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Matthew J. Ko
Centre College, matthewko1698@gmail.com

Jason Pinto
Centre College

Aaron Davis
Murray State University

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Cover Page Footnote

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New Results on Subtractive Magic Graphs

By *Aaron Davis*, *Matthew Ko*, and *Jason Pinto*

Abstract. For any edge xy in a directed graph, the subtractive edge-weight is the sum of the label of xy and the label of y minus the label of x. Similarly, for any vertex z in a directed graph, the subtractive vertex-weight of z is the sum of the label of z and all edges directed into z and all the labels of edges that are directed away from z. A subtractive magic graph has every subtractive edge and vertex weight equal to some constant k. In this paper, we will discuss variations of subtractive magic labelings on directed graphs.

1 Introduction

The concept of magic labelings originated from magic squares which have been studied since the 13^{th} century in the Yuan dynasty of China. *Magic labelings* are bijections from the integers to the vertices of a graph such that each vertex has the same magic constant. Barone [1] explored the application of magic labelings to directed graphs.

Definition 1.1. A directed graph is a graph such that every edge has a direction, always towards one vertex and away from a different vertex. If an edge is directed from vertex x to vertex y, we call x the tail, y the head, and denote the edge as xy.

Subtractive magic labelings are specialized versions of *total labelings*, which, for a graph G, we can define as a bijection from the set $\{1,2,\dots,v+e\}$ to $\mathrm{V(G)} \cup \mathrm{E(G)}$, where v is the number of vertices and e is the number of edges. Furthermore, a *total labeling* is *strong* if the vertices of G are labeled using only the labels $\{1,2,\dots,v\}$. For a total labeling, λ, Barone [1] defines the *subtractive vertex-weight* of vertex x, $wt^-(x)$, to be $\lambda(x)+\sum\lambda(yx)-\sum\lambda(xy)$ for all edges adjacent to x, and the *subtractive edge-weight* of edge xy, $wt^-(xy)$, to be $\lambda(xy)+\lambda(y)-\lambda(x)$. Figure 1 is an example of the subtractive vertex-weight of a vertex x.

From this, a *subtractive vertex-magic* labeling occurs when the subtractive vertex-weights of all vertices are equal, and a *subtractive edge-magic* labeling occurs when the subtractive edge-weights of all edges are equal. The following two definitions are from [1].

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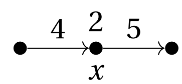


Figure 1: Subtractive Vertex-Weight, $wt^{-}(x) = 2+4-5 = 1$.

Definition 1.2. A total labeling of graph G has a subtractive vertex-magic labeling (SVML) if the vertex-weight, $wt^{-}(x) = \lambda(x) + \sum \lambda(yx) - \sum \lambda(xy)$, is equal to some integer k for all $x \in V(G)$. A graph is subtractive vertex-magic if it has a subtractive vertex-magic labeling.

Definition 1.3. A total labeling of graph G is subtractive edge-magic if the edge-weight, $wt^{-}(xy) = \lambda(xy) + \lambda(y) - \lambda(x)$, is equal to some integer k for all $xy \in E(G)$.

Subtractive magic labelings are specialized versions of *total labelings*, which, for a graph G, we can define as a bijection from the set $\{1, 2, \ldots, v+e\}$ to $V(G) \cup E(G)$. Furthermore, a *total labeling* is *strong* if the vertices of G are labeled using only the labels $\{1, 2, \ldots, v\}$. Figure 2 illustrates a strong magic labeling of a graph.

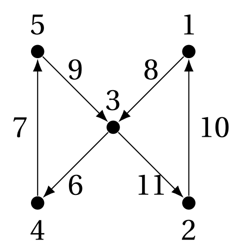


Figure 2: Strong total labeling of a graph with $v = 5$ and $k = 3$.

In this paper, we first consider subtractive vertex-magic graphs and strong subtractive vertex-magic graphs. In Section 3, we introduce subtractive vertex-antimagic graphs and show that all caterpillars can be oriented to permit a subtractive vertex-antimagic labeling. In Section 4, we introduce lemniscate graphs and provide results on the subtractive vertex-antimagic labelings of these graphs. We discuss subtractive edge-magic labelings in Section 5. Here, we show how to construct an infinite number of trees that are subtractive-edge magic, and furthermore, that when considering connected graphs, only trees may be strong subtractive edge-magic. Finally, we discuss in- and out-magic labelings. In-magic and out-magic graph labelings are another variant of magic labelings on directed graphs that calculate vertex or edge-weight by summing over edges coming into or out of each vertex. We explore an open question from Marr and Wallis that consider which orientations of trees permit in-magic labelings [2]. This question still remains elusive, as we show classes of directed trees that permit such labeling, but still seek for a broader generalization of the relationship between the two.

2 Subtractive Vertex-Magic Graphs

To begin our exploration of subtractive vertex-magic graphs, we illustrate a relationship between the sum of the vertex labels, the number of vertices, and the magic constant.

Theorem 2.1. *The vertex labels of a subtractive vertex-magic graph* G *sum to* vk*, where* k *is the magic constant and* v *is the number of vertices in* G*.*

Proof. Let G be a subtractive vertex-magic graph with magic constant k. Because the subtractive vertex-weight of each vertex is equal to k, the sum of the vertex-weights of all of the vertices of G is:

$$\sum_{x\in \mathrm{V(G)}} wt^-(x) = \sum_{x\in \mathrm{V(G)}} \left[\lambda(x) + \sum_{ix\in \mathrm{E(G)}} \lambda(ix) - \sum_{xj\in \mathrm{E(G)}} \lambda(xj)\right].$$

Every edge label in G is added once and subtracted once in the summation above, therefore,

$$\sum_{x\in \mathrm{V(G)}} \left[\lambda(x) + \sum_{ix\in \mathrm{E(G)}} \lambda(ix) - \sum_{xj\in \mathrm{E(G)}} \lambda(xj)\right] = \sum_{x\in \mathrm{V(G)}} \lambda(x) = vk.$$

Therefore the sum of all of the vertex labels is equal to vk. □

Because this result tells us the value of the magic constant, k, we can use this idea to determine bounds on k if we know the vertex labels of G.

Corollary 2.1. The magic constant, k, for a graph G with a subtractive vertex-magic labeling is bounded such that

$$\frac{v+1}{2} \le k \le \frac{v+2e+1}{2}.$$

Proof. To determine a lower bound on k, we consider the set consisting of the smallest possible values that can be assigned to the vertices. Thus $\mathrm{S} = \{1, 2, \ldots, v\}$, and by Theorem 2.1,

$$\begin{aligned} vk &= \sum_{x\in \mathrm{V(G)}} \lambda(x) \\ &\ge \sum_{i=1}^{v} i \\ &= \frac{v(v+1)}{2}. \end{aligned}$$

That is, $k \ge \frac{v+1}{2}$.

Let $\mathrm{T} = \{e+1, e+2, \ldots, e+v\}$ be the set of size v containing the largest possible assignable vertex labels for V(G). By Theorem 2.1,

$$\begin{aligned} vk &= \sum_{x \in V(G)} \lambda(x) \\ vk &\leq \sum_{i=e+1}^{e+v} i \\ &= \sum_{j=1}^{v} j + e \\ &= \frac{v(v+1)}{2} + ev \\ &= \frac{v^2 + v + 2ev}{2}. \end{aligned}$$

Thus, $k \leq \frac{v+2e+1}{2}$.

Therefore the magic constant, k, for a subtractive vertex-magic graph is bounded above by $\frac{v+2e+1}{2}$, and $\frac{v+1}{2} \leq k \leq \frac{v+2e+1}{2}$. □

Because the vertex labels of strong subtractive vertex-magic graphs must be from $\{1, 2, 3, \ldots, v\}$, we obtain the following corollary.

Corollary 2.2. For every *strong* subtractive vertex-magic graph , the magic constant $k = \frac{v+1}{2}$.

Also, in a subtractive vertex-magic graph, k must be the label on either a vertex or an edge, as k cannot be larger than $v + e$.

Corollary 2.3. The magic constant k must appear as a label in a subtractive vertex-magic graph.

Proof. By Corollary 2.1, k is bounded such that $\frac{v+1}{2} \leq k \leq \frac{v+2e+1}{2}$. Since $v \geq 1$, $\frac{v+1}{2} \geq 1$, and $2v \geq v + 1$. Thus $v + e \geq \frac{v+2e+1}{2}$. It follows that $1 \leq \frac{v+1}{2} \leq k \leq \frac{v+2e+1}{2} \leq v + e$, and $k \in \{1, 2, \ldots, v + e\}$. Therefore, k must appear as a label on a subtractive vertex-magic graph. □

Theorem 2.1 and its corollaries are helpful in seeking to understand the construction of an SVML. We first show that graphs with small degrees cannot be subtractive vertex-magic.

Theorem 2.2. *For any graph* G*, if* $\Delta(G) \leq 2$ *and* $\delta(G) = 1$*,* G *is not subtractive vertex-magic.*

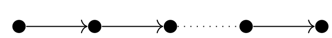
Proof. Assume for contradiction G is a subtractive vertex-magic graph with $\Delta(G) \leq 2$ and $\delta(G) = 1$ and magic constant k. From Corollary 2.3, k must appear as a label for a vertex x or directed edge e.

First, observe that if $\lambda(x) = k$, then $wt^{-}(x)$ cannot equal k. Since x only has at most two incident edges and the labels in a subtractive vertex-magic graph must be positive and unique.

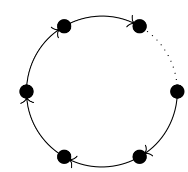
Similarly, $\lambda(e) \neq k$. Suppose e is directed out of a vertex y and towards a vertex z. Then, if $\lambda(e) = k$, $wt^{-}(z) = \lambda(z) + k$ if $\deg(z) = 1$, or if $\deg(z) = 2$, $wt^{-}(z) = \lambda(z) + k \pm \lambda(e')$, where e' is the other edge incident to z. In either case, $wt^{-}(z) \neq k$ because the labels in a subtractive vertex-magic graph must be positive and unique.

Therefore k cannot be a label for any directed edge or vertex in G. Therefore any graph with $1 \leq \Delta(\mathrm{G}) \leq 2$ cannot be subtractive vertex-magic.

□



(a) A dipath.



(b) A directed cycle.

Figure 3: Two classes of directed graphs with $\delta(\mathrm{G}) = 1$ and $\Delta(\mathrm{G}) \leq 2$.

Because all vertices of both dipaths (see Figure 3a) and dicycles (Figure 3b) have a maximum degree of 2, the following corollary follows directly from Theorem 2.2.

Corollary 2.4. There does not exist a subtractive vertex-magic labeling of a dipath or a directed cycle.

The following results lead to other families of graphs which do not permit a subtractive vertex-magic labeling.

Theorem 2.3. *If* G *is graph with* v *vertices and* e *edges that contains a vertex* x *with* $deg^{-}(x) = v - 1$ *and* $e < \frac{v^2-1}{2}$, *then* G *is not subtractive vertex-magic.*

Proof. Assume for contradiction that G is subtractive vertex-magic and $e < \frac{v^2-1}{2}$. The minimal possible subtractive vertex-weight of x is,

$$\begin{aligned} wt^{-}(x) &= \sum_{i=1}^{v} i \\ &= \frac{v(v+1)}{2}. \end{aligned}$$

Corollary 2.1 states that the magic constant k for a subtractive vertex-magic graph has an upper bound $\frac{v+2e+1}{2}$. Therefore the subtractive vertex-weight of x,

$$\begin{aligned} wt^{-}(x) &\leq \frac{v+2e+1}{2} \\ \frac{v(v+1)}{2} &\leq \frac{v+2e+1}{2} \\ v^2-1 &\leq 2e \\ \frac{v^2-1}{2} &\leq e. \end{aligned}$$

Therefore $e \geq \frac{v^2-1}{2}$ for a subtractive vertex-magic graph G that contains a vertex x with $deg^{-}(x) = v-1$, contradicting our original assumption. □

Theorem 2.4. *If* G *is a graph with v vertices and e edges that contains a vertex x with $deg^{+}(x) = v-1$ and $e < \frac{(v-1)^2}{2}$, then* G *is not subtractive vertex-magic.*

Proof. Assume for contradiction that G is subtractive vertex-magic and $e < \frac{(v-1)^2}{2}$. The maximum possible subtractive vertex-weight of x is,

$$wt^{-}(x) = v+e-\sum_{i=1}^{v-1} i = v+e-\frac{v(v-1)}{2}.$$

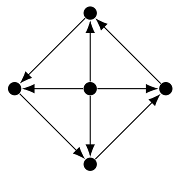
Corollary 2.1 states that the magic constant k for a subtractive vertex-magic graph has a lower bound $\frac{v+1}{2}$. Therefore the subtractive vertex-weight of x,

$$\begin{aligned} wt^{-}(x) &\geq \frac{v+1}{2} \\ v+e-\frac{v(v-1)}{2} &\geq \frac{v+1}{2} \\ e &\geq \frac{v^2+1}{2} - v \\ &= \frac{(v-1)^2}{2}. \end{aligned}$$

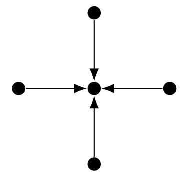
Therefore $e \geq \frac{(v-1)^2}{2}$ for a subtractive vertex-magic graph G that contains a vertex x with $deg^{+}(x) = v-1$, contradicting our original assumption. □

It follows from Theorems 2.3 and 2.4 that out-wheels, inward stars and outward stars, all shown in Figure 4, cannot be subtractive vertex magic.

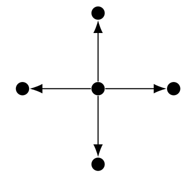
Corollary 2.5. There does not exist a subtractive vertex-magic out-wheel, inward star, or outward star graph.



(a) An out-wheel.



(b) An inward star.



(c) An outward star.

Figure 4: Examples of digraphs on five vertices that cannot have a SVML.

A di-sun graph is a class of graph that permits a subtractive vertex-magic labeling. We present a generalized labeling for a directed di-sun graph.

Definition 2.1. A di-sun graph is a directed cycle where every vertex in the cycle is adjacent to a pendant vertex. The di-sun is an outward di-sun graph if every edge not in the cycle is directed away from the vertices in the cycle.

An example of an outward di-sun is shown in Figure 5.

Theorem 2.5. *All outward di-sun graphs permit a subtractive vertex-magic labeling.*

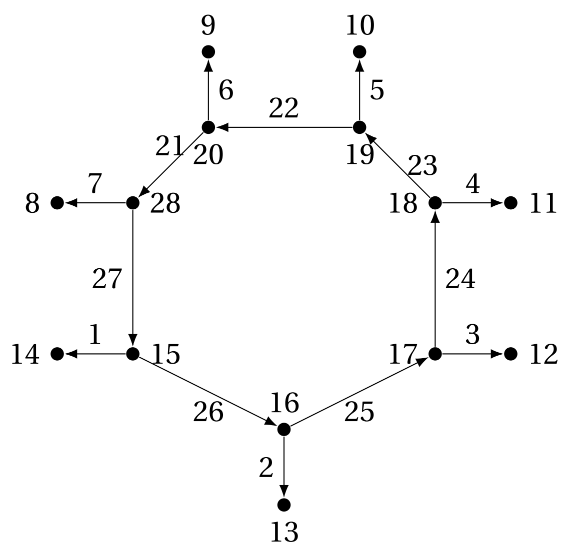


Figure 5: Subtractive vertex-magic outward di-sun graph with $v = 14$ and $k = 15$

Proof. We will first provide the labeling, then show that the labeling is subtractive vertex-magic.

Orient the outward di-sun so that the di-cycle is directed counterclockwise. Choose a pendant vertex and label it v, and label the incident edge 1. Continue labeling counterclockwise around the pendant vertices and incident edge pairs, decrementing the vertex labels by one and incriminating the edge labels by one at each step. Thus, the pendant vertices are labeled with $\{v-(\frac{v}{2}-1),\ldots,v-1,v\}$ and edges are labeled with $\{1,2,\ldots,\frac{v}{2}\}$. Next, label the vertex adjacent the pendant vertex with label v, and label this vertex $v+1$. Label the edge directed into this vertex $2v-1$. Continue labeling counterclockwise around the cycle, incrementing the vertex labels by one and decrementing the edge labels by 1 with the exception that the vertex at the tail of the edge with label $2v-1$ is given the label $2v$. Hence, the cycle vertex are labeled with $\{v+1,v+2,\ldots,v+\frac{v}{2}-1\}\cup\{2v\}$ and the cycle edges are labeled with $\{2v-\frac{v}{2},\ldots,,2v-2,2v-1\}$. We claim that the magic number k of this labeling is $v+1$.

In order to prove that we have a subtractive vertex-magic labeling, it is necessary to prove that this labeling is both a bijection and that every vertex-weight is the same. To be subtractive vertex-magic, the labeling must use every integer from the set $\{1,2,3,\ldots,e+v\}$ exactly once. From our labeling we know our labeling consists of the union of four sets, that is, $\{1,2,\ldots,\frac{v}{2}\}\cup\{v-(\frac{v}{2}-1),\ldots,v-1,v\}\cup\{\{v+1,v+2,\ldots,v+\frac{v}{2}-1\}\cup\{2v\}\}\cup\{2v-\frac{v}{2},\ldots,,2v-2,2v-1\}$. This simplifies to $\{1,2,3,\ldots,2v\}$ with no overlap. Since $e=v$ for all di-suns, this is also $\{1,2,3,\ldots,e+v\}$, making this labeling bijective.

Next, we will show that every vertex-weight is identical. First, consider the pendant vertices. Given this labeling, the vertex-weight of each pendant vertex takes the form of $wt^-(x)=(v-n)+(n+1)=v+1$ where $n\in\{0,1,2,\ldots,\frac{v}{2}-1\}$. Therefore every pendant vertex has the same vertex-weight. Now consider the vertices on the dicycle with the exception of the vertex labeled $2v$. Each vertex x on the dicycle has weight $wt^-(x)=(2v-n)+(v+n)-(n)-(2v-(n+1))$, where n is from the set $\{1,2,\ldots,\frac{v}{2}-1\}$. This simplifies to $wt^-(x)=v+1$, our magic number. Finally, we will consider the vertex labeled $2v$. We simply compute the weight which is $(2v-\frac{v}{2})+(2v)-(\frac{v}{2})-(2v-1)=v+1$. Therefore, every vertex-weight is equal to $v+1$.

We have shown that the labeling is a bijection from the set $\{1,2,3,\ldots,e+v\}$ and that every vertex-weight is equal to $v+1$. Therefore, this labeling is subtractive vertex-magic. □

Not all families of graphs that are subtractive vertex-magic can permit a strong subtractive vertex-magic labeling, such as the di-sun and dipath.

Theorem 2.6. *There does not exist a strong subtractive vertex-magic graph with a vertex of degree 1.*

Proof. Let us assume there exists a strong subtractive vertex-magic graph with a vertex x that has degree 1, where edge d is directed toward or away from x. Recall from Corollary 2.2 that $k=\frac{(v+1)}{2}$ in a strong subtractive vertex-magic graph. Because this is a strong graph, the maximum possible value of $\lambda(x)$ is v, and the minimum possible value of $\lambda(d)$ is $v+1$. This brings us to two possible cases.

Case 1: Assume x has an in-degree of 1. The subtractive vertex-weight of x is $\lambda(x)+\lambda(d)=k$. By our restrictions on labels in strong graphs, the minimum possible vertex label is 1, while the minimum possible edge label is $v+1$. Therefore, the minimum possible value of k is $1+(v+1)=v+2$. Since $v+2>\frac{(v+1)}{2}$ for all positive v, we arrived at a contradiction with Corollary 2.2.

Case 2: Assume x has an out-degree of 1. The subtractive vertex-weight of this x is $\lambda(x)-\lambda(d)=k$. The maximum value of this weight is $v-(v+1)=-1$. Since $-1<\frac{(v+1)}{2}$ for all positive v, we again arrive at a contradiction.

Hence, there does not exist a strong subtractive vertex-magic graph with a vertex of degree 1. □

At this point, multiple classes of graphs have been considered under the subtractive vertex-magic definition. This includes general properties of a subtractive vertex-magic graph and classes of graphs that do not permit SVMLs.

3 Subtractive Vertex-Antimagic Graphs

Next, we consider *subtractive vertex-antimagic graphs.* These graphs differ from subtractive vertex-magic graphs because the subtractive vertex-weight is unique for all vertices in a graph.

Definition 3.1. A graph G has a subtractive vertex-antimagic labeling (SVAL) if the subtractive vertex weights of each vertex in G is unique. If G has a SVAL, we say G is a subtractive vertex antimagic graph. Moreover, if the SVAL has vertex labels from among $\{1,2,\ldots,v\}$, then this is a strong subtractive vertex antimagic labeling, SSVAL, and G is a strong subtractive vertex-antimagic graph.

The following theorem and proof introduce a method to orient any subtractive vertex-magic graph into a subtractive vertex-antimagic graph.

Theorem 3.1. *Every subtractive vertex-magic graph can be oriented to permit a subtractive vertex-antimagic graph.*

Proof. Let λ be a subtractive vertex-magic labeling of graph G. Recall the subtractive vertex-weight of each vertex is defined by the equation

$$wt^-(x)=\lambda(x)-\sum\lambda(yx)+\sum\lambda(xz)=k.$$

Consider reversing the direction of every edge in G, resulting in a new graph, G′. Before every vertex-weight was equal to k. Let us temporarily call the new weight k'. For a vertex x in G′ we have

$$wt^-(x)=\lambda(x)+\sum\lambda(yx)-\sum\lambda(xz)=k'.$$

Observe,

$$\lambda(x) = k' - \sum \lambda(yx) + \sum \lambda(xz)$$
$$2\lambda(x) = k' + \lambda(x) - \sum \lambda(yx) + \sum \lambda(xz)$$
$$2\lambda(x) = k' + k$$
$$k' = 2\lambda(x) - k.$$

Because each vertex label in λ is unique, the value of k' is unique for each vertex. Therefore G' is subtractive vertex-antimagic. □

The following theorem focuses on a specific class of trees called a caterpillar, where all leaves are 1 edge length away from a central path called the spine. We provide a labeling that permits any caterpillar to be subtractive vertex-antimagic.

Theorem 3.2. *All caterpillar graphs can be oriented to permit a subtractive vertex-antimagic labeling.*

Proof. Let C be a caterpillar with v vertices and e edges. Let path P include the spine and leaves adjacent to the spine's end vertices, so that P has 2 vertices of degree 1 and n vertices. Orient all edges of P in the same direction. Locate the vertex on P with a single edge directed away from it. Label this vertex with $2n-1$. Label the adjacent vertex on the path with $2n-2$. Continue labeling consecutively down P, decreasing the label by 1 for each unlabeled vertex. Label the edge xy, such that $\lambda(x) = 2n-1$, with 1. Continue labeling the edges consecutively down the path, increasing each label by 1 more than the previous edge. The path is labeled from $\{1, 2, \ldots, 2n-1\}$.

Let the leaves of a caterpillar graph be the vertices adjacent P. Begin by orienting the edges adjacent to the leaves away from P. Let l be the number of unlabeled leaves, where $l = v - n$. Let j be the vertex along P with the highest degree that is adjacent to unlabeled leaves. If there is more than one, let j be the one with the smallest label. For all unlabeled leaves adjacent to j, label the first arbitrary leaf $(2v-1)-l$ and the next $(2v-1)-l-1$ and so on, decreasing each label by 1 until all adjacent leaves are labeled. Repeat this process with the next vertex adjacent to unlabeled leaves that has the highest degree and smallest label. The leaves adjacent to P are now labeled from $\{2n, 2n+1, \ldots, (2v-1)-l\}$.

Let zq be an unlabeled edge such that $\lambda(q) \in \{2n, 2n+1, \ldots, (2v-1)-l\}$. Label this edge with $\lambda(q)+l$. Repeat this process for all unlabeled edges until all labels from $\{2n+l, 2n+l+1, \ldots, 2v-1\}$ are used.

The edge with the largest label is directed toward the leaf with the largest label. The edge with the second largest label is directed toward the leaf with the second largest label, and so on. Each leaf only has at most one edge directed towards it, so these vertices have the highest subtractive vertex-weights. The subtractive vertex-weights of all leaves adjacent to P are also unique given the properties of this labeling.

This labeling permits the vertices along P of degree 1 or 2 to have a unique subtractive vertex-weights that are elements of $\{n, n+1, \ldots, 2n-1\}$. For vertices of degree 3 or more, the labeling ensures that edges directed away from these vertices have labels that are all greater than the vertex label. These vertices along P of degree 3 or more have unique subtractive vertex-weights. The vertices of highest degree, with the smallest vertex labels are adjacent to edges with the largest relative edge labels. These edges are directed away from these vertices and will have the smallest subtractive vertex-weights. Among the vertices of the smallest degree greater than or equal to 3, those with the largest vertex labels will have the largest subtractive vertex-weights.

This labeling ensures that the subtractive vertex-weights of the vertices along P are unique, where the weights of vertices of degree 3 or more are less than the weights of degree 1 or 2 and leaf vertices. The labeling ensures subtractive vertex-weights of leaf vertices adjacent to P are unique and greater than all spine vertices, together permitting a subtractive vertex-antimagic labeling. □

This section illustrates a connection between subtractive vertex-magic and subtractive vertex-antimagic graphs, showing that every graph that permits a subtractive vertex-magic labeling also permits a subtractive vertex-antimagic labeling. Specifically, caterpillars were proven to permit a subtractive vertex-antimagic labeling.

4 Lemniscate Graphs

In our search to find more families of graphs that permit a subtractive vertex-magic labeling, we created the lemniscate graph.

Definition 4.1. A lemniscate graph, ∞_v, is two directed cycles, C_v, which share a vertex and are oriented in opposite directions.

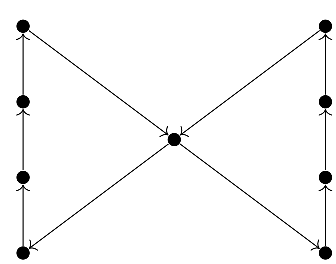


Figure 6: Lemniscate Graph, ∞_5

We provide an example of ∞_5 in Figure 6. In this section, we prove all lemniscate graphs have subtractive vertex-magic and strong subtractive vertex-antimagic labelings and find bounds on the magic constant of lemniscate graphs with v vertices.

Our first result is a corollary which extends our results from Theorem 2.1 on the range of magic constants of SVMLs to lemniscate graphs.

Corollary 4.1. For a lemniscate graph, ∞_v, with a subtractive vertex-magic labeling, the magic constant k must satisfy the following,

$$\frac{v+1}{2} \leq k \leq \frac{3(v+1)}{2}.$$

Proof. Let ∞_v be a lemniscate graph with a subtractive vertex-magic labeling. By Corollary 2.1, the least value of k is $\frac{(v+1)}{2}$.

If the vertices of ∞_v are labeled with the largest possible values in $\mathrm{T} = \{v+2, v+3, \ldots, 2v+1\}$, by Theorem 2.1,

$$(v+2)+(v+3)+\ldots+(2v+1) = (3v+3)\frac{v}{2} = vk.$$

This leads to the largest value of $k = \frac{(3v+3)}{2}$.

The range of potential magic constants for a subtractive vertex-magic lemniscate graph is $\frac{v+1}{2} \leq k \leq \frac{3(v+1)}{2}$. □

Next we will prove that for all lemniscate graphs, there exists a strong subtractive vertex-antimagic labeling, SSVAL, and then further show all lemniscate graphs permit a SSVAL in which the edge-weights form a specific type of arithmetic sequence. Figure 7 provides an illustration of the labeling associated with the following proof.

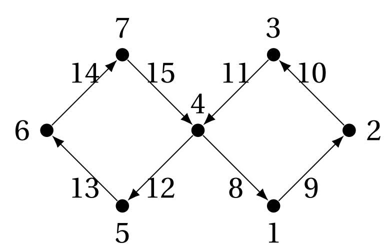


Figure 7: A lemniscate SSVAL, ∞_4

Theorem 4.1. *For any lemniscate graph, there exists a strong subtractive vertex-antimagic labeling.*

Proof. Consider a lemniscate graph ∞_v. We will prove the statement by constructing a labeling on ∞_v. First, select an edge that is directed away from the vertex of degree 4. Label the vertex at the head of that edge 1. Continue moving along edges toward unlabeled vertices, labeling each unlabeled vertex consecutively. Continue labeling onto the unlabeled cycle.

Next, label the edge directed into the vertex labeled 1 with $v+1$. Then, following the pattern used to label the vertices, label the edges.

For every vertex x of degree 2 with edges yx directed toward x and xz directed away from x, the subtractive vertex-weight is

$$\begin{aligned} wt^{-}(x) &= \lambda(x) + \lambda(yx) - \lambda(xz) \\ &= \lambda(x) + \lambda(yx) - (\lambda(yx) + 1) \\ &= \lambda(x) - 1. \end{aligned}$$

The weight of each non-center vertex is equal to the vertex label minus 1, and therefore unique because every vertex label is unique.

Now consider the center vertex, call it x_c. Because each cycle of the lemniscate is length $\frac{v-1}{2}$ and the edges are labeled consecutively, the label of the edge directed into x_c minus the edge directed out is equal to $\frac{v-1}{2}$. Because there are two cycles, the sum of incoming edges minus outgoing edges of x_c equals $v-1$, so the subtractive vertex-weight of x_c is $\frac{2(v-1)+(v+1)}{2}$. Since the largest subtractive vertex-weight of any other vertex is $v-1$, the subtractive vertex-weight of x_c is unique.

The vertices of G are labeled with 1 through v and the subtractive vertex-weight of each vertex is unique, therefore G permits an SSVAL. □

An important property of subtractive vertex-magic graphs is the magic constant, especially when finding specific labelings. The range of posssible magic constants that permit a subtractive vertex-magic labeling for lemniscate graph are given below.

Theorem 4.2. *The set of SVML magic numbers of a lemniscate graph is represented by the symmetric set* $k \in \left\{\frac{(v+1)}{2}, \ldots, \frac{3(v+1)}{2}\right\}$, *where the center of the set is* $v+1$.

Proof. Let λ be a SVML of a lemniscate graph with v vertices and magic constant k. Consider the mapping $\phi : \{1,2,\ldots,2v+1\} \to \{1,2,\ldots,2v+1\}$ defined by $\phi(x) = 2(v+1) - x$. Observe that ϕ maps each positive integer to its additive inverse in $\mathbb{Z}_{2v+2}$. Thus, ϕ is a bijection. To show that for each labeling $\lambda(k)$, there exists a labeling $\phi(\lambda)$ with magic constant $\phi(k)$, we must first show that if

$$a + b - c = k$$

then, $\phi(a) + \phi(b) - \phi(c) = \phi(k)$. Observe:

$$\begin{aligned} \phi(a) + \phi(b) - \phi(c) &= (2(v+1) - a) + (2(v+1) - b) - (2(v+1) - c) \\ &= 2(v+1) - (a+b-c) \\ &= 2(v+1) - k \\ &= \phi(k). \end{aligned}$$

Therefore, we know that every vertex under ϕ will have the same subtractive vertex-weight, implying that every SVML of a lemniscate graph, λ, with magic constant k has a complementary SVML, $\phi(\lambda)$, with magic constant $\phi(k)$. This property creates a symmetric distribution of SVMLs around magic constant $k = v + 1$. □

Conjecture 4.1. For a lemniscate graph, there exists a subtractive vertex-magic labeling with magic number $k = e$ when the central vertex is labeled with k.

The lemniscate graph has produced the most varied subtractive vertex-magic labelings when compared to most other classes of connected graphs. We would like to determine if a lemniscate graph has a generalized labeling that will always result in a subtractive vertex-magic graph.

5 Subtractive Edge-Magic Graphs

The concept of subtractive magic labelings can furthermore be extended to focus on the weights of the edges as opposed to the vertices. The *subtractive edge-weight* of a directed edge is defined as the sum of the edge label and the vertex label at the head, subtracted by the vertex label at the tail. A subtractive-edge-magic labeling, or SEML, is defined as a labeling where the subtractive edge-weight of every edge is equal. Similarly, a *subtractive edge-antimagic* labeling, SEAL, is a labeling where the subtractive edge-weights are unique.

Barone [1] introduced the following theorem that presents a relationship between gracefully labeled trees and a strong edge-magic labeling. First, we must introduce the concept of *graceful* labelings.

Definition 5.1. A labeling of the vertices of a graph G is graceful if the vertex labels are unique and make up a subset of the integers $\{0, 1, \ldots, e\}$. Furthermore, the weights of the absolute differences between adjacent vertices are also unique. A graceful labeling is considered strong if the vertex labels create a bijection to $\{1, 2, \ldots, v\}$. A graph is a graceful if it permits a graceful labeling.

Theorem 5.1. ***[Barone 2008]*** *Let* T *be a tree with n vertices. Then* T *can be oriented so as to permit a strong subtractive edge-magic labeling if and only if* T *has a graceful labeling.*

Our subsequent work extends Barone's initial approach to subtractive edge-magic labelings (called arc-magic by Barone) by first considering the implications of a graceful labeling on graphs in general and considers orientations of graceful labelings that permit a SEML and SEAL.

Theorem 5.2. *All strong, connected graceful graphs are trees.*

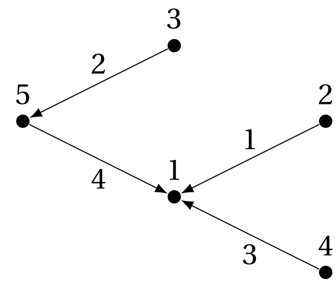


Figure 8: Strong graceful di-graph with edges labeled with the absolute differences of the vertex labels.

Proof. Let G be a simply connected graph with v vertices and e edges and suppose that G contains a cycle. If the edges of G are assigned labels $\lambda(xy) = |\lambda(x) - \lambda(y)|$, the edges of G must contain one of the labels $\{1, 2, \ldots, v-1\}$. Since $e \geq v$, and $\lambda(xy) \in \{1, 2, \ldots, v-1\}$, by the pigeon hole principle, at least two of the edges must have the same label. For a graph to be graceful with vertex labels $\lambda(x) \in \{1, 2, \ldots, v\}$, the number of edges must be less than v, therefore the graph must be a tree. □

This theorem illustrates that if a graph is strongly labeled and permits a graceful labeling, it must be a tree, extending the correlation between a strong subtractive edge-magic labeling, graceful labelings and trees that Barone initially found. The next two proofs provide orientations for existing graceful graphs that will permit a subtractive edge-magic and edge-antimagic labeling.

Theorem 5.3. *Given a graph* G *with v vertices and e edges,* G *can be oriented in a way to permit a subtractive edge-magic labeling if* G *has a strong graceful labeling.*

Proof. Let λ be a graceful labeling on G. Orient the edges such that the head is at the vertex with the larger label and the tail is at the vertex with the smaller label. Label the edges with the sum of the distinct absolute difference of the vertex label at the head and the vertex label at the tail and v.

Because any edge xy in a graph G with graceful labeling λ has a subtractive edge-weight of v, then the graph has a subtractive edge-magic labeling. □

Graphs that permit a graceful labeling λ can also be oriented similarly to present a subtractive edge antimagic labeling.

Theorem 5.4. *Given a graph* G *with v vertices and e edges,* G *can be oriented in a way to permit a subtractive edge-antimagic labeling if* G *has a strong graceful labeling.*

Proof. Let λ be a strong graceful labeling on G. Orient the edges such that the head is at the vertex with the smaller label and the tail is at the vertex with the larger label. Label the edges with the sum of the distinct absolute difference of the vertex label at the head and the vertex label at the tail and v.

Following this construction, each edge has a unique subtractive edge-weight, therefore G can be oriented to produce a subtractive edge-antimagic labeling. □

This last proof illustrates how strong subtractive edge-magic graphs can be generated from an existing strong edge-magic graph.

Theorem 5.5. *Let* G *be a graph that permits a strong subtractive edge-magic labeling with v vertices. If we add a pendant vertex x_{v+1} to the vertex with label 1, there exists a strong subtractive edge-magic labeling for the resulting graph.*

Proof. Let G be a graph with a strong subtractive edge-magic labeling λ on v vertices with $v-1$ edges and magic constant $k = v$, with vertices labeled 1 through v. Let vertex x_i be the vertex with label i.

By Theorem 5.1, a tree, T, permits a strong SEML if and only if T has a graceful labeling. To recover the graceful labeling from G, remove the edge labels and orientations. Call this graph G′. Now consider the graph G″ formed by adding a pendant vertex, x_{v+1} to x_1. The absolute difference between $v+1$ and 1 is $|(v+1)-1| = v$. This difference is unique as the maximum absolute difference in G′, the recovered graceful graph, was $|v-1|$. Therefore G″ is graceful.

Theorem 5.3 then gives an orientation of G″ that has a strong SEML. Therefore there exists a strong SEML if a pendant vertex v_{v+1} is added to a graph with a strong SEML at the vertex v_1. □

6 Results on In-/Out-Magic

In- and out-magic labelings are also related to directed graphs, but consider only the edges directed toward vertices for in-magic and edges directed away from vertices for out-magic. The following definition illustrates how the in-weight and out-weight of each vertex is determined. In this section we design a labeling that permits an in and out magic labeling for all trees.

Definition 6.1. A digraph D with v vertices and e edges is *in-magic* if there exists a bijective function $\lambda : V(D) \cup E(D) \to \{1, 2, \ldots, v+e\}$ such that for every x, the in-weight, $wt^{in}(x)$,

$$\lambda(x) + \sum_{(y,x)\in E(D)} \lambda(y,x) = k$$

for some $k \in Z$. Similarly, D is out-magic if the out-weight, $wt^{out}(x)$,

$$\lambda(x) + \sum_{(x,y)\in E(D)} \lambda(x,y) = m$$

for some $m \in Z$.

This theorem gives a generalized labeling for all trees to permit an in-magic and out-magic labeling.

Theorem 6.1. *There exists an in-magic and out-magic labeling for all trees.*

Proof. Let T be a tree with v vertices and $v-1$ edges. Label T with the following labeling λ. Select an arbitrary vertex x and give it the label $2v-1$. Now label the n vertices adjacent to x consecutively with the labels $\lambda(x)-1$, $\lambda(x)-2, \ldots$ and orient every edge away from x. Now label each edge xy, where x is the tail and y is the head, with $2v-1-\lambda(y)$. Continue to label remaining vertices and edges in the same manner, beginning with an arbitrary unlabeled vertex adjacent to a labeled vertex, with vertex labels $v-n-1$, $v-n-2, \ldots,$ i.e. the next largest available label. The in-weight of every vertex x in T, $wt^-(x)$ is now $\lambda(x)+\lambda(yx)$. Because $\lambda(yx) = 2v-1-\lambda(x)$, $wt^-(x) = \lambda(x)+2v-1-\lambda(x) = 2v-1$. Therefore every vertex has in-weight $2v-1$ and T is therefore in-magic.

It follows that reversing the orientation of each edge produces a graph where where $wt^+(x) = 2v-1$ for all $x \in V(T)$. Because each vertex has an out-weight of 2v-1 for this orientation, T is out-magic. □

The graph in Figure 9 illustrates an in-magic oriented tree, labeled in the manner described in proof of Theorem 6.1.

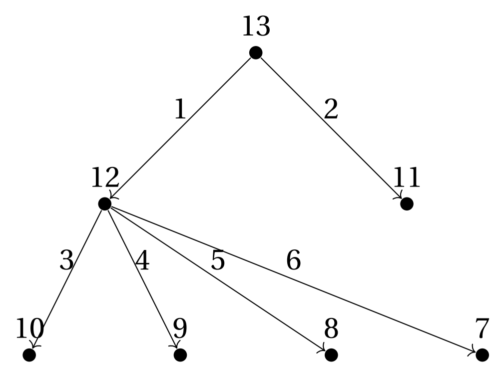


Figure 9: In-Magic orientation of a tree with 7 vertices

7 Open Questions

We have defined a lemniscate graph and discovered subtractive vertex-magic labelings for a subset of these graphs. However, there are still questions to be explored.

Question 7.1. Is there a general subtractive vertex-magic labeling for a lemniscate graph?

The more edges that are present, the more difficult it is to find a subtractive vertex-magic labeling for a graph. The lemniscate graph is a fully connected graph with many subtractive vertex-magic labelings.

Question 7.2. Are there other connected graphs that can be subtractive vertex-magic?

We have found families of graphs that permit subtractive vertex-magic and subtractive edge-magic labelings, leading us to the following question.

Question 7.3. Does there exist a graph that is both subtractive vertex and subtractive edge-magic?

Barone [1] provided a conjecture that all trees can be oriented to permit a strong subtractive edge-magic labeling. This leads us to the following question.

Question 7.4. Are there any classes of trees that do not permit a strong subtractive edge-magic labeling?

8 Conclusion

We have provided the generic labelings for classes of subtractive vertex-magic and subtractive vertex-antimagic graphs as well as subtractive edge-magic and subtractive edge-antimagic graphs. In addition, our results illustrate new properties of these graphs that aid in the discovery of specific labelings. As well as considering new classes of graphs under these subtractive conditions, we have expanded Barone's [1] original connection to graceful labelings for subtractive edge-magic.

References

[1] C. A. Barone. *Magic labelings of directed graphs.* PhD thesis, University of Victoria, 2008.

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Aaron Davis
Murray State University
adavis63@murraystate.edu

Matthew Ko
Centre College
matthewko1698@gmail.com

Jason Pinto
Centre College
jasoncpin@gmail.com