

On Consecutive Triples Of Powerful Numbers

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On Consecutive Triples of Powerful Numbers

By Edward Beckon

Abstract. A powerful number is a positive integer such that every prime that appears in its prime factorization appears there at least twice. Erdős, Mollin and Walsh conjectured that three consecutive powerful numbers do not exist. This paper shows that if they do exist, the smallest of the three numbers must have remainder 7, 27 or 35 when divided by 36.

A powerful number is a positive integer such that every prime that appears in its prime factorization appears there at least twice. Equivalently, it is a positive integer that can be expressed as the product of a perfect square and a perfect cube. Powerful numbers have been studied by Erdős and Szekeres [1], Golomb [3], and Mollin and Walsh [4]. It is an unsolved problem in mathematics whether three consecutive powerful numbers exist, although Erdős, Mollin and Walsh have conjectured that such numbers do not exist [2, 4]. This paper will show that if three consecutive powerful numbers do exist, the smallest number must be congruent to $7 \pmod{36}$, $27 \pmod{36}$ or $35 \pmod{36}$.

Theorem 0.1. *If three consecutive powerful numbers exist, they must be of the form $(36k + 7, 36k + 8, 36k + 9)$, $(36k + 27, 36k + 28, 36k + 29)$ or $(36k - 1, 36k, 36k + 1) = (36(k - 1) + 35, 36k, 36k + 1)$ for some integer k .*

Proof. Three consecutive powerful numbers must be $(\text{even}, \text{odd}, \text{even})$ or $(\text{odd}, \text{even}, \text{odd})$. However, in the $(\text{even}, \text{odd}, \text{even})$ case, either the smallest number is congruent to $2 \pmod{4}$ or the smallest number is congruent to $0 \pmod{4}$, in which case the largest number is congruent to $2 \pmod{4}$. Whichever number is congruent to $2 \pmod{4}$ will be divisible by 2 but not by 4, so 2 will appear in the number's prime factorization only once. However, by the definition of a powerful number, every prime that appears in the prime factorization must appear there at least twice, so this number cannot be powerful, and we have a contradiction because all three numbers have to be powerful. Therefore, the $(\text{even}, \text{odd}, \text{even})$ case is impossible, so the powerful numbers must be $(\text{odd}, \text{even}, \text{odd})$. The middle number is even, so 2 must appear in that number's prime factorization at least twice by the definition of a powerful number. The middle number is therefore divisible by 4. Also, the three powerful numbers are consecutive, so one of

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them will be divisible by 3. By the definition of a powerful number, 3 must appear in that number's prime factorization at least twice, so the number is divisible by 9. Therefore, the middle number is divisible by 4 and one of the numbers is divisible by 9. There are now three cases depending on which number is divisible by 9.

Case 1:

The smallest number is divisible by 9.

We already know the middle number is divisible by 4. The smallest number will therefore be congruent to $3 \pmod{4}$. We will let the smallest number be x . Because x is divisible by 9, x can be written as $9j$, when j is an integer. We have $x \pmod{4} = 3$, so $(9j) \pmod{4} = 3$ (since $x = 9j$). Thus, $3 = (9j) \pmod{4} = ((9 \pmod{4})(j \pmod{4})) \pmod{4} = (j \pmod{4}) \pmod{4} = j \pmod{4}$.

Therefore, $j \pmod{4} = 3$, so j can be written as $4k + 3$, when k is an integer. Therefore, $x = 9j = 9(4k + 3) = 36k + 27$, so the three consecutive powerful numbers are of the form $(36k + 27, 36k + 28, 36k + 29)$.

Case 2:

The middle number is divisible by 9.

We already know the middle number is divisible by 4. Because the middle number is also divisible by 9, the middle number is divisible by 36. Since the middle number is divisible by 36, it can be written as $36k$, when k is an integer. Therefore, the three consecutive powerful numbers are of the form $(36k - 1, 36k, 36k + 1) = (36(k - 1) + 35, 36k, 36k + 1)$.

Case 3:

The largest number is divisible by 9.

We already know the middle number is divisible by 4, so the largest number is congruent to $1 \pmod{4}$. We will let the largest number be x . Because x is divisible by 9, x can be written as $9j$, when j is an integer. We have $x \pmod{4} = 1$, so $(9j) \pmod{4} = 1$ (since $x = 9j$). Thus, $1 = (9j) \pmod{4} = ((9 \pmod{4})(j \pmod{4})) \pmod{4} = (j \pmod{4}) \pmod{4} = j \pmod{4}$.

Therefore, $j \pmod{4} = 1$, so j can be written as $4k + 1$, when k is an integer. Therefore, $x = 9j = 9(4k + 1) = 36k + 9$, so the three consecutive powerful numbers are of the form $(36k + 7, 36k + 8, 36k + 9)$.

Summarizing, we have shown that if three consecutive powerful numbers exist, they must be of the form $(36k + 7, 36k + 8, 36k + 9)$, $(36k + 27, 36k + 28, 36k + 29)$ or $(36k - 1, 36k, 36k + 1) = (36(k - 1) + 35, 36k, 36k + 1)$, when k is an integer. \square

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