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Linear Combinations of Harmonic Univalent Mappings

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Abstract. Many properties are known about analytic functions, however the class of harmonic functions which are the sum of an analytic function and the conjugate of an analytic function is less understood. We wish to find conditions such that linear combinations of univalent harmonic functions are univalent. We focus on functions whose image is convex in one direction i.e. each line segment in that direction between points in the image is contained in the image. M. Dorff proved sufficient conditions such that the linear combination of univalent harmonic functions will be univalent on the unit disk. The conditions are: the mappings must be locally univalent, their images must be convex in the imaginary direction and they must satisfy a normalization which states that the right and left extremes of the image are the image of 1 and -1 respectively. In this paper we generalize this existing theorem. The conditions of this theorem are geometric, and we would like to maintain this feature in the generalization. We show that the image may be convex in any direction and that any points on the boundary of the domain, which no longer must be the unit disk, can be the points that are mapped to the extrema, which now must be in the direction perpendicular to the direction of convexity.

1 Introduction

A function $f(z) = u + iv$ is complex **harmonic**, where $z = x + iy$ is a complex variable, if it satisfies Laplace's equation:

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0.$$

A function is complex analytic at a point if it is differentiable at that point and an open neighborhood around that point. For the remainder of this article analytic will be understood to mean complex analytic, and the word "complex" will be dropped. Since any function that is analytic satisfies the Cauchy-Riemann equations, all analytic functions

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satisfy Laplace's equation. So the analysis of harmonic functions is a natural extension of the study of analytic functions. Harmonic functions show up in many branches of physics where Laplace's equation appears, which include electricity and magnetism, and quantum mechanics. Therefore, mathematical research into these functions is motivated not only by their mathematical relevance but also their applications to physical systems. Any harmonic function f is of the form $f = h+\overline{g}$ where h and g are analytic. A function is **univalent** in domain D if for all $z_1, z_2 \in$ D such that $z_1 \neq z_2$, $f(z_1) \neq f(z_2)$. Univalent functions are also known as one to one, injective, or schlicht. A bijection is a function that relates each point in the domain to a single point in the co-domain and vice versa. A function is **locally univalent** at a point if there exists a neighborhood of that point that f is univalent on. Note that a locally univalent function is not necessarily (globally) univalent; it is in general much easier to show local univalence at every point in a domain than to show univalence in that domain. The unit disk $\{z \mid |z| < 1\}$ will be notated $\mathbb{D}$. One technique for solving Laplace's equation with given boundary conditions is to form a linear combination of other known solutions to Laplace's equation (harmonic functions); the linear combination of harmonic functions is always harmonic. A linear combination of two complex functions f_1 and f_2 is $f_3 = tf_1 + (1-t)f_2$ where $0 \leq t \leq 1$. If we do this, it will be helpful to know the conditions under which various properties are preserved. In this paper we seek to generalize a theorem which gives sufficient conditions for a linear combination of harmonic univalent functions to be univalent. This gives us a way to generate new univalent harmonic functions which could provide valuable insight into their behavior, along with the insight we gain from the theorems themselves. To limit the scope of our work–and to provide a useful result–we will restrict our research to functions with the unit disk as their domain. We will also restrict the image of the function by convexity.

A region is convex if the line segment connecting any two points in that region is contained within the region. A region is convex in the θ direction if any line segment parallel to the ray $re^{i\theta}$ between two points in the region is contained in the region. Specifically a region is convex in the real or imaginary direction if it is convex in the θ = 0 or $\pi/2$ directions respectively. It is obvious that a convex region is convex in every direction. Finally, a function f on a domain E is convex in the θ direction if its image, f(E) is convex in the θ direction.

Part of our motivation for focusing on functions convex in one direction is the following result.

Theorem 1.1. *Every function convex in one direction is univalent.*

Much of the existing work on univalent harmonic mappings focuses on functions convex in the horizontal direction. The most important theorem in this field is the shearing theorem, which converts a harmonic function to an analytic function. This allows us to use results on analytic functions, which are much better understood and easier to work with. We state this theorem and a useful corollary.

Theorem 1.2 (Clunie, Sheil-Small Shearing Theorem [1]). *A harmonic function $f = \overline{g} + h$ which is locally univalent in $\mathbb{D}$ is a univalent mapping of $\mathbb{D}$ onto a domain convex in the horizontal direction if and only if, $h - g$ is a conformal univalent mapping of $\mathbb{D}$ onto a domain convex in the horizontal direction.*

Corollary 1.3. *A harmonic function $f = \overline{g} + h$ which is locally univalent in $\mathbb{D}$ is a univalent mapping of $\mathbb{D}$ onto a domain convex in the θ direction if and only if $h - e^{2i\theta} g$ is a conformal univalent mapping of $\mathbb{D}$ onto a domain convex in the θ direction.*

In order to use this powerful theorem we need to know whether f is locally univalent. We can determine this by finding what is called the analytic dilation which is denoted $\omega(z) = g'(z)/h'(z)$. A function is **sense preserving** if, when you take the image of the counterclockwise oriented boundary of the domain, the image will be on the left side of the boundary of the image

Theorem 1.4 (Lewy [2]). *A harmonic function $f = \overline{g} + h$ is locally univalent and sense preserving in domain D if and only if $|\omega(z)| < 1$ for all $z \in \mathbb{D}$ where $\omega(z) = g'(z)/h'(z)$*

Finally we will make use of the following theorem by Hengartner and Schober [3].

Theorem 1.5. *Suppose f is analytic and non-constant in $\mathbb{D}$. Then $\mathrm{Re}\{(1 - z^2) f'(z)\} \geq 0, z \in \mathbb{D}$ if and only if*

1. *f is univalent in $\mathbb{D}$,*
2. *f is convex in the imaginary direction*
3. *f satisfies normalization 1.*

Where normalization 1 is:

Normalization 1. Suppose f is complex-valued harmonic and non-constant in $\mathbb{D}$. There exists sequences z'_n, and z''_n converging to $z = 1$, $z = -1$, respectively, such that

$$\lim_{n\to\infty} \mathrm{Re}\{f(z'_n)\} = \sup\{\mathrm{Re}\{f(z)\} | z \in \mathbb{D}\}$$

$$\lim_{n\to\infty} \mathrm{Re}\{f(z''_n)\} = \inf\{\mathrm{Re}\{f(z)\} | z \in \mathbb{D}\}$$

This condition says that f maps the horizontal extremes $1, -1$ in the input domain to the horizontal extremes of the image; the limits and extrema are necessary for cases when the image is unbounded or undefined at the boundary.

We can now state and prove the result by Dorff which we will seek to generalize [2].

Theorem 1.6. *Let $f_1 = \overline{g}_1 + h_1$, and $f_2 = \overline{g}_2 + h_2$ be univalent harmonic mappings convex in the imaginary direction. Let $f_3 = t f_1 + (1 - t) f_2$ and $|\omega_3| < 1$ on $\mathbb{D}$. If f_1, f_2 satisfy normalization 1 then f_3 is convex in the imaginary direction and univalent* $(0 \leq t \leq 1)$

The conditions of this theorem are highly restrictive, especially in terms of directionality. The convexity of the domain is confined to one direction and the normalization forces two specific points in the domain to map to two specific points in the image. This is the motivation for generalizing the theorem.

Proof. Since $|\omega_3| < 1$, f_3 is locally univalent. By **Corollary 1.3** with $\theta = \pi/2$, $h_1 + g_1$ and $h_2 + g_2$ are univalent and convex in the imaginary direction. $h_1 + g_1$ and $h_2 + g_2$ satisfy normalization 1, since $\mathrm{Re}\{h_1 + g_1\}$=$\mathrm{Re}\{f_1\}$ and $\mathrm{Re}\{h_2 + g_2\}$=$\mathrm{Re}\{f_2\}$.
So we can apply **Theorem 1.5** to both, giving us:

$$\mathrm{Re}\{(1-z^2)(g_1'(z)+h_1'(z))\} \geq 0$$

$$\mathrm{Re}\{(1-z^2)(g_2'(z)+h_2'(z))\} \geq 0$$

Then,

$$\begin{aligned}
&\mathrm{Re}\{(1-z^2)(g_3'(z)+h_3'(z))\}\\
&\qquad = \mathrm{Re}\{(1-z^2)(t(g_1'(z)+h_1'(z))+(1-t)(g_2'(z)+h_2'(z)))\}\\
&\qquad = t\mathrm{Re}\{(1-z^2)(g_1'(z)+h_1'(z))\}+(1-t)\mathrm{Re}\{(1-z^2)(g_2'(z)+h_2'(z))\}\\
&\qquad \geq 0
\end{aligned}$$

By **Theorem 1.5**, $h_3 + g_3$ is univalent and convex in the imaginary direction. So by **Corollary 1.3**, f_3 is univalent.

□

This result makes sense in the context of our understanding of real functions; the combination of two one-to-one real functions is one-to-one if they are both increasing or both decreasing. This theorem corresponds to the case where both are ”increasing" in the positive real direction. By symmetry we expect that univalence is preserved no matter which direction the function is “increasing" in. Additionally we suspect that this theorem should apply to functions convex in any direction, once again by symmetry.

2 New Results

We devised and proved a theorem that contains the aforementioned generalizations and several others.

Theorem 2.1. *Let $f_1 = \overline{g}_1 + h_1$, and $f_2 = \overline{g}_2 + h_2$ be univalent harmonic mappings convex in the direction θ on a domain E, which can be analytically deformed to the unit disk. $f_3 = tf_1 + (1-t)f_2$ is convex in the θ direction and univalent $(0 \leq t \leq 1)$, if $|\omega_3| < 1$ on E, if f_1, f_2 reach their unique extremes in the direction $\theta - \pi/2$ at the same two points in* E.

The proof relies on a generalization by Royster and Ziegler [4] of **Theorem 1.5**. We will also use a Lemma which is itself a generalization of **Theorem 1.6**.

Theorem 2.2 (Royster and Ziegler)**.** *Suppose f is analytic and non-constant in $\mathbb{D}$. Then f maps $\mathbb{D}$ univalently onto a domain convex in the imaginary direction if and only if there are numbers m and n, with $0 \le m < 2\pi$ and $0 \le n \le \pi$ such that,*

$$\mathrm{Re}\{-ie^{im}(1-2\cos(n)e^{-im}z+e^{-2im}z^2)f'(z)\} \ge 0. \ \ z \in \mathbb{D}$$

Additionally, $f(e^{i(m-n)})$ and $f(e^{i(m+n)})$ are the right and left extremes, respectively of the image.

Observe that, if we already know that f maps $\mathbb{D}$ univalently onto a domain convex in the imaginary direction and we identify the values of m and n such that $f(e^{i(m-n)})$ and $f(e^{i(m+n)})$ are the unique right and left extremes, then those values for m and n will necessarily satisfy the inequality given in **Theorem 2.2**. Further research ought to attempt to remove the uniqueness condition. This observation is used several times.

Lemma 2.3. *Let $f_1 = \overline{g}_1 + h_1$, and $f_2 = \overline{g}_2 + h_2$ be univalent harmonic mappings convex in the imaginary direction. Let $f_3 = tf_1 + (1-t)f_2$ and $|\omega_3| < 1$ on $\mathbb{D}$. If $f_j(e^{i(m-n)})$ and $f_j(e^{i(m+n)})$, where $0 \le m < 2\pi$ and $0 \le n \le \pi$, are the unique right and left extremes respectively of the image of f_j for $j = 1,2$, then f_3 is convex in the imaginary direction and univalent* $(0 \le t \le 1)$

The proof of this lemma parallels the proof of **Theorem 1.6**.

Proof. Since $|\omega_3| < 1$, f_3 is locally univalent. By **Corollary 1.3** with $\theta = \pi/2$, $h_1 + g_1$ and $h_2 + g_2$ are univalent and convex in the imaginary direction. $h_1 + g_1$ and $h_2 + g_2$ achieve their right and left extremes at $f(e^{i(m-n)})$ and $f(e^{i(m+n)})$ respectively, since $\mathrm{Re}\{h_1 + g_1\}$=$\mathrm{Re}\{f_1\}$ and $\mathrm{Re}\{h_2 + g_2\}$=$\mathrm{Re}\{f_2\}$.

We apply **Theorem 2.2** to both functions, giving us:

$$\mathrm{Re}\{(-ie^{im}(1-2\cos(n)e^{-im}z+e^{-2im}z^2))(g_1'(z)+h_1'(z))\} \ge 0$$

$$\mathrm{Re}\{(-ie^{im}(1-2\cos(n)e^{-im}z+e^{-2im}z^2))(g_2'(z)+h_2'(z))\} \ge 0$$

Then we can manipulate these inequalities as was done in the proof of **Theorem 1.6**.

$$\begin{aligned}
&\mathrm{Re}\{(-ie^{im}(1-2\cos(n)e^{-im}z+e^{-2im}z^2))(g_3'(z)+h_3'(z))\} \\
&\quad = \mathrm{Re}\{(-ie^{im}(1-2\cos(n)e^{-im}z+e^{-2im}z^2)) \\
&\qquad \cdot (t(g_1'(z)+h_1'(z))+(1-t)(g_2'(z)+h_2'(z)))\} \\
&\quad = t\mathrm{Re}\{(-ie^{im}(1-2\cos(n)e^{-im}z+e^{-2im}z^2))(g_1'(z)+h_1'(z))\} \\
&\qquad + (1-t)\mathrm{Re}\{(-ie^{im}(1-2\cos(n)e^{-im}z+e^{-2im}z^2))(g_2'(z)+h_2'(z))\} \\
&\quad \ge 0
\end{aligned}$$

By **Theorem 2.2**, $h_3 + g_3$ is univalent and convex in the imaginary direction. So by **Corollary 1.3**, f_3 is univalent.

$\square$

Finally, we are able to prove **Theorem 2.1**.

*Proof of **Theorem 2.1**.* Since E can be analytically deformed to $\mathbb{D}$, there exists $\sigma(z) : \mathbb{D} \to$ E such that σ is a analytic bijection. Consider $f_j^*(z) = e^{-i(\theta-\pi/2)} f_j(\sigma(z))$, for $j = 1,2$, and $f_3^* = t f_1^* + (1-t) f_2^*$. All three of these are functions on the unit disk and $f_1^*(z)$, and $f_2^*(z)$ are convex in the imaginary direction. Since $f_1(z)$, and $f_2(z)$ have the same points in E which map to the image extrema in the direction $\theta - \pi/2$ on E. If we apply σ^{-1}, which exists since σ is univalent, to each of these points, the two new points in $\mathbb{D}$ will be the image extremes in the $\theta - \pi/2$ direction of $f_1(\sigma(z))$, and $f_2(\sigma(z))$. Therefore, those two points must map to the extrema in the horizontal direction of $f_1^*(z)$ and $f_2^*(z)$. Since both these points are on the boundary of $\mathbb{D}$, let us write the point which maps to the right extreme as $e^{i\phi_1}$ and the point which maps to the left extreme as $e^{i\phi_2}$ where $0 \le \phi_1, \phi_2 \le 2\pi$. Consider $m = (\phi_1 + \phi_2)/2$ and $n = (\phi_2 - \phi_1)/2$. Then $\phi_1 = m - n$ and $\phi_2 = m + n$.

We merely need to prove local univalence to apply **Lemma 2.3**.

$$\begin{aligned}
\omega_3^*(z) &= \frac{t e^{-i(\theta-\pi/2)} \sigma'(z) g_1'(\sigma(z)) + (1-t) e^{-i(\theta-\pi/2)} \sigma'(z) g_2'(\sigma(z))}{t e^{-i(\theta-\pi/2)} \sigma'(z) h_1'(\sigma(z)) + (1-t) e^{-i(\theta-\pi/2)} \sigma'(z) h_2'(\sigma(z))} \\
&= \frac{t g_1'(\sigma(z)) + (1-t) g_2'(\sigma(z))}{t h_1'(\sigma(z)) + (1-t) h_2'(\sigma(z))} \\
&= \omega_3(\sigma(z))
\end{aligned}$$

Since $\sigma(z) \in \mathbb{D}$, $|\omega_3^*(z)| = |\omega_3(\sigma(z))| < 1$. We can now apply **Lemma 2.3** which says that f_3^* is convex in the imaginary direction and univalent. Then $e^{i(\theta-\pi/2)} f_3^*(z) = f_3(\sigma(z))$ is convex in the θ direction and univalent on $\mathbb{D}$. Since the inner term does not affect the image, and is a bijection, $f_3(z)$ is convex in the θ direction and univalent. $\square$

This new theorem is much broader than the original as we desired. It was true that the functions could be convex in any direction. Moreover, we were able to generalize the other conditions more than we expected. We can now use many more domains than just the unit disk, and the conditions on the extrema have been generalized to allow their pre-image to be any two points. The beauty of this theorem is that it provides geometric conditions which can be quickly verified for any mappings.

For example, two functions that we can combine using this theorem but not the previous theorem are $f_1(z) = z + \frac{i}{2}\overline{z}$ and $f_2(z) = z - \frac{1}{3}\overline{z}^3$ on $\mathbb{D}$. They both are convex in the $\pi/4$ direction and map $-\pi/4$ and $3\pi/4$ to their extremes in the $-\pi/4$ direction.

These two functions are fairly simple; our generalization applies to a broad swath of functions which the original theorem misses.

It is likely possible to generalize this result to close-to-convex functions, which contain the set of functions convex in one direction. A theorem for those functions could help to subdivide the set of close-to-convex functions, similarly to how **Theorem 2.1** divides functions convex in one direction into each individual direction. While that subdivision is easy to see, a subdivision for close to convex functions is not immediately obvious. It might also be generalizable to a wider class of domains, perhaps all simply connected domains. These would be areas for future research.

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