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Maxine E. Calle
Reed College, callem@reed.edu

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Putting the "k" in Curvature: k-Plane Constant Curvature Conditions

By *Maxine Calle*

Abstract. This research generalizes the two invariants known as constant sectional curvature (csc) and constant vector curvature (cvc). We use k-plane scalar curvature to investigate the higher-dimensional analogues of these curvature conditions in Riemannian spaces of arbitrary finite dimension. Many of our results coincide with the known features of the classical $k = 2$ case. We show that a space with constant k-plane scalar curvature has a uniquely determined tensor and that a tensor can be recovered from its k-plane scalar curvature measurements. Through two example spaces with canonical tensors, we demonstrate a method for determining constant k-plane vector curvature values, as well as the possibility of a connected set of values. We also generate loose bounds for candidate values based on sectional curvatures. By studying these k-plane curvature invariants, we can further characterize model spaces by generating basis-independent numbers for various subspaces.

1 Introduction and Background

Differential geometry uses the tools of calculus to study local behaviors of manifolds, which (as topological entities) do not have an intrinsic 'shape.' Consequently, we are interested in developing invariants that can characterize a space independently from any choice of coordinate system, and so be able to tell when two spaces are 'the same.' Curvature provides some of the most fundamental tools with which we can describe spaces. Gaussian curvature, as studied extensively by the famous mathematician Carl Friedrich Gauss in the 19th century, is an intrinsic property independent of the isometric embedding of the surface in Euclidean space. Bernhard Riemann extended this notion of instrinsic curvature for higher-dimensional Riemannian manifolds, which are (smooth) manifolds equipped with an inner product on the tangent space at each point. A Riemannian curvature tensor, arising out of the family of inner products, can be used to define

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familiar geometric notions on Riemannian manifolds. This tensor acts as a measure of the local isometric difference between the inner product at that point and the standard inner product of Euclidean space. In this sense, the surface will be locally 'flat' when the tensor vanishes. The interested reader can look to [10, 12] for further reading on differential geometry and modern Riemannian geometry.

This work concerns two curvature invariants, known as constant sectional curvature and constant vector curvature. Classically, the former idea is very well-understood: spaces with this condition are the so-called space forms (see, for example, [12, §7] and [10, §4.3, §5.4]). Moreover, the constancy of the sectional curvature uniquely determines the Riemannian metric, and so there is only one example of constant sectional curvature (up to reasonable equivalence). Constant vector curvature uses sectional curvature to develop a less restrictive condition, as first introduced in 2011 by Schmidt and Wolfson [14]. Since their original work, constant vector curvature has been studied extensively in 3-dimensional spaces [13, 15].

Since sectional curvature measures can only provide information about subspaces of dimension 2, previous literature had primarily utilized these invariants to study manifolds of relatively low dimension. It seems natural to generalize these conditions to consider k-dimensional planes in a space of arbitrary finite dimension, and this work does so by utilizing the k-plane scalar curvature, as presented in [3]. These generalized invariants, called k-plane constant sectional curvature (k-csc) and k-plane constant vector curvature (k-cvc), exhibit many of the same features as their 2-dimensional counterparts. In particular, we show that spaces with the k-csc condition have a uniquely determined algebraic curvature tensor (see **Corollary 4.3**), and consequently the classical case and the generalized case are equivalent (see **Corollary 4.4**). We use specific examples to demonstrate the variety of k-cvc structure (see **Example 5.2** and **Example 5.4**).

Outline.

Section 2 is dedicated to discussing some central concepts from differential geometry. Section 3 introduces the definitions of k-csc and k-cvc and gives some immediate generalizations of known results for the classical 2-csc and 2-cvc conditions. Section 4 focuses on the k-csc condition. **Theorem 4.1** shows that a model space with k-csc(0) (for $2 \le k \le n-2$) must have the zero tensor. This result gives several corollaries, including that including that there is a unique tensor that gives rise to the k-csc condition, which coincides with the tensor known to give 2-csc (see **Corollary 4.3**). Section 5 investigates k-cvc in the context of model spaces with canonical tensors and we present two different examples of k-cvc spaces. **Example 5.2** demonstrates a method for calculating possible k-cvc values, and shows that a model space can have multiple values for a given k. **Example 5.4** has k-cvc for any value in $[0,1]$, and we can also bound other possible values based on the sectional curvatures measurements. Section 6 presents more general k-cvc results for model spaces with canonical tensors and discusses open areas for further research.

2 Background: Curvature

Working locally, we can create algebraic representations of the manifold at a point. Such a *model space* is denoted by $\mathscr{M} = (\mathrm{V}, \langle\ ,\ \rangle, \mathrm{R})$ and defined as a triple of an n-dimensional vector space V, a non-degenerate inner product $\langle\ ,\ \rangle$ on V, and an algebraic curvature tensor R. Given a manifold, metric, and point on the manifold, we can build a model space from the tangent space, the metric, and the Riemannian curvature tensor at that point. If we are interested in the behavior of a manifold at a point, we can study the properties of the representative model space. In this more abstract setting, algebraic curvature tensors are a central tool to understanding and investigating properties of the space.

Definition 2.1. An **Algebraic Curvature Tensor (ACT)** is a multilinear function from four tangent vectors in V to a scalar,

$$\mathrm{R} : \mathrm{V} \times \mathrm{V} \times \mathrm{V} \times \mathrm{V} \to \mathbb{R}$$

with the following properties, for all $x, y, z, w \in \mathrm{V}$:

1. *Skew-symmetry in the first two slots:* $\mathrm{R}(x, y, z, w) = -\mathrm{R}(y, x, z, w)$,
2. *Interchange symmetry:* $\mathrm{R}(x, y, z, w) = \mathrm{R}(z, w, x, y)$,
3. *The Bianchi identity:* $\mathrm{R}(x, y, z, w) + \mathrm{R}(z, x, y, w) + \mathrm{R}(y, z, x, w) = 0$.

For an orthonormal set of vectors $\{e_i, e_j, e_k, e_l\}$, let R_{ijkl} denote $\mathrm{R}(e_i, e_j, e_k, e_l)$.

As algebraic entities, ACTs are subject to quite a bit of algebraic structure. In particular, the ACTs over an n-dimensional space V form an $\frac{n^2(n^2-1)}{12}$-dimensional vector space, denoted $\mathscr{A}(\mathrm{V})$. Thus we can add and scale ACTs as we would any other object from linear algebra, so

$$(\lambda \mathrm{R}_1 + \mathrm{R}_2)(x, y, z, w) = \lambda \mathrm{R}_1(x, y, z, w) + \mathrm{R}_2(x, y, z, w)$$

for any $\mathrm{R}_1, \mathrm{R}_2 \in \mathscr{A}(\mathrm{V})$, $x, y, z, w \in \mathrm{V}$, and $\lambda \in \mathbb{R}$. Based on the work of Fiedler [6, 7], Gilkey showed that $\mathscr{A}(\mathrm{V})$ is spanned by *canonical* tensors [8, Theorem 1.8.2]. A canonical ACT R_ϕ is defined with respect to a symmetric, bilinear form ϕ, where

$$\mathrm{R}_\phi(x, y, z, w) = \phi(x, w)\phi(y, z) - \phi(x, z)\phi(y, w).$$

For example, we can consider the canonical ACT with respect to the inner product, denoted R_*. Recall that the inner product $\langle\ ,\ \rangle$ is a symmetric, bilinear form such that

$\langle e_i, e_j\rangle = \delta_{ij}$ for a pair of orthonormal vectors $e_i,\ e_j$. For the purposes of this research, we assume that inner products are positive-definite, although it is likely that this work could be extended to Lorentzian spaces (for merely non-degenerate inner products). Thus, for a pair of orthonormal vectors e_i, e_j, $(R_*)_{ijji} = \delta_{ii}\delta_{jj} - \delta_{ij}^2 = 1$.

The connection between canonical ACTs and linear algebra makes the study of these tensors particularly appealing. Given a symmetric, bilinear form ϕ on a vector space equipped with a non-degenerate inner product, there is a unique self-adjoint linear transformation A (with a corresponding matrix representation) such that $\phi(x, y) = \langle Ax, y\rangle$. For this reason, we can discuss aspects of ϕ (such as rank, kernel, or eigenvalues) in terms of the same aspects of A.

In particular, the *kernel* of a tensor R is defined by

$$\ker(R) = \{v \in V \mid R(v, y, z, w) = 0, \text{ for any } y, z, w \in V\}.$$

By the various symmetry properties of ACTs, it is clear that we need not restrict $v \in \ker(R)$ to appear only in the first slot, as discussed in [5]. For canonical tensors, we can equate the kernel of the tensor with the kernel of its form.

Proposition 2.2. *[9, Lemma 1.6.3] If* $\operatorname{rank}(\phi) \geq 2$ *for a symmetric, bilinear form* ϕ*, then* $\ker(\phi) = \{v \in V|\phi(v, w) = 0,\ \forall\, w \in V\} = \ker(R_\phi)$.

Using algebraic curvature tensors, we can develop and compute different curvature invariants. As manifolds rely heavily on a choice of coordinate systems, we are interested in finding characterizations independent of this choice. An invariant central to our study is the *sectional curvature*.

Definition 2.3. Let $x, y \in V$ be tangent vectors. Let $\pi = \operatorname{span}\{x, y\}$ be a non-degenerate 2-plane. The **sectional curvature** is a form $\kappa : V \times V \to \mathbb{R}$, where

$$\kappa(\pi) = \frac{R(x, y, y, x)}{\langle x, x\rangle\langle y, y\rangle - \langle x, y\rangle^2}.$$

This measurement is a curvature invariant since $\kappa(\pi)$ is independent of the chosen basis for π [8, Lemma 1.6.4]. Note that (under the assumption of a positive-definite inner product) $\kappa_{ij} = R_{ijji}$, where κ_{ij} denotes the sectional curvature of the plane spanned by orthonormal vectors e_i and e_j.

A closely related measurement is the *scalar curvature* (or *Ricci scalar*), which can be defined as the average of sectional curvatures scaled by $n(n-1)$. Given an orthonormal basis $\{e_1, \ldots, e_n\}$ for V, the scalar curvature is given by $\tau = \sum_{i,j} R_{ijji}$. Taking V to be the tangent space at a point of a manifold, the scalar curvature assigns a real number based on the local intrinsic geometry of the manifold at that point. In two dimensions, the scalar curvature completely characterizes the curvature of a surface.

This measurement can also be defined in terms of the *Ricci tensor*, which is a symmetric, bilinear form given by $\mathrm{Ric}(x, y) = \sum_{i=1}^{n} R(x, e_i, e_i, y)$. The scalar curvature is thus the trace of the Ricci tensor. In lower dimensions, the Ricci curvature is completely determined by sectional curvatures, and in turn can completely determine the full curvature tensor. For a more in-depth treatment of these concepts, see [8] or [10, Chapter 4].

Our research uses a generalized version of sectional curvature to investigate two curvature invariants, known as constant sectional curvature and constant vector curvature. A space has *constant sectional curvature* (csc) when $\kappa(\pi) = \varepsilon$ for some $\varepsilon \in \mathbb{R}$ and every 2-plane π. This property results in spaces that are locally homogeneous, meaning that there is a (local) distance-preserving map between any two points, and consequently such spaces are well-behaved and well-understood. The weaker condition of *constant vector curvature* (cvc) implies that every non-zero $v \in \mathrm{V}$ lies in some 2-plane with sectional curvature ε, and was developed by Schmidt and Wolfson in 2011 in their work with three-manifolds [14].

Since the seminal paper on cvc, the condition has been completely resolved for 3-dimensional model spaces. Through a combined effort, it has been shown that all 3-dimensional Riemannian model spaces have cvc(ε) for some $\varepsilon \in \mathbb{R}$, however this is not necessarily the case for Lorentzian model spaces [13, 15]. Additionally, as noted by the referee, Schmidt and Wolfson have since altered the definition of cvc. According to the definition given in [16], a manifold (or model space) has CVC(ε) if it has cvc(ε) and ε is *extremal*, meaning that ε is a bound on all other sectional curvatures. While this paper utilizes the original definition of cvc, future research might extend our work to include the updated definition.

3 k-plane Curvature: General Results

While both csc and cvc are well-studied in the three-dimensional case, higher dimensional model spaces are relatively unexplored. This research builds upon previous work in this area to generalize both curvature conditions for model spaces of arbitrary finite dimension. To do so, we require a higher-dimensional analogue of sectional curvature, known as *k-plane scalar curvature*. This measurement, like 2-plane sectional curvature, is a geometric invariant in the sense of generating representative numbers that are independent of a chosen basis for the k-plane.

Definition 3.1. [3, Section 2] Let $\mathscr{M} = (\mathrm{V}, \langle\ ,\ \rangle, \mathrm{R})$ with an orthonormal basis $\{e_1, \ldots, e_n\}$ for V. Define $\mathrm{M_L} = (\mathrm{L}, \langle\ ,\ \rangle, \mathrm{R_L})$ with an orthonormal basis $\{f_1, \ldots, f_k\}$ for $\mathrm{L} \subseteq \mathrm{V}$, $\langle\ ,\ \rangle_\mathrm{L} = \langle\ ,\ \rangle|_\mathrm{L}$, and $\mathrm{R_L} = \mathrm{R}|_\mathrm{L} \in \mathscr{A}(\mathrm{L})$. Define the ***k*-plane scalar curvature** of L as a mapping $\mathscr{K}_\mathrm{R} : \mathrm{Gr}(k, \mathscr{M}) \to \mathbb{R}$ given by

$$\mathscr{K}_\mathrm{R}(\mathrm{L}) = \sum_{j>i=1}^{k} \kappa(f_i, f_j).$$

Here $\mathrm{Gr}(k, \mathscr{M})$ is the Grassmannian, the space of k-dimensional linear subspaces of $\mathscr{M}$. For $\mathrm{L} = \mathrm{span}\{e_1, \ldots, e_k\}$, if it is clear we are computing $\mathscr{K}_\mathrm{R}(\mathrm{L})$ with respect to a certain tensor, we let $\mathscr{K}_{\mathrm{R}}(\mathrm{L}) = \mathscr{K}(\mathrm{L}) = \mathscr{K}(e_1,\ldots,e_k)$ for the ease of notation. Similarly, although it is understood we are evaluating $\mathscr{K}(\mathrm{L})$ with respect to the restricted space $\mathscr{M}_{\mathrm{L}}$, we will discuss $\mathscr{K}(\mathrm{L})$ mostly in terms of the given model space $\mathscr{M}$.

Note that on an orthonormal basis, calculating $\mathscr{K}(\mathrm{L})$ amounts to summing over certain R_{ijji} terms (assuming a positive-definite inner product). If L is a 2-plane, $\mathscr{K}(\mathrm{L}) = \kappa(\mathrm{L})$, and when $k = n$, we have $\mathscr{K}(\mathrm{L}) = \frac{\tau}{2}$. Since the only n-plane curvature is characterized by the curvature of the entirity of V, we are mostly interested in k-planes for $2 \le k \le n-1$.

Given this tool for calculating k-plane curvatures, we can study constant curvature conditions on model spaces of any finite dimension. Intuitive adaptations of the csc and cvc conditions follow:

Definition 3.2. A model space $\mathscr{M}$ has k**-plane constant sectional curvature** ε, denoted k-csc(ε), if $\mathscr{K}(\mathrm{L}) = \varepsilon$ for all non-degenerate k-planes L.

Definition 3.3. A model space $\mathscr{M}$ has k**-plane constant vector curvature** ε, denoted k-cvc(ε), if for all $v \in \mathrm{V}$ where $v \neq \vec{0}$ there is some non-degenerate k-plane L containing v such that $\mathscr{K}(\mathrm{L}) = \varepsilon$.

Some intuitive properties follow immediately from the construction of the two conditions. In particular, it is clear from the definitions that the k-csc condition implies the k-cvc condition, by simply choosing a k-plane L that contains the vector under examination. Most of the following propositions are natural generalizations of results from the classical case, as given in [1, Section 2].

Proposition 3.4. *Let* $\mathrm{M}_1 = (\mathrm{V}, \langle\ ,\ \rangle, \mathrm{R}_1)$ *have* k*-csc(*ε*) and* $\mathrm{M}_2 = (\mathrm{V}, \langle\ ,\ \rangle, \mathrm{R}_2)$ *have* k*-cvc(*δ*). Then* $\mathrm{M} = (\mathrm{V}, \langle\ ,\ \rangle, \mathrm{R} = \mathrm{R}_1 + \mathrm{R}_2)$ *has* k*-cvc(*$\varepsilon + \delta$*).*

Proof. Let $\mathscr{M}_1, \mathscr{M}_2$ and $\mathscr{M}$ be given as above. Let $v \in \mathrm{V}$ be non-zero and L be a k-plane containing v such that $\mathscr{K}_{\mathrm{R}_2}(\mathrm{L}) = \delta$. Note also that $\mathscr{K}_{\mathrm{R}_1}(\mathrm{L}) = \varepsilon$. Then by the vector space properties of $\mathscr{A}(\mathrm{V})$,

$$\mathscr{K}_{\mathrm{R}}(\mathrm{L}) = \sum_{j>i=1}^{k} \mathrm{R}_{ijji} = \sum_{j>i=1}^{k} (\mathrm{R}_1 + \mathrm{R}_2)_{ijji} = \sum_{j>i=1}^{k} (\mathrm{R}_1)_{ijji} + \sum_{j>i=1}^{k} (\mathrm{R}_2)_{ijji} = \varepsilon + \delta.$$

□

Proposition 3.5. *Suppose* $\mathrm{M} = (\mathrm{V}, \langle\ ,\ \rangle, \mathrm{R})$ *has* k*-cvc(*ε*), and let* $c \in \mathbb{R}$*. Then* $\mathrm{M}_c = (\mathrm{V}, \langle\ ,\ \rangle, c\mathrm{R})$ *has* k*-cvc(*$c\varepsilon$*).*

Proof. Let $\mathscr{M} = (\mathrm{V}, \langle\ ,\ \rangle, \mathrm{R})$ have k-cvc(ε) and consider $\mathscr{M}_c = (\mathrm{V}, \langle\ ,\ \rangle, c\mathrm{R})$ for some $c \in \mathbb{R}$. Let $v \in \mathrm{V}$ be non-zero and L be a k-plane containing v such that $\mathscr{K}_{\mathrm{R}}(\mathrm{L}) = \varepsilon$. Again using the vector space properties of $\mathscr{A}(\mathrm{V})$,

$$\mathscr{K}_{c\mathrm{R}}(\mathrm{L}) = \sum_{j>i=1}^{k} (c\mathrm{R})_{ijji} = \sum_{j>i=1}^{k} c(\mathrm{R}_{ijji}) = c \sum_{j>i=1}^{k} \mathrm{R}_{ijji} = c\epsilon.$$

□

Proposition 3.6. *Let* $\mathrm{M} = (\mathrm{V}, \langle\ ,\ \rangle, \mathrm{R})$ *with* $\dim(\ker(\mathrm{R})) \geq k-1$. *Then* $\mathcal{M}$ *has* k*-cvc(0).*

Proof. Let $\mathcal{M}$ be a model space and suppose the nullity of R is $\alpha \geq k-1$. Take $\{e_1, \ldots, e_\alpha\}$ as an orthonormal basis for ker(R) and extend to $\{e_1, \ldots, e_n\}$ an orthonormal basis for V. Take $v = \sum_{i=1}^n x_i e_i$ to be non-zero, and construct a k-plane L spanned by

$$f_1 = e_1, \ldots, f_{k-1} = x_{k-1}e_{k-1} + \cdots + x_\alpha e_\alpha / \sqrt{x_{k-1}^2 + \cdots + x_\alpha^2},$$

$$\text{and } f_k = x_{\alpha+1}e_{\alpha+1} + \cdots + x_n e_n / \sqrt{x_{\alpha+1}^2 + \cdots + x_n^2}.$$

Note that since $\alpha \geq k-1$, $f_{k-1} \neq 0$. In the event that $x_{k-1} = \cdots = x_\alpha = 0$, set $f_{k-1} = e_{k-1}$. Similarly, if $x_{\alpha+1} = \cdots = x_n = 0$, set $f_k = e_n$. So when $\mathcal{K}(\mathrm{L}) = \sum_{j>i=1}^k \mathrm{R}(f_i, f_j, f_j, f_i)$ is written in terms of the e_i's, an $e_i \in \ker(\mathrm{R})$ appears in each R_{ijkl} term. So $\mathcal{K}(\mathrm{L}) = 0$ and $\mathcal{M}$ has k-cvc(0). □

This final proposition generalizes a previously-known result which states that ACTs with non-trivial kernels give 2-cvc(0) and only 2-cvc(0).

Proposition 3.7. *[1, Theorem 2.1] For a given model space* $\mathcal{M}$*, if* ker(R) *is non-trivial, then* $\mathcal{M}$ *has* 2*-cvc(*0*) and not* 2*-cvc(*ε*) for any other value of* ε.

Note that the relationship between the nullity and the value k generalizes to higher dimensions, although it is not necessary that we get *only* k-cvc(0) for $k > 2$, as we will see in later examples.

4 k-plane Constant Sectional Curvature

Having defined a general notion of k-plane constant sectional curvature, we can present our main result. We generalize the well-known fact that an ACT whose sectional curvatures vanishes must itself be the zero tensor. The classical $k = 2$ result follows since a tensor R can be written in terms of sectional curvatures (see [10, Lemma 4.3.3] or [12, Theorem 6.5]). We employ a method inspired by Klinger [11] to isolate tensor components as 0 by considering particular 'skewed' k-planes.

Theorem 4.1. *Take* $2 \leq k \leq n-2$ *and let* $\mathrm{M} = (\mathrm{V}, \langle\ ,\ \rangle, \mathrm{R})$ *be a model space. If* $\mathcal{K}(\mathrm{L}) = 0$ *for all* k*-planes* L*, then* $\mathrm{R} \equiv 0$.

Proof. Let $\mathcal{M}$ be a model space and take $e_1, e_2 \in \mathrm{V}$ an arbitrary pair of orthonormal vectors. Extend to an orthnormal basis $\{e_1, \ldots, e_n\}$ for V. Since the result is already known for $k = 2$, suppose $\mathcal{M}$ has k-csc(0) for some $3 \leq k \leq n-2$. First we will prove that the ACT entries of the form R_{ijjk} are 0 (for $i \neq k$), which then implies that the sectional curvature vanishes. Since R can be written in terms of sectional curvatures, it follows that R must be the zero tensor.

Consider the k-plane $\mathrm{L} = \mathrm{span}\{\cos\theta e_1 + \sin\theta e_2, e_3, \ldots, e_{k+1}\}$. For the sake of notation, call $e_{k+1} := e_\alpha$. By hypothesis, the k-plane scalar curvature of L is 0, regardless of the value of θ, so

$$\begin{aligned}
0 =& \mathscr{K}(\cos\theta e_1 + \sin\theta e_2, e_3, \ldots, e_\alpha) \\
=& \cos^2\theta\,(\mathrm{R}_{1331} + \cdots + \mathrm{R}_{1\alpha\alpha1}) + \sin^2\theta(\mathrm{R}_{2332} + \cdots + \mathrm{R}_{2\alpha\alpha2}) \\
&+ 2\cos\theta\sin\theta(\mathrm{R}_{1332} + \cdots + \mathrm{R}_{1\alpha\alpha2}) + \sum_{j>i=3}^{\alpha} \mathrm{R}_{ijji} \\
=& \cos^2\theta\left(\sum_{j=3}^{\alpha} \mathrm{R}_{1jj1} + \sum_{j>i=3}^{\alpha} \mathrm{R}_{ijji}\right) - \cos^2\theta\left(\sum_{j>i=3}^{\alpha} \mathrm{R}_{ijji}\right) \\
&+ \sin^2\theta\left(\sum_{j=3}^{\alpha} \mathrm{R}_{2jj2} + \sum_{j>i=3}^{\alpha} \mathrm{R}_{ijji}\right) - \sin^2\theta\left(\sum_{j>i=3}^{\alpha} \mathrm{R}_{ijji}\right) \\
&+ \sum_{j>i=3}^{\alpha} \mathrm{R}_{ijji} + 2\cos\theta\sin\theta(\sum_{j=3}^{\alpha} \mathrm{R}_{1jj2}).
\end{aligned}$$

But observe that

$$\sum_{j=3}^{\alpha} \mathrm{R}_{1jj1} + \sum_{j>i=3}^{\alpha} \mathrm{R}_{ijji} = \mathscr{K}(e_1, e_3, \ldots, e_\alpha) \quad \text{and} \quad \sum_{j=3}^{\alpha} \mathrm{R}_{2jj2} + \sum_{j>i=3}^{\alpha} \mathrm{R}_{ijji} = \mathscr{K}(e_2, e_3, \ldots, e_\alpha).$$

Since both are 0 by supposition,

$$\begin{aligned}
0 =& -\cos^2\theta\left(\sum_{j>i=3}^{\alpha} \mathrm{R}_{ijji}\right) - \sin^2\theta\left(\sum_{j>i=3}^{\alpha} \mathrm{R}_{ijji}\right) + \sum_{j>i=3}^{\alpha} \mathrm{R}_{ijji} \\
&+ 2\cos\theta\sin\theta\left(\sum_{j=3}^{\alpha} \mathrm{R}_{1jj2}\right) \\
=& (\cos^2\theta + \sin^2\theta)\left(-\sum_{j>i=3}^{\alpha} \mathrm{R}_{ijji}\right) + \sum_{j>i=3}^{\alpha} \mathrm{R}_{ijji} + 2\cos\theta\sin\theta\left(\sum_{j=3}^{\alpha} \mathrm{R}_{1jj2}\right) \\
=& 2\cos\theta\sin\theta\left(\sum_{j=3}^{\alpha} \mathrm{R}_{1jj2}\right).
\end{aligned}$$

Since the equation must hold true for all values of θ,

$$0 = \sum_{j=3}^{\alpha} \mathrm{R}_{1jj2}. \tag{1}$$

Since we were considering $k \le n-2$, we can construct a new k-plane by setting $f_k = e_n$ and keeping all other f_i basis vectors the same. We repeat the process above to get

$$0 = \sum_{n-1\neq j=3}^{n} \mathrm{R}_{1jj2}. \tag{2}$$

Now, subtracting (3.2) from (3.1):

$$0 = \mathrm{R}_{1\alpha\alpha 2} - \mathrm{R}_{1nn2},$$

hence $\mathrm{R}_{1\alpha\alpha 2} = \mathrm{R}_{1nn2}$. Since our choices of e_α and e_n were arbitrary, we could permute any e_i, e_j basis vectors to get $\mathrm{R}_{1ii2} = \mathrm{R}_{1jj2}$. Since $0 = \sum_{j=3}^{\alpha} \mathrm{R}_{1jj2} = (\alpha-2)\mathrm{R}_{1332}$, and since $\alpha > 2$, we can conclude that $\mathrm{R}_{1332} = \mathrm{R}_{1jj2} = 0$ for all $j \in \{1,\dots,n\}$. Since e_1 and e_2 were arbitrary, we have $\mathrm{R}_{ijjk} = \mathrm{R}_{jikj} = 0$ for any distinct $i, j, k \in \{1,\dots,n\}$.

To show that $\mathrm{R}_{1221} = 0$, set $f_1 = e_i$, $f_2 = \cos\theta e_j + \sin\theta e_k$, $f_3 = \cos\theta e_k - \sin\theta e_j$ for some $e_i, e_j, e_k \in \{e_1,\dots,e_n\}$. Extend $\{f_1, f_2, f_3\}$ to an orthonormal basis for V. Since $\mathrm{R}_{ijjk} = 0$ on any orthonormal basis, we get

$$\begin{aligned}
0 &= \mathrm{R}(f_2, f_1, f_1, f_3)\\
&= \cos^2\theta \mathrm{R}_{ijki} - \sin\theta\cos\theta \mathrm{R}_{ijji} - \sin^2\theta \mathrm{R}_{ikji} + \sin\theta\cos\theta \mathrm{R}_{ikki}\\
&= -\sin\theta\cos\theta \mathrm{R}_{ijji} + \sin\theta\cos\theta \mathrm{R}_{ikki}.
\end{aligned}$$

Again this equation must hold for all θ, so $0 = \mathrm{R}_{ikki} - \mathrm{R}_{ijji}$, meaning that $\mathrm{R}_{ijji} = \mathrm{R}_{ikki}$. Since $0 = \mathscr{K}(\mathrm{L})$ for all k-planes L, including the coordinate plane spanned by $e_1,\dots,e_k$, we get that

$$\sum_{j>i=1}^{k} \mathrm{R}_{ijji} = \frac{k(k-1)}{2}\mathrm{R}_{1221} = 0,$$

and hence $\mathrm{R}_{1221} = \mathrm{R}_{ijji} = 0$ for all $i, j \in \{1,\dots,n\}$. Since the sectional curvature vanishes and R can be written in terms of sectional curvatures, $\mathrm{R} \equiv 0$. □

This result immediately gives some pleasing corollaries. Just as a tensor can be recovered from its sectional curvatures, it can also be recovered from k-plane scalar curvatures. Further, there is only one example (up to reasonable equivalence) of an ACT with the k-csc condition.

Corollary 4.2. *Set* $2 \le k \le n-2$. *Let* $\mathscr{M}_1 = (\mathrm{V}, \langle\ ,\ \rangle, \mathrm{R}_1)$ *and* $\mathscr{M}_2 = (\mathrm{V}, \langle\ ,\ \rangle, \mathrm{R}_2)$ *be model spaces. Suppose* $\mathscr{K}_{\mathrm{R}_1}(\mathrm{L}) = \mathscr{K}_{\mathrm{R}_2}(\mathrm{L})$ *for all* k*-planes* L. *Then* $\mathrm{R}_1 = \mathrm{R}_2$.

Proof. Let $\mathscr{M}_1 = (\mathrm{V}, \langle\ ,\ \rangle, \mathrm{R}_1)$ and $\mathscr{M}_2 = (\mathrm{V}, \langle\ ,\ \rangle, \mathrm{R}_2)$ be model spaces. Construct $\mathscr{M} = (\mathrm{V}, \langle\ ,\ \rangle, \mathrm{R})$ and set $\mathrm{R} = \mathrm{R}_1 - \mathrm{R}_2$. Then any $\mathrm{L} = \mathrm{span}\{e_1,\dots,e_k\}$ has k-plane scalar curvature

0, since

$$\begin{aligned}
\mathscr{K}_{\mathrm{R}}(\mathrm{L}) &= \sum_{j>i=1}^{k} \mathrm{R}(e_i, e_j, e_j, e_i) \\
&= \sum_{j>i=1}^{k} (\mathrm{R}_1 - \mathrm{R}_2)(e_i, e_j, e_j, e_i) \\
&= \sum_{j>i=1}^{k} \mathrm{R}_1(e_i, e_j, e_j, e_i) - \sum_{j>i=1}^{k} \mathrm{R}_2(e_i, e_j, e_j, e_i) \\
&= \mathscr{K}_{\mathrm{R}_1}(\mathrm{L}) - \mathscr{K}_{\mathrm{R}_2}(\mathrm{L}) \\
&= 0.
\end{aligned}$$

Since $\mathscr{K}(\mathrm{L}) = 0$ for any k-plane, $0 = \mathrm{R} = \mathrm{R}_1 - \mathrm{R}_2$ by **Theorem 4.1**. $\square$

Recalling that R_* denotes the canonical tensor with respect to the inner product, we get the following corollary.

Corollary 4.3. *The ACT* $\mathrm{R} = \frac{2\varepsilon}{k(k-1)}\mathrm{R}_*$ *is the unique ACT that has k-csc(ε) for* $2 \leq k \leq n-1$.

Proof. Let $\mathscr{M}$ be a model space with $\mathrm{R} = \frac{2\varepsilon}{k(k-1)}\mathrm{R}_*$. Let $\mathrm{L} = \mathrm{span}\{e_1, \ldots, e_k\}$ for some set of orthonormal vectors. Then

$$\begin{aligned}
\mathscr{K}(\mathrm{L}) &= \sum_{j>i=1}^{k} \mathrm{R}_{ijji} \\
&= \sum_{j>i=1}^{k} \frac{2\varepsilon}{k(k-1)} (\mathrm{R}_*)_{ijji} \\
&= \frac{2\varepsilon}{k(k-1)} \sum_{j>i=1}^{k} 1 \\
&= \frac{2\varepsilon}{k(k-1)} (\frac{1}{2}) \sum_{i=1}^{k} \sum_{j=1}^{k-1} 1 && \text{since } j > i \text{ and } \mathrm{R}_{iiii} = 0, \\
&= \frac{\varepsilon}{k(k-1)} (k-1) \sum_{i-1}^{k} 1 \\
&= \frac{\varepsilon}{k} k \\
&= \varepsilon.
\end{aligned}$$

By Corollary 3.1.1, any other tensor that gives the same k-plane scalar curvature measurements must in fact be R itself. $\square$

An equivalent statement is that $\mathscr{M}$ has k-csc(ε) if and only if $\mathrm{R} = \frac{2\varepsilon}{k(k-1)}\mathrm{R}_*$. This result coincides with the known fact that $\mathscr{M}$ has 2-csc(γ) if and only if $\mathrm{R} = \gamma\mathrm{R}_*$.

Corollary 4.4. *$\mathscr{M}$ has k-csc(ε) if and only if it has j-csc(δ), where* $\delta = \varepsilon\frac{j(j-1)}{k(k-1)}$.

Proof. Suppose $\mathscr{M}$ has k-csc(ε) for some $2 \le k \le n-2$ and $\varepsilon \in \mathbb{R}$. By **Corollary 4.2**, $\mathrm{R} = \frac{2\varepsilon}{k(k-1)}\mathrm{R}_*$. So $\mathscr{M}$ has 2-csc(γ) where $\gamma = \frac{2\varepsilon}{k(k-1)}$. Set $\delta = \varepsilon\frac{j(j-1)}{k(k-1)}$ for some $2 \le j \le n-2$. Then $\gamma = \frac{2\delta}{j(j-1)}$, so $\gamma\mathrm{R}_* = \frac{2\delta}{j(j-1)}\mathrm{R}_*$ and hence $\mathscr{M}$ has j-csc(δ). The argument from j-csc(δ) to k-csc(ε) merely swaps k and ε with j and δ. □

The only case that escapes these methods is $k = n-1$, as our approach relied on the extra codimensions to give a certain degree of freedom in isolating curvature components. In the case where the $(n-1)$-plane scalar curvature vanishes, some interesting phenomenon arise. In particular, the Ricci curvature vanishes (**Theorem 4.5**), and we can equate the k-plane scalar curvature of a plane with the $(n-k)$-plane scalar curvature of its orthogonal complement (**Theorem 4.6**).

Theorem 4.5. *Suppose a model space $\mathscr{M}$ has $(n-1)$-csc(0). Then the Ricci scalar* $\tau = 0$ *and the Ricci tensor* $\mathrm{Ric} = 0$.

Proof. Suppose $\mathscr{M}$ has k-csc(0) for $k = n-1$. In particular, $0 = \sum_{j>i=1}^{n-1}\mathrm{R}_{ijji}$ since $\sum_{j>i=1}^{n-1}\mathrm{R}_{ijji} = \mathscr{K}(\mathrm{L})$ for L spanned by $\{e_1,\ldots,e_{n-1}\}$. Then

$$\begin{aligned}\frac{\tau}{2} &= \sum_{j>i=1}^{n}\mathrm{R}_{ijji}\\ &= \sum_{j>i=1}^{n-1}\mathrm{R}_{ijji} + \sum_{j>i=1}^{n-1}\mathrm{R}_{inni}\\ &= \sum_{j>i=1}^{n-1}\mathrm{R}_{inni}\\ &= \mathrm{Ric}(\mathrm{e_n},\mathrm{e_n}).\end{aligned}$$

By permuting the e_i, we get $n > 2$ equations $\frac{\tau}{2} = \mathrm{Ric}(\mathrm{e_i},\mathrm{e_i})$. Summing over the equations, we get $\frac{n\tau}{2} = \sum_{i=1}^{n}\mathrm{Ric}(\mathrm{e_i},\mathrm{e_i})$. But then $\tau = \sum_{i=1}^{n}\mathrm{Ric}(\mathrm{e_i},\mathrm{e_i}) = \frac{\mathrm{n}}{2}\tau$, and since $n > 2$ it must be that $0 = \tau = \mathrm{Ric}(\mathrm{e_i},\mathrm{e_i})$ for all $i \in \{1,\ldots,n\}$ on any orthonormal basis. Since Ric is a symmetric, bilinear form, there is a particular orthonormal basis upon which Ric is diagonalized, so the $\mathrm{Ric_{ii}}$ entries are the only possible non-zero entries. But $\mathrm{Ric_{ii}} = 0$ on this basis as well, and hence $\mathrm{Ric} \equiv 0$. □

Theorem 4.6. *Suppose a model space $\mathscr{M}$ has $(n-1)$-csc(0). If* L *is a k-plane for some* $2 \le k \le n-2$, *then* $\mathscr{K}(\mathrm{L}) = \mathscr{K}(\mathrm{L}^\perp)$.

Proof. Given a model space $\mathscr{M}$ with $(n-1)$-csc(0), consider a 2-plane L spanned by some orthonormal vectors $\{e_1, e_2\}$. Extend this set to an orthonormal basis for V, $\{e_1, \dots, e_n\}$. Let $\mathrm{L}^{\perp} = \operatorname{span}\{e_3, \dots, e_n\}$ be an $(n-2)$-plane. Now consider the $(n-1)$-plane L' spanned by $\{e_1, e_3, \dots, e_n\}$. By assumption,

$$0 = \mathscr{K}(\mathrm{L}') = \sum_{j=3}^{n} \mathrm{R}_{1jj1} + \sum_{j>i=3}^{n} \mathrm{R}_{ijji} = (\mathrm{Ric}(\mathrm{e}_1, \mathrm{e}_1) - \mathrm{R}_{1221}) + \sum_{\mathrm{j>i=3}}^{\mathrm{n}} \mathrm{R}_{\mathrm{ijji}},$$

but the final expression is just $= -\kappa(e_1, e_2) + \mathscr{K}(e_3, \dots, e_n)$. This shows that the sectional curvature of the 2-plane spanned by two arbitrary orthonormal vectors is equal to the k-plane scalar curvature of the $(n-2)$-plane that is the remaining subspace of V. Induction on the original number of orthonormal vectors shows that the result holds for any $2 \le k \le n-2$. Given a k-plane L spanned by some set of orthonormal vectors $\{e_1, \dots, e_k\}$, we again extend this set to be an orthonormal basis for V and consider $\mathrm{L}^{\perp} = \operatorname{span}\{e_{k+1}, \dots, e_n\}$. Supposing that $\mathscr{K}(\mathrm{L}) = \mathscr{K}(\mathrm{L}^{\perp})$,

$$\begin{aligned}
-\mathscr{K}(e_1, \dots, e_{k+1}) + \mathscr{K}(e_{k+2}, \dots, e_n) &= -\left(\sum_{j>i=1}^{k+1} \mathrm{R}_{ijji}\right) + \sum_{j>i=+2}^{n} \mathrm{R}_{ijji} \\
&= -\left(\sum_{j>i=1}^{k} \mathrm{R}_{ijji} + \sum_{j=1}^{k} \mathrm{R}_{k+1,j,j,k+1}\right) \\
&\quad + \sum_{j>i=k+2}^{n} \mathrm{R}_{ijji} \\
&= -\sum_{j>i=1}^{k} \mathrm{R}_{ijji} + \mathrm{Ric}(\mathrm{e}_{\mathrm{k+1}}, \mathrm{e}_{\mathrm{k+1}}) \\
&\quad - \sum_{j=1}^{k} \mathrm{R}_{k+1,j,j,k+1} + \sum_{j>i=k+2}^{n} \mathrm{R}_{ijji} \\
&= -\sum_{j>i=1}^{k} \mathrm{R}_{ijji} + \sum_{j=k+2}^{n} \mathrm{R}_{k+1,j,j,k+1} \\
&\quad + \sum_{j>i=k+2}^{n} \mathrm{R}_{ijji} \\
&= -\sum_{j>i=1}^{k} \mathrm{R}_{ijji} + \sum_{j>i=k+1}^{n} \mathrm{R}_{ijji} \\
&= -\mathscr{K}(e_1, \dots, e_k) + \mathscr{K}(e_{k+1}, \dots, e_n) \\
&= 0,
\end{aligned}$$

by the induction hypothesis. Since we have shown the desired equality for $(k+1)$-planes and $(n-(k+1))$-planes, the result follows by induction. $\square$

Despite these structural conditions, we could not prove that such a model space must have $\mathrm{R} \equiv 0$ and in fact suspect the opposite. The proposed tensor R (on an orthonormal basis) takes the values $\mathrm{R}_{1221} = \mathrm{R}_{3443} = 1$, $\mathrm{R}_{1331} = \mathrm{R}_{2442} = -1$, and otherwise is 0. The high degree of symmetry in this tensor seems sufficient to satisfy all the requirements given by the theorems above. In order to prove or disprove this conjecture, our current method requires searching for particular $(n-1)$-planes that would allow us to isolate tensor components, although perhaps another method could be employed more effectively.

5 k-plane Constant Vector Curvature: Examples

Having fully characterized model spaces with k-plane constant sectional curvature (for $2 \le k \le n-2$), we now investigate model spaces with k-plane constant vector curvature (for $3 \le k \le n-1$). Although it follows immediately that a k-csc model space also has k-cvc, there are many model spaces with only the second, weaker condition.

Given an arbitrary vector $v \in \mathrm{V}$, we seek to construct a k-plane L such that $v \in \mathrm{L}$ and $\mathscr{K}(\mathrm{L})$ does not depend on the components of v. That is, the construction of L should work indiscriminately for v with non-zero components in any number of dimensions. The following examples consider model spaces with canonical tensors, and we utilize the eigenspaces of the associated symmetric, bilinear form ϕ to appropriately decompose v.

Given a linear transformation $\mathrm{A} : \mathrm{V} \to \mathrm{V}$, recall that an *eigenvector* of A is a non-zero $v \in \mathrm{V}$ with *eigenvalue* $\lambda \in \mathbb{R}$ such that $\mathrm{A}v = \lambda v$. Further, the eigenvalues of a linear transformation are basis independent, as λ is an eigenvalue if and only if $\det(\mathrm{A} - \lambda \mathrm{I}_n) = 0$, and this calculation is independent of a chosen basis.

We can also discuss eigenvalues in the context of symmetric, bilinear forms, defining the eigenvalues of ϕ as the eigenvalues of A. So for any eigenvector v_i and $w \in \mathrm{V}$, we know $\phi(v_i, w) = \langle \mathrm{A}v_i, w\rangle = \lambda_i \langle v_i, w\rangle$. Given an eigenvalue λ_i, there is an *eigenspace* spanned by the associated eigenvector v_i, denoted E_i. Finally, recall that the *spectrum* of ϕ, denoted $\mathrm{spec}(\phi)$, is the collection of eigenvalues of ϕ repeated according to multiplicity.

Investigating the k-cvc condition in model spaces with canonical tensors, we express the k-plane scalar curvature in terms of eigenvalues of the associated form ϕ. We can relate these two objects through sectional curvatures.

Theorem 5.1. *Let $\mathscr{M} = (\mathrm{V}, \langle\ ,\ \rangle, \mathrm{R}_\phi)$ be a model space. For $i \neq j$, if f_i, f_j are unit vectors in the eigenspaces for λ_i, λ_j, respectively, then the sectional curvature is $\kappa(f_i, f_j) = \lambda_i \lambda_j$.*

Proof. By the Spectral Theorem, there is some change of basis that diagonalizes ϕ, so $\phi(e_i, e_j) = 0$ for $i \neq j$. For unit vectors $f_i \in \mathrm{E}_i$ and $f_j \in \mathrm{E}_j$ (with $i \neq j$), we have

$$\kappa(f_i, f_j) = \mathrm{R}_\phi(f_i, f_j, f_j, f_i) = \phi(f_i, f_i)\phi(f_j, f_j) - \phi(f_i, f_j)^2 = \lambda_i \lambda_j.$$

□

This theorem allows us to easily calculate k-plane scalar curvatures with respect to canonical tensors R_ϕ, particularly for diagonalized ϕ. We can see this result in action in the following example.

Example 5.2. Consider the model space $\mathcal{M} = (V, \langle\ ,\ \rangle, R)$ with the orthonormal basis $\{e_1, \ldots, e_6\}$ for V, a positive-definite inner product $\langle\ ,\ \rangle$, a canonical tensor $R = R_\phi$, where ϕ is represented by

$$\begin{bmatrix} I_2 & 0_2 & 0_2 \\ 0_2 & -I_2 & 0_2 \\ 0_2 & 0_2 & 0_2 \end{bmatrix}$$

where I_2 is the 2×2 identity matrix and 0_2 is the 2×2 matrix whose entries are all 0. Note that we have three two-dimensional eigenspaces, $E_1 = \text{span}\{e_1, e_2\}$ with $\lambda_1 = 1$, $E_2 = \text{span}\{e_3, e_4\}$ with $\lambda_2 = -1$, and $E_3 = \text{span}\{e_5, e_6\} = \ker(R)$. We will show that $\mathcal{M}$ has multiple k-cvc values for different choices of k. One way to approach this problem is to decompose the given vector into compo that are contained entirely in some eigenspace. Such a construction is often possible in multiple ways for a given k.

Proposition 5.3. *The model space $\mathcal{M}$ of **Example 5.2** has the following properties:*

(i) k-cvc(0) for $k = 2, 3$,

(ii) k-cvc(-1) for $k = 3, 4, 5$,

(iii) k-cvc(-2) for $k = 5$.

Note that (i) follows immediately from **Proposition 3.6**, since $\dim(\ker(R)) \geq 2$. Further, $\mathcal{M}$ cannot have 2-cvc(ε) for any $\varepsilon \neq 0$ by **Proposition 3.7**. Given a non-zero vector $v \in V$, we take the decomposition $v = a_1 v_1 + a_2 v_2 + a_3 v_3$ where $v_i \in E_i$ are unit vectors and $a_i \in \mathbb{R}$. Using such a decomposition, we can construct planes spanned by combinations of the v_i and e_i that have the desired k-plane scalar curvature. For example, considering the 3-plane spanned by $\{v_1, v_2, v_3\}$, we have

$$\mathcal{K}(v_1, v_2, v_3) = \kappa(v_1, v_2) + \kappa(v_1, v_3) + \kappa(v_2, v_3) = -1$$

by **Theorem 5.1** and since $v_3 \in \ker(R)$. In the case where some $a_i = 0$, we can replace v_i with an appropriate e_j. For example, if $v = a_1 v_1$, we can consider the 3-plane spanned by $\{v_1, e_4, e_6\}$ which has 3-plane scalar curvature -1 (by a similar calculation). Since such a 3-plane exists for any $v \in V$, we can say our model space has 3-cvc(-1).

We can take a similar approach for $k = 4, 5$. In fact, there are multiple possible constructions of 4-planes that give the desired result. One of these possibilities is the 4-plane spanned by $\{e_1, e_2, v_2, v_3\}$. Similarly, the 5-plane spanned by $\{v_1, e_3, \ldots, e_6\}$ shows

$\mathscr{M}$ has 5-cvc(-1), while the plane spanned by $\{e_1,\ldots,e_4,v_3\}$ shows 5-cvc(-2). Even if some $a_i = 0$, as before we can make suitable adjustments to these planes without changing the result.

This first model space exemplifies how studying the k-cvc condition can generate a wide variety of representative numbers, which can aid in the characterization of the model space. Now we consider and example that has the interesting property of giving k-cvc values for a connected interval.

Example 5.4. Let $\mathrm{M} = (\mathrm{V}, \langle\ ,\ \rangle, \mathrm{R})$ be a model space such that $\dim(\mathrm{V}) = n \geq 4$, the inner product $\langle\ ,\ \rangle$ is positive definite, and $\mathrm{R} = \mathrm{R}_\phi$ where ϕ is represented by

$$\begin{bmatrix} \mathrm{I}_2 & 0_2 & \ldots & 0_2 \\ 0_2 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0_2 & 0 & \ldots & 0 \end{bmatrix}$$

where I_2 and 0_2 are expressed as in the previous example. Here we have two eigenspaces, a two-dimensional eigenspace $\mathrm{E}_1 = \mathrm{span}\{e_1, e_2\}$ with $\lambda_1 = 1$, and $\mathrm{E}_2 = \mathrm{span}\{e_3,\ldots,e_n\} = \ker(\mathrm{R})$ of dimension $n-2$.

Proposition 5.5. *The model space $\mathscr{M}$ in **Example 5.4** has the following properties:*

(i) k*-cvc*(0) *for* $k \geq 2$*,*

(ii) k*-cvc*(1) *for* $k \geq 3$*,*

(iii) 3*-cvc*$([0,1])$ *and only* 3*-cvc*$([0,1])$*,*

(iv) *at least* k*-cvc*$([0,1])$ *for* $k \geq 4$*,*

(v) *For* $k \geq 4$*, if* $\mathscr{M}$ *has* k*-cvc*(ε) *then* $\varepsilon \in [0, k-1)$*,*

As before, (i) follows from **Proposition 3.6** and the nullity of R. More explicitly, we can take the same approach as in the previous example and decompose a given $v \in \mathrm{V}$ into $v = a_1 v_1 + a_2 v_2$. Since $\dim(\mathrm{E}_2) = n-2$, we can create an orthonormal basis for E_2 by finding vectors $w_1,\ldots,w_{n-3}$ perpendicular to v_2. Then $\{v_1, v_2, w_1,\ldots w_{k-2}\}$ spans a k-plane L_0 with $\mathscr{K}(\mathrm{L}_0) = 0$, since all the vectors besides v_1 are in the kernel of R. For (ii), instead consider the span of $\{e_1, e_2, v_2, w_1,\ldots,w_{k-3}\}$. Call this plane L_1. So $\mathscr{K}(\mathrm{L}_1) = \kappa(e_1, e_2) = 1$ since $v_2, w_1,\ldots,w_{k-3} \in \ker(\mathrm{R})$.

Results (iii)–(v) involve generating a connected set of k-cvc values and bounding possible k-cvc values, both of which are rich areas for future research.

The claim that $\mathscr{M}$ has k-cvc($[0,1]$) is to say that for nonzero $v \in \mathrm{V}$ and $\varepsilon \in [0,1]$, we can construct a k-plane that contains v and has k-plane scalar curvature ε. We do so by rotating $(k-1)$-planes in $v^{\perp}$ via a linear transformation represented by a matrix in $\mathrm{SO}(n)$, the group of orthogonal matrices with determinant 1. Let the linear transformation $\mathrm{A}_\theta : [0, \frac{\pi}{2}] \to \mathrm{SO}(n)$ be represented by

$$\mathrm{A}_\theta = \begin{bmatrix} \mathrm{I}_{k-1} & 0 & 0 \\ 0 & \mathrm{R} & 0 \\ 0 & 0 & \mathrm{I}_{n-k-1} \end{bmatrix}$$

where

$$\mathrm{R} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

This matrix rotates the k^{th} vector into the $(k+1)^{th}$, and the $(k+1)^{th}$ into the $-k^{th}$. We can construct two k-planes $\mathrm{L}_0, \mathrm{L}_1$ containing v such that $\mathscr{K}(\mathrm{L}_0) = 0$ and $\mathscr{K}(\mathrm{L}_1) = 1$, and use the above special orthogonal matrix to rotate between the two. Thus we can obtain a connected set of k-cvc values, that is to say, for every $\varepsilon \in [0,1]$, there is some k-plane containing v such that $\mathscr{K}(\mathrm{L}) = \varepsilon$. To show (iv) and the first part of (iii), we take $k \geq 3$, $n \geq 4$.

We decompose V into E_1 and $\mathrm{E}_2 = \ker(\mathrm{R})$, where E_1 is the eigenspace associated with $\lambda_1 = 1$. Then take $v = av_1 + bv_2$, where $v_i \in \mathrm{E}_i$ are unit vectors. Since E_1 is 2-dimensional, $\mathrm{span}\{v_1, u\} = \mathrm{E}_1$ for some unit vector $u \perp v_1$. Similarly, we can construct an orthonormal basis $\{v_2, w_1, \ldots, w_{n-3}\}$ that spans the $(n-2)$-dimensional $\ker(\mathrm{R})$. Now, set $f_1 = v_1,\ f_2 = v_2,\ f_3 = w_1,\ \ldots,\ f_k = w_{k-2}, f_{k+1} = u$. Extend this to an orthonormal basis for all of V. Then for the k-plane $\mathrm{L} = \mathrm{span}\{f_1, \ldots, f_k\}$,

$$\begin{aligned}
\mathscr{K}(\mathrm{A}_\theta \mathrm{L}) &= \mathscr{K}(\mathrm{A}_\theta f_1, \ldots, \mathrm{A}_\theta f_k) \\
&= \mathscr{K}(v_1, v_2, w_1, \ldots, w_{k-3}, \cos\theta w_{k-2} + \sin\theta u) \\
&= \mathrm{R}(v_1, \cos\theta w_{k-2} + \sin\theta u, \cos\theta w_{k-2} + \sin\theta u, v_1 \\
&= \mathrm{R}(v_1, \sin\theta u, \sin\theta u, v_1) \\
&= \sin^2\theta \mathrm{R}(v_1, u, u, v_1) \\
&= \sin^2\theta,
\end{aligned}$$

since $\mathrm{span}\{v_1, u\} = \mathrm{E}_1$ has sectional curvature 1. If $a = 0$, then $v \in \mathrm{E}_2$. So $\mathrm{E}_1 = \mathrm{span}\{e_1, e_2\}$. Set $f_1 = e_1$ and $f_{k+1} = e_2$, and keep all other f_i the same. Extend this to an orthonormal basis for V. Then, as before,

$$\mathscr{K}(\mathrm{A}_\theta \mathrm{L}) = \mathrm{R}(e_1, \sin\theta e_2, \sin\theta e_2, e_1) = \sin^2\theta \mathrm{R}_{1221} = \sin^2\theta.$$

If $b = 0$, then $v \in \mathrm{E}_1$. So $\mathrm{V}_1 = \mathrm{span}\{e_3, \ldots, e_n\}$. Now, set $f_2 = e_3, \ldots, f_k = e_{k+1}$ and keep all other f_i the same. Extend this to an orthonormal basis for V. Then, as before, $\mathscr{K}(\mathrm{A}_\theta \mathrm{L}) = \mathrm{R}(v_1, \sin\theta u, \sin\theta u, v_1) = \sin^2\theta$.

Note $\mathscr{K}(A_0L) = 0$ and $\mathscr{K}(A_{\frac{\pi}{2}}L) = 1$. Further, $L \mapsto \mathscr{K}(L)$ is continuous, and so by the Intermediate Value Theorem, for all $\varepsilon \in [0,1]$, there is some θ such that $\mathscr{K}(A_\theta L) = \varepsilon$, where L contains an arbitrary $v \in V$. Further, we explicitly determine this θ to be $\arcsin(\sqrt{\varepsilon})$. Let $\varepsilon \in [0,1]$, and set $\theta = \arcsin(\sqrt{\varepsilon})$. Then

$$\mathscr{K}(A_\theta L) = \sin^2\theta = \sin^2(\arcsin(\sqrt{\varepsilon})) = \sqrt{\varepsilon}^2 = \varepsilon.$$

To bound the k-cvc values, we suppose $\mathscr{M}$ has k-cvc(ε) for some $\varepsilon \in \mathbb{R}$ and $k \geq 3$. Take $v = e_1$ and let L be a k-plane containing v such that $\mathscr{K}(L) = \varepsilon$. Without loss of generality, since v is contained in L, we can set $f_1 = v$, and extend to an orthonormal basis $\{f_1, \ldots, f_k\}$ for L. So each $f_i = a_{i2}e_2 + \cdots + a_{in}e_n$ for some $a_{ij} \in \mathbb{R}$ where $\sum_{j=2}^n a_{ji}^2 = 1$. Since $e_3, \ldots, e_n \in \ker(R)$,

$$\varepsilon = \mathscr{K}(L) = \sum_{i=2}^k a_{i2}^2 R_{1221} = \sum_{i=2}^k a_{i2}^2 < \sum_{i=2}^k 1 = k-1.$$

The strict inequality holds since f_i are orthonormal, so if any of the $a_{i2}^2 = 1$ then $a_{j2} = 0$ for all $j \neq i$. We get a lower bound of 0, as $\mathscr{K}(L)$ is a sum of squares of real numbers and so cannot be negative. Hence $0 \leq \mathscr{K}(L) = \varepsilon < k-1$, which proves (v).

Finally to finish (iii), we suppose $\mathscr{M}$ has 3-cvc(ε) for some $\varepsilon \in \mathbb{R}$ and consider $v = e_3$. Let $L = \operatorname{span}\{f_1, f_2, f_3\}$ be a 3-plane containing v such that $\mathscr{K}(L) = \varepsilon$. As before, we can construct an orthonormal basis for L such that $f_1 = v$. Let $f_2 = a_1e_1 + \cdots + a_ne_n$ and $f_3 = b_1e_1 + \cdots + b_ne_n$. Since $f_1 \in \ker(R)$, $\mathscr{K}(L) = R(f_2, f_3, f_3, f_2)$. By [2, Theorem 1.1], the sectional curvature values are bounded by the products of eigenvalues, so for $\pi = \operatorname{span}\{f_2, f_3\}$, $0 \leq \kappa(\pi) = R(f_2, f_3, f_3, f_2) \leq 1$. Hence $0 \leq \mathscr{K}(L) = \varepsilon \leq 1$.

6 k-plane Constant Vector Curvature: General Discussion

Based on the work done in these examples, we can obtain results that apply more generally to model spaces with canonical curvature tensors. Given our previous results, we can construct a model space with the k-cvc condition for any connected interval of values, and further we can establish loose bounds on possible values based on eigenvalues.

Theorem 6.1. *For any interval* $[a,b]$ *in* $\mathbb{R}$, *there exists* $M = (V, \langle\ ,\ \rangle, R)$ *such that* $\mathscr{M}$ *has* k*-cvc*$([a,b])$ *for* $k \geq 3$.

Proof. Let $[a,b] \in \mathbb{R}$ be an interval. Let $M_1 = (V, \langle\ ,\ \rangle, R_1)$ have k-csc(a), and let $M_2 = (V, \langle\ ,\ \rangle, R_2)$ where $R_2 = (b-a)R_\phi$ for R_ϕ as in **Example 5.4**. So by **Proposition 3.5** and **Example 5.4**, M_2 has k-cvc($(b-a)[0,1]$), in other words, it has k-cvc($[0, b-a]$). Construct $M = (V, \langle\ ,\ \rangle, R)$ such that $R = R_1 + R_2$. Then by **Proposition 3.4**, $\mathscr{M}$ has k-cvc($a + [0, b-a]$) which is to say $\mathscr{M}$ has k-cvc($[a,b]$). □

Theorem 6.2. *Suppose a model space* $\mathcal{M} = (V, \langle\ ,\ \rangle, R_\phi)$ *has* k*-cvc*(ε) *for some* $\varepsilon \in \mathbb{R}$*. Then*

$$\binom{k}{2}\min\{\lambda_i\lambda_j | i \neq j\} \leq \varepsilon \leq \binom{k}{2}\max\{\lambda_i\lambda_j | i \neq j\}.$$

Proof. Let $\mathcal{M}$ has k-cvc(ε) for some $\varepsilon \in \mathbb{R}$. By [2, Theorem 1.1], the sectional curvatures are bounded by products of eigenvalues of ϕ. Since calculating the k-plane scalar curvature amounts to summing over $\binom{k}{2}$ sectional curvatures, we know we could sum at least $\binom{k}{2}$ minimal sectional curvatures and at most $\binom{k}{2}$ maximal sectional curvatures. □

Clearly for model spaces with a canonical ACT there is some relationship between the eigenvalues of ϕ and the possible k-cvc values. We suspect that the multiplicity of eigenvalues can determine whether a model space has k-cvc(ε) for some $\varepsilon \in \mathbb{R}$. In particular, if there are no more than k distinct eigenvalues, we conjecture that $\mathcal{M}$ has k-cvc(ε) for some $\varepsilon \in \mathbb{R}$. There is a lot of room for further research in this area, particularly in sharpening bounds on possible values.

Another interesting problem is in the realm of geometric realization, and creating good examples of realizable k-cvc model spaces. A manifold is said to be a *geometric realization* of R at a point in the manifold if there is an isometry from the tangent space at the point to V that appropriately relates R to the Riemannian curvature tensor (see [8, Section 1.12] for a more precise description). Although every ACT is geometrically realizable (see [8, Theorem 1.12.2]), such realizations are quite complicated in general. As noted by the referee, many known examples of of cvc(ε) spaces (for $\varepsilon \neq 0$) are symmetric spaces such as $\mathbb{CP}^n$ and $\mathbb{S}^2 \times \mathbb{S}^2$. It would be both challenging and interesting to develop a method of creating k-cvc examples that can be realized across a manifold.

Finally, the k-plane sectional curvature measurement is adopted from work by Chen on submanifolds [3, Section 2, (2.6)]. Using a similar tool, the k-Ricci curvature, Chen goes on to develop the notion of a k-Einsteinian space, where a k-dimensional subspace of V is k-Einsteinian if the k-Ricci curvature is constant across the space. This curvature condition is similar (but not equivalent) to our generalized notions of csc and cvc. Future research might build upon the work presented here by incorporating this curvature condition, or by studying related curvature conditions such as Schmidt and Wolfson's updated k-CVC conditon [16]. The full potential of these curvature conditions has yet to be explored, and further studies could yield exciting and interesting results.

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Maxine Calle
Reed College
callem@reed.edu