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Combinatorial Identities on Multinomial Coefficients and Graph Theory

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Combinatorial Identities on Multinomial Coefficients and Graph Theory

By *Seungho Lee*

Abstract. We study combinatorial identities on multinomial coefficients. In particular, we present several new ways to count the connected labeled graphs using multinomial coefficients.

1 Introduction

The number of ways to put k distinct items into n distinct bins, with bin number 1 holding k_1 of the items (without considering the order of the items there), bin number 2 holding k_2 of the items, and so on, is given by

$$\frac{k!}{k_1!k_2!\cdots k_n!},$$

which is denoted by

$$\binom{k}{k_1\ k_2\ \ldots\ k_n},$$

called a *multinomial coefficient*. This also counts the number of ways you can permute k items, with k_1 of them being identical to each other, k_2 of them being identical to each other, and so on, into a sequence of lengh k (with the order of the items being considered). Here, $k_1 + k_2 + \cdots + k_n = k$. See Roberts and Tesman [7] for more detail.

Given how useful these multinomial coefficients are in counting, it is not surprising to see them frequently in combinatorial identities. In this paper, we present a few combinatorial identities involving multinomial coefficients. Section 2 of our paper states how to write a power of a natural number as a sum of multinomial coefficients. This will serve as a warm-up that introduces the reader to multinomial coefficients and to combinatorial proofs. We present three proofs for the identity: two different combinatorial proofs, and a purely algebraic proof. In Section 3, we consider how to count the number of connected labeled graphs. After briefly reviewing some previous results, we present new recursive ways to count these graphs.

Mathematics Subject Classification. 05C30
Keywords. Multinomial coefficients, Labeled graphs, Connected graphs

2 Rewriting a power of a natural number

Let's take a look at how to write a power of a natural number as a sum of multinomial coefficients. This section will serve as a warm-up that introduces the reader to multinomial coefficients and to combinatorial proofs. Let $\mathbb{N}_0$ be the set of whole numbers, that is, the set of zero and natural numbers.

Theorem 2.1. *For a natural number n and a whole number* $k \geq 0$*, we have*

$$n^k = \sum_{\substack{(k_1,k_2,\ldots,k_n)\in\mathbb{N}_0^n \\ k_1+k_2+\cdots+k_n=k}} \frac{k!}{k_1!k_2!\cdots k_n!}. \tag{1}$$

We will prove **theorem 2.1** in three different ways.

First Proof. We will count the number of ways to do the following: From n distinct items, we select k of them with repetition and then put them in a sequence of length k (therefore, the order of the items matters). For each selection, we have n items to choose from, and we are making k selections. So we have $n \times n \times \cdots \times n = n^k$.

On the other hand, we may start by selecting the first item k_1 times, the second item k_2 times, and so on, where $k_1 + k_2 + \cdots + k_n = k$. Then we have k_1 identical items, k_2 identical items, and so on, to be arranged into a sequence of length k. We can do this in $\binom{k}{k_1\ k_2\ \ldots\ k_n}$ different ways. In order to count the number of sequences that we are considering, we add these multinomials over all possible such $k_1, k_2, \ldots, k_n$. □

Second Proof. Recall that the binomial theorem states that

$$(x+y)^k = \sum_{i=0}^{k} \binom{k}{i} x^{k-i} y^i$$

for variables x and y and for a whole number k. Similarly, for variables $x_1, x_2, \ldots, x_n$, and a whole number k, we have

$$(x_1 + x_2 + \cdots + x_n)^k = \sum_{\substack{(k_1,k_2,\ldots,k_n)\in\mathbb{N}_0^n \\ k_1+k_2+\cdots+k_n=k}} \binom{k}{k_1\ k_2\ \ldots\ k_n} x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n}. \tag{2}$$

You can see the above by the following argument: Clearly, $k_1 + k_2 + \cdots + k_n = k$ since we are expanding $(x_1 + x_2 + \cdots + x_n)^k$ to obtain terms of the form $x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n}$. In

$$(x_1 + x_2 + \cdots + x_n)^k = (x_1 + x_2 + \cdots + x_n) \cdots (x_1 + x_2 + \cdots + x_n),$$

everytime you multiply $(x_1 + x_2 + \cdots + x_n)$ and expand, you are basically deciding which one to choose from $x_1, x_2, \ldots, x_n$ to distribute to other factors, in order to form $x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n}$. For given $k_1, k_2, \ldots, k_n$, there are $\binom{k}{k_1\ k_2\ \ldots\ k_n}$ ways to form $x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n}$.

Now, we set $x_1 = x_2 = \cdots = x_n = 1$ in (2) to prove **theorem 2.1**. □

Third Proof. We will use induction on n. The claim is clearly true for $n = 1$ for any whole number k. Now, we assume that

$$n^j = \sum_{\substack{(j_1, j_2, \dots, j_n) \in \mathbb{N}_0^n \\ j_1 + j_2 + \cdots + j_n = j}} \frac{j!}{j_1! j_2! \cdots j_n!}$$

is true for $n-1$ and any j, $0 \le j \le k$.

Then, for any t, $0 \le t \le k$, we have

$$\begin{aligned} n^t = (1 + (n-1))^t &= \sum_{j=0}^{t} \binom{t}{j} 1^{t-j} (n-1)^j \\ &= \sum_{j=0}^{t} \binom{t}{j} \sum_{\substack{(j_1, j_2, \dots, j_{n-1}) \in \mathbb{N}_0^{n-1} \\ j_1 + j_2 + \cdots + j_{n-1} = j}} \frac{j!}{j_1! j_2! \cdots j_{n-1}!} \\ &= \sum_{j=0}^{t} \sum_{\substack{(j_1, j_2, \dots, j_{n-1}) \in \mathbb{N}_0^{n-1} \\ j_1 + j_2 + \cdots + j_{n-1} = j}} \binom{t}{j} \frac{j!}{j_1! j_2! \cdots j_{n-1}!}. \end{aligned}$$

However,

$$\begin{aligned} \binom{t}{j} \frac{j!}{j_1! j_2! \cdots j_{n-1}!} &= \frac{t!}{j!(t-j)!} \cdot \frac{j!}{j_1! j_2! \cdots j_{n-1}!} \\ &= \frac{t!}{j_1! j_2! \cdots j_{n-1}! (t-j)!} \\ &= \frac{t!}{j_1! j_2! \cdots j_{n-1}! j_n!} \end{aligned}$$

where $j_n = t - j$. Therefore,

$$\begin{aligned} n^t &= \sum_{j=0}^{t} \sum_{\substack{(j_1, j_2, \dots, j_{n-1}) \in \mathbb{N}_0^{n-1} \\ j_1 + j_2 + \cdots + j_{n-1} = j}} \binom{t}{j} \frac{j!}{j_1! j_2! \cdots j_{n-1}!} \\ &= \sum_{j_n=0}^{t} \sum_{\substack{(j_1, j_2, \dots, j_{n-1}) \in \mathbb{N}_0^{n-1} \\ j_1 + j_2 + \cdots + j_{n-1} = j}} \frac{t!}{j_1! j_2! \cdots j_{n-1}! j_n!} \\ &= \sum_{\substack{(j_1, j_2, \dots, j_n) \in \mathbb{N}_0^n \\ j_1 + j_2 + \cdots + j_n = t}} \binom{t}{j_1 \ j_2 \ \dots \ j_n} \end{aligned}$$

for any t, $0 \le t \le k$. Thus, the induction hypothesis is true for every n. □

The first proof is obtained by answering a question in two different ways, giving us the identity. This idea of answering one question in two different ways can be quite useful for producing combinatorial identities. See Benjamin and Quinn [1] for more detail.

Example 2.2. Let $n = 2$ and $k = 3$. Then $n^k = 8$. For **theorem 2.1**, the corresponding (k_1, k_2) are $(3,0), (2,1), (1,2)$ and $(0,3)$. Each contributes $\frac{3!}{3!0!}, \frac{3!}{2!1!}, \frac{3!}{1!2!}$, and $\frac{3!}{0!3!}$, which add up to 8.

Example 2.3. Let $n = 3$ and $k = 5$. Then $n^k = 243$. For **theorem 2.1**, the corresponding (k_1, k_2, k_3) are $(5,0,0), (4,1,0), (4,0,1), (3,2,0), (3,1,1), (3,0,2), (2,3,0), (2,2,1), (2,1,2), (2,0,3), (1,4,0), (1,3,1), (1,2,2), (1,1,3), (1,0,4), (0,5,0), (0,4,1), (0,3,2), (0,2,3), (0,1,4)$, and $(0,0,5)$. Their multinomial coefficients add up to 243.

3 The number of connected labeled graphs

Now we consider how to count the number of connected labeled graphs. After briefly reviewing some previous results, we will see new recursive ways to count these graphs. We only consider graphs that do not have any loops or parallel edges here. A *graph of order p* means a graph that has exactly p vertices. We consider a *labeled graph* of order p, which is a graph whose vertices are assigned with integers from 1 through p. When constructing a labeled graph of order p, there are $\binom{p}{2}$ possible edges between its vertices, and we choose whether or not to include each possible edge in the graph. Thus the number of labeled graphs of order p, denoted by G_p, is $2^{\binom{p}{2}}$.

A *connected graph* is a graph in which any two vertices are joined by a path within the graph. If a graph is not connected, the graph is called *disconnected*. A *component* of a graph is a maximal connected subgraph. A *rooted subgraph* has one of its vertices, called the *root*, distinguished from the others.

It turns out that you can count the number of connected labeled graphs of order p. The following theorem appears in Harary and Palmer [3].

Theorem 3.1. *The number* C_p *of connected labeled graphs of order p satisfies*

$$\mathrm{C}_p = 2^{\binom{p}{2}} - \frac{1}{p}\sum_{k=1}^{p-1} k\binom{p}{k} 2^{\binom{p-k}{2}}\mathrm{C}_k. \tag{3}$$

Here, we reproduce the proof that appears in Harary and Palmer [3].

Proof. We observe that a different rooted labeled graph is obtained when a labeled graph is rooted at each of its vertices. Hence the number of rooted labeled graphs of order p is $p\mathrm{G}_p$.

Next, we consider the number of rooted labeled graphs in which the root is in a component of exactly k vertices. First, there are $\binom{p}{k}$ ways to choose the vertices for the component. Over these vertices, the component can happen in C_k ways, so the rooted component can happen in $k\mathrm{C}_k$ ways over the chosen k vertices. And the remaining $p-k$ vertices can form other parts of the graph in G_{p-k} ways. Thus, the number of rooted labeled graphs in which the root is in a component of exactly k vertices is $k\mathrm{C}_k\binom{p}{k}\mathrm{G}_{p-k}$. On summing from $k=1$ to p, we arrive again at the number of rooted labeled graphs, that is, $\sum_{k=1}^{p} k\binom{p}{k}\mathrm{C}_k\mathrm{G}_{p-k}$. So, we have

$$\begin{aligned} p\mathrm{G}_p &= \sum_{k=1}^{p} k\binom{p}{k}\mathrm{C}_k\mathrm{G}_{p-k} \\ &= \sum_{k=1}^{p-1} k\binom{p}{k}\mathrm{C}_k\mathrm{G}_{p-k} + p\mathrm{C}_p. \end{aligned}$$

So,

$$p\mathrm{C}_p = p\mathrm{G}_p - \sum_{k=1}^{p-1} k\binom{p}{k}\mathrm{C}_k\mathrm{G}_{p-k},$$

which means,

$$\begin{aligned} \mathrm{C}_p &= \mathrm{G}_p - \frac{1}{p}\sum_{k=1}^{p-1} k\binom{p}{k}\mathrm{C}_k\mathrm{G}_{p-k} \\ &= 2^{\binom{p}{2}} - \frac{1}{p}\sum_{k=1}^{p-1} k\binom{p}{k}\mathrm{C}_k 2^{\binom{p-k}{2}}. \end{aligned}$$

□

See Wilf [9] for an alternative proof.

The values of C_p are listed in The On-Line Encyclopedia of Integer Sequences [5, sequence A001187]. Here are the first few terms:

p	1	2	3	4	5	6	7
C_p	1	1	4	38	728	26704	1866256

Although (3) is the standard reference for C_p, other expressions of C_p can be useful. Using an exponential generating function, Flajolet and Sedgewick [2, pp. 138] wrote C_p as

$$\begin{aligned} \mathrm{C}_p = 2^{\binom{p}{2}} - \frac{1}{2}\sum \binom{p}{p_1\ p_2} 2^{\binom{p_1}{2}+\binom{p_2}{2}} \\ + \frac{1}{3}\sum \binom{p}{p_1\ p_2\ p_3} 2^{\binom{p_1}{2}+\binom{p_2}{2}+\binom{p_3}{2}} - \cdots \end{aligned}$$

where the kth term is a sum over $p_1 + \cdots + p_k = p$, with $0 < p_j < p$. Although quite complicated, this expression is useful in determining that almost all labeled graphs of order p are connected. See Flajolet and Sedgewick [2] for more details.

We can also write C_p as

$$\mathrm{C}_p = \sum_{k=1}^{p-1} \binom{p-2}{k-1} (2^k - 1)\mathrm{C}_k \mathrm{C}_{p-k} \tag{4}$$

which was obtained by Riordan using a generating function, as mentioned in Harary and Palmer [3]. Nijenhuis and Wilf [4] proved (4) combinatorically. Nijenhuis and Wilf [4] found (4) useful since it provided them with a recursive recipe for the construction of connected graphs.

Motivated by these alternative expressions of C_p and their usefulness, we derive other expressions of C_p. Let $n_1 + \cdots + n_k = p$, $n_1 \geq n_2 \geq \cdots \geq n_k$, where the largest value n_1 repeats m_1 times in the sum, the next largest value repeats m_2 times, and so on. We use $m!$ to denote $\prod m_i!$. As an example, in $33 = 6+6+6+5+4+4+2$, $m_1 = 3, m_2 = 1, m_3 = 2, m_4 = 1$, and $m! = 12$.

Lemma 3.2. *The number of disconnected labeled graphs of order p is given by*

$$\sum_{\substack{n_1+\cdots+n_k=p\\ k\geq 2\\ n_1\geq\cdots\geq n_k>0}} \frac{p!}{n_1!n_2!\cdots n_k!} \cdot \frac{1}{m!} \cdot \mathrm{C}_{n_1}\mathrm{C}_{n_2}\cdots\mathrm{C}_{n_k}$$

where the sum is taken over k and $n_1, n_2, \cdots, n_k$.

Proof. A disconnected graph has two or more components. Let $k \geq 2$ be the number of components. Given k, let n_1 be the order of the largest component, n_2 be the order of the second largest component, and so on, with $n_1 \geq n_2 \geq \cdots \geq n_k$ and $n_1 + \cdots + n_k = p$. Given the order of each component, there are $\binom{p}{n_1 \cdots n_k}\frac{1}{m!}$ ways to arrange vertices into these components. Here we need to divide by $m!$ because the multinomial coefficient counts the number of ways to put vertices into *distinct* components, whereas we do not differentiate between components of equal order. Once vertices are decided for all the components, then there are $\mathrm{C}_{n_1}\mathrm{C}_{n_2}\cdots\mathrm{C}_{n_k}$ ways to actually form components. □

The following Theorem is immediate.

Theorem 3.3.

$$\mathrm{C}_p = 2^{\binom{p}{2}} - \sum_{\substack{n_1+\cdots+n_k=p\\ k\geq 2\\ n_1\geq\cdots\geq n_k>0}} \frac{p!}{n_1!n_2!\cdots n_k!} \cdot \frac{1}{m!} \cdot \mathrm{C}_{n_1}\mathrm{C}_{n_2}\cdots\mathrm{C}_{n_k}. \tag{5}$$

Remark 3.4. There is an interesting way to interpret **theorem 3.3**. Let $C(x)$ be the exponential generating function for $\{C_p\}$, and let $G(x)$ be the exponential generating function for $\{G_p\}$, where G_p is the number of labeled graphs of order p, $2^{\binom{p}{2}}$. Riddell [6] found that $C(x) = \log(1+G(x))$, which implies that $\{C_p\}$ and $\{G_p\}$ can be interpreted as sequences of cumulants and moments. Let $\lambda = n_1 + \cdots + n_k$ be a partition of p, denoted $\lambda \vdash p$, and let $\lambda! = n_1! n_2! \cdots n_k!$ and $C_\lambda = C_{n_1} C_{n_2} \cdots C_{n_k}$. Rewriting (5) as

$$G_p = \sum_{\lambda \vdash p} \frac{C_\lambda}{\lambda!} \frac{p!}{m!},$$

we recover identities relating moments and cumulants. See Rota and Shen [8] for more on cumulants. However, note that our proof here is entirely combinatorial, contrary to Rota and Shen [8].

Comparing **theorem 3.3** with **theorem 3.1**, we obtain

Corollary 3.5.

$$\frac{1}{p}\sum_{k=1}^{p-1} k\binom{p}{k} 2^{\binom{p-k}{2}} C_k = \sum_{\substack{n_1+\cdots+n_k=p \\ k\geq 2 \\ n_1\geq\cdots\geq n_k>0}} \frac{p!}{n_1! n_2! \cdots n_k!} \cdot \frac{1}{m!} \cdot C_{n_1} C_{n_2} \cdots C_{n_k}.$$

Example 3.6. a. Let $p=3$. Then 3 is either $2+1$ or $1+1+1$ with $m_1 = 3$. Thus,

$$C_3 = 2^{\binom{3}{2}} - \frac{3!}{2!} C_2 C_1 - \frac{3!}{1} \frac{1}{3!} C_1 C_1 C_1 = 4.$$

b. Let $p=4$. Then 4 is either $3+1, 2+2, 2+1+1$, or $1+1+1+1$. Thus,

$$C_4 = 2^{\binom{4}{2}} - \frac{4!}{3!} C_3 C_1 - \frac{4!}{2!2!} \frac{1}{2!} C_2 C_2 - \frac{4!}{2!} \frac{1}{2!} C_2 C_1 C_1 - 4! \frac{1}{4!} C_1^4 = 38.$$

c. Let $p=5$. Then 5 is either $4+1, 3+2, 3+1+1, 2+2+1, 2+1+1+1$, or $1+1+1+1+1$. Thus,

$$\begin{aligned} C_5 = 2^{\binom{5}{2}} &- \frac{5!}{4!} C_4 C_1 - \frac{5!}{3!2!} C_3 C_2 - \frac{5!}{3!} \frac{1}{2!} C_3 C_1 C_1 \\ &- \frac{5!}{2!2!} \frac{1}{2!} C_2 C_2 C_1 - \frac{5!}{2!} \frac{1}{3!} C_2 C_1^3 - 5! \frac{1}{5!} C_1^5 = 728. \end{aligned}$$

d. Let $p=6$. Then 6 is either $5+1, 4+2, 4+1+1, 3+3, 3+2+1, 3+1+1+1, 2+2+2, 2+2+1+1, 2+1+1+1+1$, or $1+1+1+1+1+1$. Thus,

$$\begin{aligned} C_6 = 2^{\binom{6}{2}} &- \frac{6!}{5!} C_5 C_1 - \frac{6!}{4!2!} C_4 C_2 - \frac{6!}{4!} \frac{1}{2!} C_4 C_1 C_1 \\ &- \frac{6!}{3!3!} \frac{1}{2!} C_3 C_3 - \frac{6!}{3!2!} C_3 C_2 C_1 - \frac{6!}{3!} \frac{1}{3!} C_3 C_1^3 - \frac{6!}{2!2!2!} \frac{1}{3!} C_2^3 \\ &- \frac{6!}{2!2!} \frac{1}{2!2!} C_2^2 C_1^2 - \frac{6!}{2!} \frac{1}{4!} C_2 C_1^4 - 6! \frac{1}{6!} C_1^6 = 26704. \end{aligned}$$

We can count C_p in a different way, building up from subgraphs.

Theorem 3.7.

$$C_{p+1} = \sum_{\substack{n_1+\cdots+n_k=p\\ k\geq 1\\ n_1\geq\cdots\geq n_k>0\\ 1\leq j_i\leq n_i,\ i=1,\ldots,k}} \frac{p!}{n_1!n_2!\cdots n_k!}\cdot\frac{1}{m!}\cdot\binom{n_1}{j_1}\cdots\binom{n_k}{j_k}C_{n_1}C_{n_2}\cdots C_{n_k}.$$

Proof. In order to obtain a connected labeled graph of order $p+1$, we first construct its subgraph H formed by vertices 1 through p, which is a labeled graph of order p. Similar to the proof of **lemma 3.2**, but now with $k\geq 1$ since it could be connected, the number of labeled graphs of order p is

$$\sum_{\substack{n_1+\cdots+n_k=p\\ k\geq 1\\ n_1\geq\cdots\geq n_k>0}} \frac{p!}{n_1!n_2!\cdots n_k!}\cdot\frac{1}{m!}\cdot C_{n_1}C_{n_2}\cdots C_{n_k}.$$

To this subgraph H, we join the vertex $p+1$. For it to be connected, the vertex $p+1$ has to have at least one edge to every component of H. For the component of order n_i, there are n_i possible edges between the component and the vertex $p+1$. Let $j_i, 1\leq j_i\leq n_i$ be the number of edges between the component and the vertex $p+1$. Then there are $\binom{n_i}{j_i}$ ways to choose the edges between them. □

Example 3.8. Let $p=5$. Then 5 is either $5, 4+1, 3+2, 3+1+1, 2+2+1, 2+1+1+1$, or $1+1+1+1+1$. Thus,

$$\begin{aligned}
C_6 \ = \ & \frac{5!}{5!}\left[\binom{5}{1}+\binom{5}{2}+\binom{5}{3}+\binom{5}{4}+\binom{5}{5}\right]C_5\\
&+\frac{5!}{4!}\left[\binom{4}{1}+\binom{4}{2}+\binom{4}{3}+\binom{4}{4}\right]C_4C_1\\
&+\frac{5!}{3!2!}\left[\binom{3}{1}\binom{2}{1}+\binom{3}{1}\binom{2}{2}+\binom{3}{2}\binom{2}{1}\right.\\
&\qquad\left.+\binom{3}{2}\binom{2}{2}+\binom{3}{3}\binom{2}{1}+\binom{3}{3}\binom{2}{2}\right]C_3C_2\\
&+\frac{5!}{3!}\frac{1}{2!}\left[\binom{3}{1}+\binom{3}{2}+\binom{3}{3}\right]C_3C_1C_1\\
&+\frac{5!}{2!2!}\frac{1}{2!}\left[\binom{2}{1}\binom{2}{1}+\binom{2}{1}\binom{2}{2}+\binom{2}{2}\binom{2}{1}+\binom{2}{2}\binom{2}{2}\right]C_2C_2C_1\\
&+\frac{5!}{2!}\frac{1}{3!}\left[\binom{2}{1}+\binom{2}{2}\right]C_2C_1^3+5!\frac{1}{5!}C_1^5 = 26704.
\end{aligned}$$

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