, b must be congruent to either 1 or 3 modulo 4. These provide the two cases we split our argument into. In each case, we first remove all σ_3 using a similar process to Proposition 4.1, then after some technical manipulations remove all σ_2 in two consecutive stages.

Lemma 4.9 . *If k is a positive integer, then* $T(2,6k+3) \leq_g T(4,4k+1)$.

Proof. By definition, we have

$$\begin{aligned}
&T(4,4k+1)\\
&= (\sigma_3\sigma_2\sigma_1)^{4k+1}\\
&= (\Delta_4^2)^k\sigma_3\sigma_2\sigma_1\\
&= (\Delta_3^2\sigma_3\sigma_2\sigma_1\sigma_1\sigma_2\sigma_3)^k\sigma_3\sigma_2\sigma_1\\
&= (\Delta_3^2)^k(\sigma_3\sigma_2\sigma_1\sigma_1\sigma_2\sigma_3)^k\sigma_3\sigma_2\sigma_1\\
&\to (\Delta_3^2)^k\sigma_3(\sigma_2\sigma_1\sigma_1\sigma_2)^k\sigma_2\sigma_1 \text{ (applying } k \text{ CC}s)\\
&= (\Delta_3^2)^k(\sigma_2\sigma_1\sigma_1\sigma_2)^k\sigma_2\sigma_1\\
&= (\sigma_2\sigma_1\sigma_2\sigma_1\sigma_2\sigma_1)^k(\sigma_2\sigma_1\sigma_1\sigma_2)^k\sigma_2\sigma_1\\
&= (\sigma_2\sigma_1\sigma_1\sigma_2\sigma_1\sigma_1)^k(\sigma_2\sigma_1\sigma_1\sigma_2)^k\sigma_2\sigma_1\\
&= (\sigma_2\sigma_1\sigma_1\sigma_2\sigma_1\sigma_1\sigma_2\sigma_1\sigma_1\sigma_2)^k\sigma_2\sigma_1\\
&\to \sigma_2(\sigma_1\sigma_1\sigma_2\sigma_1\sigma_1\sigma_2\sigma_1\sigma_1)^k\sigma_2\sigma_2\sigma_1 \text{ (applying } k-1 \text{ CC}s)\\
&= (\sigma_1\sigma_1\sigma_2\sigma_1\sigma_1\sigma_2\sigma_1\sigma_1)^k\sigma_1\sigma_2\sigma_1\sigma_1\\
&= (\sigma_1\sigma_1)^k(\sigma_2\sigma_1\sigma_1\sigma_2)^k(\sigma_1\sigma_1)^k\sigma_1\sigma_2\sigma_1\sigma_1\\
&= (\sigma_1\sigma_1)^{2k+1}(\sigma_2\sigma_1\sigma_1\sigma_2)^k\sigma_2\sigma_1\\
&\to (\sigma_1\sigma_1)^{2k+1}\sigma_2(\sigma_1\sigma_1)^k\sigma_1 \text{ (} k \text{ CC}s)\\
&= (\sigma_1)^{2(2k+1)+2(k)+1}\\
&= (\sigma_1)^{6k+3}\\
&= T(2,6k+3).
\end{aligned}$$

Note that $u(\mathrm{T}(4,4k+1)) - u(\mathrm{T}(2,6k+3)) = 6k-(3k+1) = 3k-1$, which is exactly how many crossing changes we have used. Therefore, the torus knot $\mathrm{T}(2,6k+3) \leq_g \mathrm{T}(4,4k+1)$. □

Lemma 4.10 . *If k is a positive integer, then* $\mathrm{T}(2,6k+5) \leq_g \mathrm{T}(4,4k+3)$.

Proof. By definition, we have

$$\begin{aligned}
&\mathrm{T}(4,4k+3)\\
&= (\sigma_3\sigma_2\sigma_1)^{4k+3}\\
&= (\Delta_4^2)^k\sigma_3\sigma_2\sigma_1\sigma_3\sigma_2\sigma_1\sigma_3\sigma_2\sigma_1\\
&= (\Delta_4^2)^k\sigma_3\sigma_2\sigma_1\sigma_2\sigma_3\sigma_2\sigma_1\sigma_2\sigma_1\\
&= (\Delta_3^2\sigma_3\sigma_2\sigma_1\sigma_1\sigma_2\sigma_3)^k\sigma_1\sigma_2\sigma_3\sigma_2\sigma_1\sigma_2\sigma_1\sigma_2\sigma_1\\
&= (\Delta_3^2)^{k+1}(\sigma_3\sigma_2\sigma_1\sigma_1\sigma_2\sigma_3)^k\sigma_1\sigma_2\sigma_3\\
&= (\Delta_3^2)^{k+1}\sigma_1\sigma_2(\sigma_3\sigma_2\sigma_1\sigma_1\sigma_2\sigma_3)^k\sigma_3\\
&\to (\Delta_3^2)^{k+1}\sigma_1\sigma_2\sigma_3(\sigma_2\sigma_1\sigma_1\sigma_2)^k \text{ (applying } k \text{ CC}s)\\
&= (\Delta_3^2)^{k+1}\sigma_1\sigma_2(\sigma_2\sigma_1\sigma_1\sigma_2)^k\\
&= (\sigma_2\sigma_1\sigma_2\sigma_1\sigma_2\sigma_1)^{k+1}\sigma_1\sigma_2(\sigma_2\sigma_1\sigma_1\sigma_2)^k\\
&= (\sigma_2\sigma_1\sigma_1\sigma_2\sigma_1\sigma_1)^{k+1}\sigma_1\sigma_2(\sigma_2\sigma_1\sigma_1\sigma_2)^k\\
&= (\sigma_2\sigma_1\sigma_1\sigma_2\sigma_1\sigma_1)^k\sigma_2\sigma_1\sigma_1\sigma_2\sigma_1\sigma_1\sigma_1\sigma_2(\sigma_2\sigma_1\sigma_1\sigma_2)^k\\
&= (\sigma_2\sigma_1\sigma_1\sigma_2\sigma_1\sigma_1)^k\sigma_1\sigma_1\sigma_1\sigma_2\sigma_1\sigma_1\sigma_2\sigma_2(\sigma_2\sigma_1\sigma_1\sigma_2)^k\\
&\to (\sigma_2\sigma_1\sigma_1\sigma_2\sigma_1\sigma_1)^k\sigma_1\sigma_1\sigma_1\sigma_2\sigma_1\sigma_1(\sigma_2\sigma_1\sigma_1\sigma_2)^k \text{ (applying one CC)}\\
&= (\sigma_2\sigma_1\sigma_1\sigma_2\sigma_1\sigma_1\sigma_2\sigma_1\sigma_1\sigma_2)^k\sigma_2(\sigma_1)^5\\
&\to \sigma_2(\sigma_1\sigma_1\sigma_2\sigma_1\sigma_1\sigma_2\sigma_1\sigma_1)^k(\sigma_1)^5 \text{ (applying } k \text{ CC}s)\\
&= \sigma_2(\sigma_2\sigma_1\sigma_1\sigma_2)^k(\sigma_1\sigma_1\sigma_1\sigma_1)^k(\sigma_1)^5\\
&\to (\sigma_1\sigma_1)^k\sigma_2(\sigma_1\sigma_1\sigma_1\sigma_1)^k(\sigma_1)^5 \text{ (applying } k \text{ CC}s)\\
&= (\sigma_1\sigma_1)^k(\sigma_1\sigma_1\sigma_1\sigma_1)^k(\sigma_1)^5\\
&= (\sigma_1)^{2(k)+4(k)+5}\\
&= (\sigma_1)^{6k+5}\\
&= \mathrm{T}(2,6k+5).
\end{aligned}$$

Note that $u(\mathrm{T}(4,4k+3)) - u(\mathrm{T}(2,6k+5)) = 6k+3-(3k+2) = 3k+1$, which is exactly how many crossing changes we have used. Therefore, the torus knot $\mathrm{T}(2,6k+5) \leq_g \mathrm{T}(4,4k+3)$. □

Theorem 4.5 . *Let a and b be odd positive integers. If $a \leq \frac{3b+3}{2}$, then* $\mathrm{T}(2,a) \leq_g \mathrm{T}(4,b)$.

Proof. This proof follows the same structure as the proof of Theorem 4.4. For Lemma 4.9, $b = 4k+1$, so

$$\mathrm{T}(3,6k+3) = \mathrm{T}\left(3,6\left(\frac{b-1}{4}\right)+3\right) = \mathrm{T}\left(3,\frac{3b+3}{2}\right) \leq_g \mathrm{T}(4,b).$$

For Lemma 4.10, $b = 4k+3$, so

$$\mathrm{T}(3,6k+5) = \mathrm{T}\left(3,6\left(\frac{b-3}{4}\right)+5\right) = \mathrm{T}\left(3,\frac{3b+1}{2}\right) \leq_g \mathrm{T}(4,b).$$

Note that in the second case, $\frac{3b+3}{2}$ is even, so as before we only need to show the number one lower. By Feller's Theorem 2 (Theorem 1.1), we have that the torus knot $\mathrm{T}(2,a) \leq_g \mathrm{T}(4,b)$ if $a \leq \frac{3b+3}{2}$. □

While we conjecture that the converse of Theorem 4.4 is true, we know that the converse of Theorem 4.5 is not: we have found that $\mathrm{T}(2,13) \leq_g \mathrm{T}(4,7)$.

So far, all of the Gordian adjacencies presented have been between torus knots. We next consider which positive braid w we can concatenate another positive braid β' in B_n with. In doing so, we want the concatenation to result in a longer positive braid β in B_n whose closure is Gordian adjacent to the closure of β'. Note that w must always have even length for both $\hat{\beta}$ and $\hat{\beta}'$ to be knots, as the unknotting formula $\frac{\ell-n+1}{2}$ must result in an integer. If w satisfies the following properties, Gordian adjacency can be guaranteed.

Theorem 4.6 . *Let β in B_n be a positive braid where $\beta = \beta' w$ and $\hat{\beta}$ and $\hat{\beta}'$ are knots. If $\hat{w}$ is a link with n components, then $\hat{\beta}' \leq_g \hat{\beta}$.*

Proof. By Lemma 3.2, we can use the five rules to make $\beta' w$ into a new braid word $\beta' w'$ where w' has no more than one σ_{n-1}. As none of the five rules change the number of components in the braid closure of the subword we perform the rules on, $\hat{w}'$ must have n components. If w' had only one σ_{n-1}, we could delete the σ_{n-1} using the Markov Destabilization Rule and express w' on $n-1$ strands; since $\hat{w}'$ has n components, this cannot be done. Therefore, w' has no σ_{n-1}. Using Lemma 3.2 on $\beta' w'$ again leads to a w'' with no σ_{n-2}. By repeating this process n times, we can turn $\beta' w$ into β' using the five rules. Since there is a sequence using the five rules that goes from β to β', $\hat{\beta}' \leq_g \hat{\beta}$. □

Theorem 4.6 tells us that we can add a braid word whose closure is an n-component link (for instance, Δ_n^2) to any positive braid β in B_n to get a new knot K such that that $\hat{\beta} \leq_g$ K. In the other direction, deleting any subword whose closure is an n-component link preserves Gordian adjacency to the original braid word. This theorem also provides an alternate proof for a weakened version of Feller's Theorem 2 (Theroem 1.1), as it shows $\mathrm{T}(a,c) \leq_g \mathrm{T}(a,d)$ if $c = d - ka$ for some positive integer k.

5 Positive Paths

The concept of Gordian adjacency is closely tied to the concept of unknotting sequences, as a knot $K_1 \leq_g K_2$ if and only if K_1 is contained in an unknotting sequence of K_2 [6]. In finding Gordian adjacencies between positive braid knots, we are really describing unknotting sequences that contain positive braid knots. The five rules give us further tools to explore these unknotting sequences.

Definition 5.1 . *A positive path from a positive braid knot* $\hat{\beta}_2$ *to a positive braid knot* $\hat{\beta}_1$ *is a subsequence of an unknotting sequence of* $\hat{\beta}_2$, $\hat{\beta}_2 \to \hat{\alpha}_1 \to \ldots \to \hat{\alpha}_n \to \hat{\beta}_1$, *such that each* $\hat{\alpha}_i$ *is a positive braid knot.*

Note that every Gordian adjacency we show in this paper is shown through a positive path.

Theorem 5.1 . *Every positive braid knot has an unknotting sequence that is a positive path.*

Proof. The combination of Lemmas 3.1 and 3.3 tells us that every positive braid word that represents a knot can be unknotted using the five rules, and that this procedure forms an unknotting sequence. As the application of any one of our rules to any positive braid results in a positive braid, this unknotting sequence contains only positive braid knots. □

The next question we consider is how many unknotting sequences of positive braid knots are positive paths. Since every knot with an unknotting number greater than one has infinetly many unknotting sequences [1], we naturally ask ourselves whether there are an infinite number of unknotting sequences of a positive braid knot $\hat{\beta}$ that are positive paths. Our next theorem implies that any positive braid knot has only finitely many positive paths.

Lemma 5.2 . *If* β *is a positive braid in* B_n *whose closure is a knot, then each generator* σ_i *for* $1 \leq i \leq n-1$ *must appear at least once in* β.

Proof. Assume β is a positive braid in B_n that has no σ_i for some $1 \leq i \leq n-1$. Note that $\beta = \beta_1\beta_2$ for positive braids β_1 and β_2, where $j < i$ for all σ_j in β_1 and $j > i$ for all σ_j in β_2. We know this can be done via the Distant Generators Rule because any generator that ends up in β_2 has subscript at least two greater than the subscript of any generator that ends up in β_1. As shown by the figure below, we see that $\hat{\beta}$ must be a link, as it has at least two components $\hat{\beta}_1$ and $\hat{\beta}_2$. By the contrapositive, if $\hat{\beta}$ is a knot, then there are no σ_i that β does not contain. □

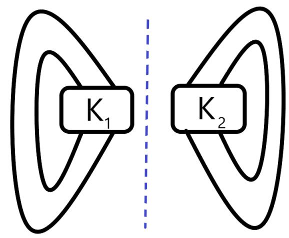


Figure 8: A Braid with No σ_i

Lemma 5.3 . *Let $\hat{\beta}$ be a positive braid knot such that β is in B_n. If there exists $1 \leq i \leq n-1$ such that σ_i appears exactly once in β, then $\hat{\beta}$ can be expressed as the closure of a positive braid β' in B_{n-1}.*

Proof. We first note that $\beta = \beta_1\sigma_i\beta_2$ for positive braids β_1 and β_2, where $j < i$ for all generators σ_j in β_1 and $j > i$ for all σ_j in β_2. Similarly to the previous lemma, each generator can be moved to the correct side of the braid word using the Distant Generators Rule and the Conjugation Rule, as the generator gets moved around the end of the braid word while avoiding σ_i. This works because any generator that ends up in β_2 has subscript at least two greater than the subscript of any generator that ends up in β_1.

After expressing β in this form, in Figure 9 below, we see that we can isotopy the closure of $\beta_1\sigma_i\beta_2$ into the closure of $\beta_1\beta_3$, where β_3 is the same braid word as β_2, but with each subscript decreased by one. Since β_1 and β_3 are in B_{n-1}, this completes the proof. □

Lemma 5.4 . *If $\hat{\beta}$ is a positive braid knot with unknotting number m, then $\hat{\beta}$ can be expressed as the closure of a positive braid on $2m+1$ strands or fewer.*

Proof. Let n be the fewest number of strands on which $\hat{\beta}$ can be expressed as a positive braid. By [7], we have

$$u(\hat{\beta}) = m = \frac{\ell - n + 1}{2},$$

implying $\ell = 2m + n - 1$. Since we have $n-1$ generators $\sigma_1, \cdots, \sigma_{n-1}$, Lemmas 5.2 and 5.3 give us that each of these generators must appear at least twice. Therefore, the length of our braid word must be at least twice the number of generators, so $\ell \geq 2(n-1)$. Substituting, we get $2m + n - 1 \geq 2(n-1)$, implying $n \leq 2m+1$. This tell us that $2m+1$ is greater than or equal to the fewest number of strands on which $\hat{\beta}$ can be expressed as a positive braid. □

Theorem 5.2 . *For each positive integer m, there are a finite number of positive braid knots with unknotting number m.*

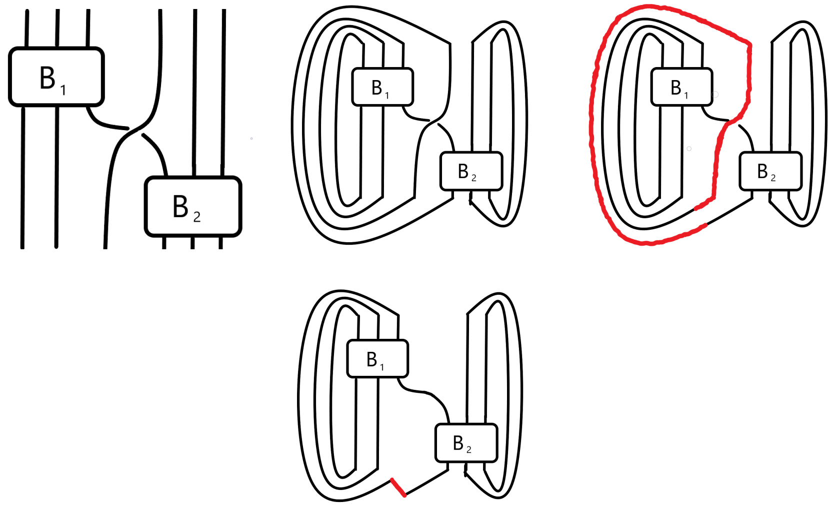


Figure 9: A Braid with Only One σ_i

Proof. Let $\hat{\beta}$ be a positive braid knot with unknotting number m. By Lemma 5.4, we know that $brd(\hat{\beta}) \leq 2m+1$. This means that $\hat{\beta}$ can be expressed on $2m+1$ strands, as any knot with a lower index can simply add each generator between its index and $2m$ inclusive to the end of its braid word to get a braid on $2m+1$ strands with an isotopic closure. If a positive braid with unknotting number m is expressed on $2m+1$ strands, by the unknotting number formula for positive braids, $\ell = 2m+n-1 = 2m+(2m+1)-1 = 4m$. So we have $4m$ letters in our braid word, each of which must be one of $2m$ generators. So there are no more than $(2m)^{4m}$ positive braid knots of unknotting number m. □

Naturally, most of these $(2m)^{4m}$ possible braid words have unknotting number less than m, are links, or form isotopic closures, so the actual number of positive braid knots of unknotting number m is drastically lower. The goal here was simply to set some finite bound on the number of positive braid knots of a given unknotting number. This is not possible for knots in general, it is known that that there are an infinite number of knots of any unknotting number greater than zero. [1].

Conjecture 5.5 . *If* $\hat{\beta}_1$ *and* $\hat{\beta}_2$ *are positive braid knots such that* $\hat{\beta}_1 \leq_g \hat{\beta}_2$*, then there exists a positive path from* $\hat{\beta}_2$ *to* $\hat{\beta}_1$.

This conjecture is true for all positive braid knots we show in this paper to be Gordian adjacent. It is also a statement that shows the value of Theorem 5.2. If we have two positive braid knots known to be Gordian adjacent, we only need to look at a finite number of knots to determine whether a positive path from one to the other exists. If the conjecture is true, then there is only a finite number of possible positive paths to check before we can determine that some positive braid knot is not Gordian adjacent to another.

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