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Cover Page Footnote

We would like to thank Dr. Jason Elsinger for coaching our team and revising our drafts. We would also like to thank Dr. Brian Winkel for hosting us at the Joint Mathematics Meetings and providing helpful discussions. Finally, we thank SCUDEM for the challenge and experience.

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Using Differential Equations to Model Predator-Prey Relations as Part of SCUDEM Modeling Challenge

By *Zachary Fralish*, *Bernard Tyson III*, and *Anthony Stefan*

Abstract. Differential equation modeling challenges provide students with an opportunity to improve their mathematical capabilities, critical thinking skills, and communication abilities through researching and presenting on a differential equations model. This article functions to display an archetype summary of an undergraduate student team's response to a provided prompt. Specifically, the provided mathematical model estimates how certain stimuli from a predator are accumulated to trigger a neural response in a prey. Furthermore, it tracks the propagation of the resultant action potential and the physical flight of the prey from the predator through the analysis of larval zebrafish as a model organism. This article also shares personal testimonies to highlight the benefits of these kinds of challenges for students.

1 Introduction

In the Spring 2018 SIMIODE Challenge Using Differential Equations Modeling (SCUDEM) [2], teams of three students were given three situations to model from different disciplines: life sciences, physical sciences, and social sciences. Each team was given a week to choose one problem, and prepare both an executive summary and an oral presentation on the model they built. The executive summary is a two page report describing the problem, as well as the assumptions, implications, and limitations of their model created to solve the problem. On the day of the challenge all teams meet at a central location to give a short presentation that explains their model and its implications. Also, an additional issue to consider is given and teams are provided with an hour to incorporate this issue into their presentations. The need to develop clear assumptions which simplify a real-world phenomenon forces students to have a practical outlook when developing a model. Students also gain a deeper understanding of mathematical principles and develop their communication skills for their model. Finally, a discussion

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of the limitations and future directions of their model allows students to think beyond their chosen phenomenon to more general settings.

Often, working equations in a classroom does not provide students with enough problem solving practice necessary to apply their technical skills to more realistic or abstract issues found outside of the college campus. The process of synthesizing ideas into a coherent executive summary, as well as presenting to others, requires students to have a deeper understanding of a particular area of mathematics and develops their communication skills. Students also work in teams, where each student contributes different experiences and backgrounds to benefit the group. In addition, an unknown consideration on the day of the competition forces students to adapt to unforeseen circumstances, which requires a thorough understanding of their model. Furthermore, they can connect with other student teams and build a network that allows for opportunities to further develop their work.

The problem we chose [3], as shown in the appendix, generally asked "How can two or more inputs be incorporated together to enable a simple organism to decide whether or not to flee?" Provided below is background information we gathered from articles released to each team on predator-prey relations, in addition to our executive summary. Furthermore, three personal testimonies are presented that explain how the competition influenced and benefited each of us.

2 Executive Summary

In order to survive in the wild, prey must optimize their limited energy. In doing so, they only flee when absolutely necessary, and swiftly respond when under impending danger. This has evolutionarily molded the neural networking of some species of prey to accurately and rapidly respond to threats, as epitomized by the contralateral system of two Mauthner cells (M-cells) [1]. Kinetic, spatial, auditory, electrical, and tensive stimuli are recorded by these two neurons to create an assessment of peril associated with the presence of another organism. When a threshold is reached within the M-cell, activation of a signaling cascade through connected reticulospinal and motor neurons initiates movement away from the stimuli. These cells are located bilaterally in the head of larval zebrafish, a model organism that was in an article provided with our prompt, and are responsible for signaling a ballistic escape response called a C-start.[1] The C-start response is an escape reflex that is exhibited by fish and amphibians. This mathematically models how multiple outputs from a predator are accumulated to trigger a neural response in prey. The propagation of the resultant action potential and the physical fleeing from the predator will also be modelled through the analysis of larval zebrafish as a model organism.

2.1 Models and Assumptions

In order to succinctly model this natural phenomenon, the following assumptions were made. We developed a model in two-dimensions, simply to stay out of the complexity of three-dimensions. We assumed the size of the predator was equal to the greatest dimension of the predator (height, width, diagonal, etc.) and we defined the size as the "image" of the predator. The perceived size of the predator by the prey was assumed to be proportional to the subtended angle (β) whose origin is at the eye of the prey and is formed by the rays from the origin to the left and right or top and bottom extreme points of the predator's image. Figure 1 depicts the angle β that is formed by the rays to the endpoints of the predator's image.

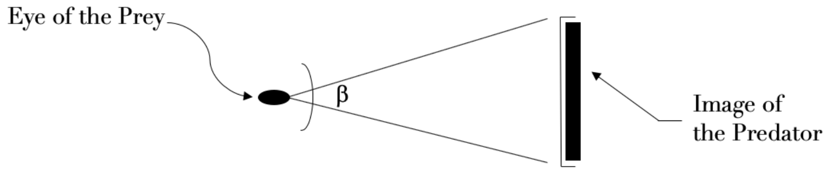

Figure 1: The relationship between the perceived size and the image of the predator.

Continuing to use the larval zebrafish as a model organism, it was assumed the physical response of the prey was caused purely by M-cell stimulation. Figure 2 is a visual representation of the location of M-cells in zebrahfish.

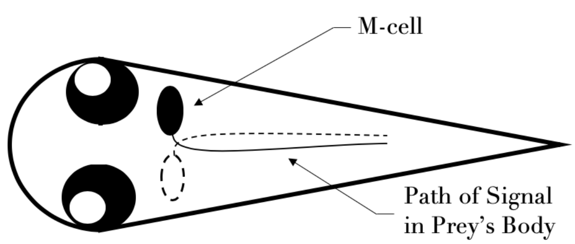

Figure 2: Bilateral M-cells and contralateral signal paths in larval zebrafish.

It was assumed that once the threshold potential ($\hat{R}$) of these cells is reached, a signal is sent that only stimulates a C-start response. We further assumed that the neuronal system of the stimulated M-cell is a properly functioning neural system with little to no resistance. The electrical potential required to stimulate a C-start response was assumed to be the summation of a "baseline" (R_i) electrical potential in addition to five stimuli. The baseline electrical potential was assumed to be independent of the predator and could be caused by pre-existing stress, memory or exhaustion in the prey. The five dependent stimuli were kinetic ($\frac{d\beta}{dt}$), spatial (β), auditory (A), electrical (E), and tensive

(T) stimuli. All of these stimuli contributed to the generation of an action potential (R) within the M-cells of the prey. These factors were assumed to each have a varying impact on potential for response, described by the following equation,

$$\mathrm{R} = \mathrm{R}_i + \mathrm{K}_v \frac{d\beta}{dt} + \mathrm{K}_s\beta + \mathrm{K}_e\mathrm{E} + \mathrm{K}_a\mathrm{A} + \mathrm{K}_t\mathrm{T}, \tag{1}$$

where $\mathrm{K}_v, \mathrm{K}_s, \mathrm{K}_e, \mathrm{K}_a, \mathrm{K}_t$ are proportionality constants. These constants can be set to zero when the predator is incapable of sending a stimuli, or the prey is incapable of receiving a stimuli due to impairments such as blindness or deafness. See Table 1 for definitions of the variables in (1).

Symbol	Meaning
R	Potential for response
$d\beta/dt$	Approach rate of predator
R_i	Baseline potential of prey
β	Perceived size of the predator
A	Magnitude of sound produced by the predator
E	Magnitude of electrical output from the predator
T	Magnitude of tension from predator

Table 1: Variables in Determining Potential for Response

Auditory stimuli could include sounds from bodily functions or contact with the environment. An example of electrical stimuli could be electrical discharge from the nervous system such as that of an eel. Tensive stimuli are changes in pressure surrounding the prey, such as the disturbance caused by an accelerating predator displacing water. Each stimuli is registered in the form of electric potentials in a single M-cell. When the additive value of these electric potentials, R, is greater than $\hat{\mathrm{R}}$, the action potential produced in the M-cell causes an excitatory signal. If this threshold electric potential is not reached, then no signal is propagated.

Since we assumed that the prey has a fully functioning neural system and there is little to no resistance in the excitatory signal's propagation, the response signal can be modelled as a lossless transmission line. Voltage (V) and current (I) of the propagated electrical signal is given as a function of the self-inductance (L) and the capacitance (C), respectively, within the neurons and can be defined as

$$\frac{\partial \mathrm{V}}{\partial x} = -\mathrm{L}\frac{\partial \mathrm{I}}{\partial t} \qquad \text{and} \qquad \frac{\partial \mathrm{I}}{\partial x} = -\mathrm{C}\frac{\partial \mathrm{V}}{\partial t}.$$

Using these definitions and the following standard wave equation [4] that incorporates the propagation speed of the electrical signal down the neuron (u),

$$\frac{\partial^2 V}{\partial^2 t} - u^2\left(\frac{\partial^2 V}{\partial^2 x}\right) = \frac{\partial^2 I}{\partial^2 t} - u^2\left(\frac{\partial^2 I}{\partial^2 x}\right), \tag{2}$$

(2) can be solved for u, yielding

$$u = \frac{1}{\sqrt{LC}}.$$

The stimulated M-cell also causes feedback inhibition in the mirrored M-cell, exponentially suppressing signal transduction as shown in the following equation,

$$\frac{dS}{dt} = -KS,$$

where S represents the electrical signal and K is a proportionality constant. Due to the contralateral nature of the M-cells, the stimulated M-cell triggers muscle contraction opposite of the stimulus. Meanwhile, the inhibition of the other M-cell through the repression of signaling leads to relaxation in muscles on the side of the stimulation from the predator. This antagonistic muscular contraction motivates a ballistic escape mechanism, called a C-start.

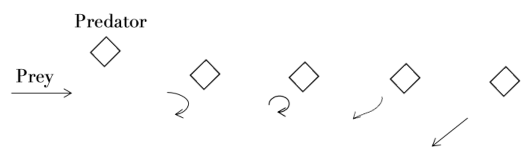

Figure 3: Stages of the C-start response exhibited by larval zebrafish.

As shown in Figure 3, the C-start occurs in essentially two primary actions. First, the head of the prey rotates away from the stimulus, forming the body of the prey into a shape resembling the letter "C." Then, surrounding water is forced towards the stimulus as the prey straightens out, accelerating the prey in the opposite direction.

The first step of the C-start can be modelled through the following angular kinematics equation,

$$\frac{d\theta}{dt} = \omega_{initial} + \alpha t,$$

where $\omega_{initial}$ is the initial rate of angular rotation and α is the angle of deviation from the initial position. The second step involves the propulsion of water away from the prey allowing the system of the prey and surrounding water to function in a mechanism similar to rocket propulsion. If hydrodynamic and gravitational forces are ignored, the law of conservation of momentum can be applied to Newton's second law of motion to create the following equation,

$$m_s\left(\frac{dv_{prey}}{dt}\right) = u\left(\frac{dm_s}{dt}\right),$$

where v_{prey} is the velocity of the prey, u is the velocity that the water is propelled away from the straightened prey, and m_s is the system's variable mass. This system can be solved for the velocity of the prey by integrating both sides of the equation with respect to time to create the following solution,

$$v_{prey} = u\ln\left(\frac{m_s}{m_{pocket}}\right),$$

where m_{pocket} is the mass of the water propelled from the system, similar to a rocket burning fuel. Here, we have modeled the speed at which the prey is physically escaping the predator using a C-start response. Altogether, these equations model the response of the prey to the predator in three parts: (1) the formation of an action potential from stimuli, (2) the tracking of the action potential down a neuron, and (3) the physical flight of the prey.

2.2 Limitations and Future Directions

We recognize three ways our model can be improved upon while remaining in two dimensions: (1) the requirement that all possible stimuli are limited to the five previously mentioned stimuli, (2) the fact that the model does not account for a simultaneous encounter of multiple predators from different directions, and (3) the requirement that there will not be anything blocking the path of the C-start response. Moreover, there is great potential for growth in the exploration of the baseline electrical potential variable (R_i). It would be very interesting to explore how this baseline variable is biologically determined and to model a function that would describe it more explicitly.

3 Additional Issue

On the day of the competition, students were given an hour to incorporate an additional issue into their model and describe this inclusion during their presentation. The addition to our problem was as follows: "Determine the best strategies that a predator can use to successfully catch a prey animal that uses your model to determine when to flee [1]." In evaluating a variety of attacks on the prey, we concluded that either the predator could initiate a response ($R \geq \hat{R}$), or the predator could not trigger a response ($R < \hat{R}$). We thought that predators would utilize the C-start response to their advantage if they hunted in pairs. With this strategy, one predator could trigger the response causing the prey to flee in the opposite direction towards the predator's hunting partner. Furthermore, we concluded that if a predator paralyzed the prey's nervous system or used camouflage to approach the prey by stealth, the flight reaction of the prey may be prevented.

4 Personal Testimonies

"Competitions like SCUDEM are the breath of mathematics. They challenge and expose students to life-like situations of modeling differential equations. Peers learn from each other because of the competition's oral and writing components, and have the opportunity to work with a professor as a coach. This exposure to life-like situations is different than being in a classroom because of the problems posed. Students are forced to work with what knowledge they have prior to the competition to solve these problems. Moreover, I was out of my comfort zone when we held meetings to write the executive summary. Throughout my years in college, I realized my writing style faded and my grammar became messy. Fortunately, with the help .of my teammates, most of my skills were refreshed so that I could write scientifically, concisely, and correctly. Furthermore, working with a professor before and after the competition is an inspiring experience. I believe there is a difference between being coached in your studies, compared to being tutored or lectured. A coach pushes and challenges your skills with high enthusiasm. This competition offers diverse opportunities: from networking to learning new skills in addition to the subject of differential equations itself. I believe all STEM majors should consider participating in similar modeling challenges during their college career."

-Anthony Stefan, Mathematics Major

"Prior to competing in SCUDEM, Calculus II was the highest level math course I had taken. Participating in this competition allowed me to develop an understanding of differential equations and how they are used to model real-world phenomena. It also offered me the opportunity to understand the varied learning styles of my peers and how these different styles of learning mesh together to form a productive team. Undergraduate college has become an environment that exposes students to many disciplines, without allowing them to achieve a thorough understanding of any one in particular. Challenges like SCUDEM force students to gain a deep and thorough understanding and exploration of a particular subject so that they can offer a meaningful solution to a problem. Techniques such as this where the relationship between knowledge and objective is clearly delineated and that objective involves the creation of something that can actually be utilized in the real-world is a technique that teachers should use in their classrooms everyday."

- Bernard Tyson III, Chemistry Major

“This competition made me see mathematics as more than just problems to solve from a book. I had always enjoyed mathematics, but often failed to see its applications in my classes. After I was able to connect material that I learned in biology and physics classes to a mathematical model, I saw how fundamental mathematics is to any scientific area of study. In-depth analysis on a single project with a small team exposed me to multiple techniques of problem analysis and provided me with new methodologies to approach difficult scenarios. Beyond this, this competition allowed me to present at the 2019 Joint Mathematics Meeting and attend several lectures that introduced me to a wide variety of other applications of mathematics. I also met some great people and really had a fun time. Students should take the opportunity to join the SCUDEM challenge or similar events.”

-Zachary Fralish, Biochemistry and Molecular Biology Major

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Appendix

SCUDEM 2018 Problem B - Alarm Bells

Prey animals have to strike a balance when deciding whether or not to flee a potential predator. Moving away in a hurry can expend a great deal of energy, and some prey animals only have a limited ability to detect a larger animal’s intentions. As an example, this dilemma was explored in a recent paper [1]. The researchers in this particular study examined the response of larval zebra fish and found that both the size of the potential predator and the rate the size changed influenced how the larval fish responded to a potential threat.

We ask that you explore the general phenomenon and develop a system of ordinary differential equations that mimics this behavior. The basic idea is that relatively simple organisms must make complex decisions and do so with the least possible resources. Is it possible for an organism to incorporate a relatively small amount of information, such as the size and he rate of change of the size of a potential threat, and then make this decision based on a simple model of ordinary differential equations? If so, what does your model imply about repeated exposures? Does the frequency of those exposures in a short time have an impact on prey response?

A good starting point for understanding the basic ideas behind these models can be found in a paper by Tyson [2]. The models in this paper demonstrate how a response can be determined from a single input. The question we ask is, "How can two or more inputs be incorporated together to enable a simple organism to decide whether or not to flee?"

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   This article is available as a pdf file at the SCUDEM 2018 Modeling Problems web site `https://www.simiode.org/resources/4430` under the title "2017-Bhattaacharyya EtAl - Visual Threat Etc. Current Biology.pdf" and is used with kind permission of the publisher, Elsevier Ltd.

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