Rose-Hulman Undergraduate Mathematics Journal

Volume 21
Issue 1

Article 3

On the Enumeration of Shapes

May Cai
Georgia Institute of Technology, mayhcai@gmail.com

Nicholas Liao
Georgia Institute of Technology, nliao7@gatech.edu

Follow this and additional works at: https://scholar.rose-hulman.edu/rhumj

Part of the Discrete Mathematics and Combinatorics Commons

Recommended Citation
Cai, May and Liao, Nicholas (2020) "On the Enumeration of Shapes," *Rose-Hulman Undergraduate Mathematics Journal*: Vol. 21 : Iss. 1 , Article 3.
Available at: https://scholar.rose-hulman.edu/rhumj/vol21/iss1/3

On the Enumeration of Shapes

Cover Page Footnote
This line of examination comes from questions posed by the Shape Computation Lab at the College of Design at Georgia Tech. As such we would like to thank Athanassios Economou, and his group Kurt Hong, Heather Ligler, and James Park of the Shape Computation Lab for their advice and suggestions throughout the work. We also owe our wonderful advisor Josephine Yu thanks for her invaluable advice and support.

This article is available in Rose-Hulman Undergraduate Mathematics Journal: https://scholar.rose-hulman.edu/rhumj/vol21/iss1/3

ℝose-Hulman Undergraduate Mathematics Journal
Volume 21, Issue 1, 2020

On the Enumeration of Shapes

By *May Cai* and *Nicholas Liao*

Abstract. We define a shape as a union of finitely many line segments. Given an arrangement of lines on a plane, we count the number of shapes in the arrangement by examining the symmetries of the arrangement and applying Burnside's lemma. We further establish a generating function for the number of distinct line segments on a line with k distinguished points. We list all affine line arrangements of four and five line segments, together with the corresponding number of shapes on them.

1 Introduction

Consider a particular arrangement of n lines on the plane. We define a shape on arrangement as the union of non-zero length line segments, one from each line. Our task is to consider how many distinct shapes can be generated from a given line arrangement.

There are infinitely many possible line segments on a given line. For this question to be meaningful, we must define what it means for two shapes to be equivalent. Any given line in an arrangement will have up to $n-1$ intersections with the other lines in the arrangement. Say our given line has k intersection points. We will consider each intersection point to be a distinguished point on a line that divides it into $2k+1$ zones, k of which are 0-length zones that correspond to the points themselves. If two line segments span the same zones then we will consider them equivalent.

Then we can consider two shapes on a given line arrangement equivalent if they contain equivalent line segments. For example, in Figure 1, shapes (a) and (b) would be considered the same arrangement while shape (c) would be a different shape.

Shape (c) in Figure 1 differs from the other two in that the upper-left line segment touches the intersection point between two lines of the arrangement (and so includes that zone).

Furthermore we will also consider two shapes equivalent if one is a reflection or rotation of another shape with equivalent line segments as the other (and so equivalent in the previous definition after rotation or reflection). In Figure 2, we consider shapes (a) and (b) to be equivalent because a simple 180 degree rotation (or a reflection across

Mathematics Subject Classification. 05A15
Keywords. enumeration, Polya theory, line arrangements

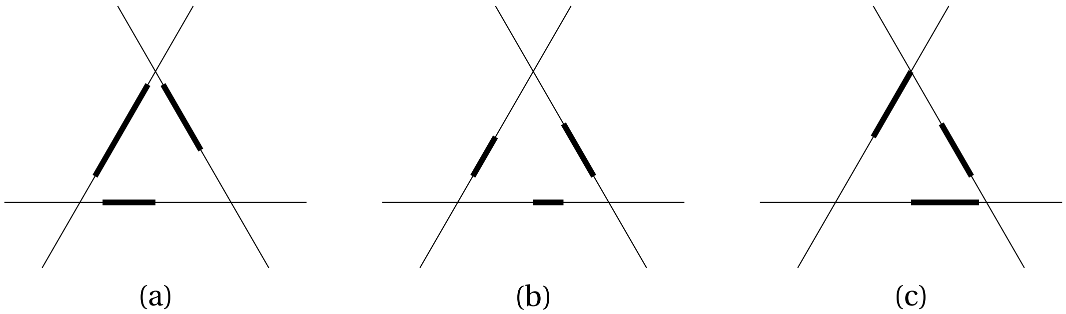


Figure 1: 3 shapes on 3 lines

the horizontal axis) leaves the two arrangements equivalent to one another (with the previous definition), while shape (c) has a distinct line segment arrangement.

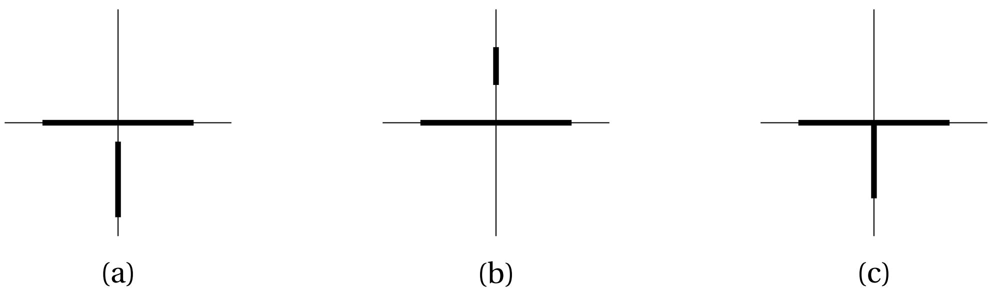


Figure 2: 3 shapes on two lines

The natural question is, then, how many such shapes can there be on any given line arrangement? In Section 2 we count the number of distinct ways to place line segments on a line with k distinguishing points. We establish a bijection between that sequence and the crystal ball numbers on a cubic lattice. Then in Section 3 we discuss the total number of distinct shapes on a line arrangement, and in Section 4 we use Burnside's lemma to reduce that number to the total number of shapes distinct under symmetry. Finally Section 5 contains an enumeration of the total number of line arrangements in four and five lines, and an explanation of how they were generated.

Our motivation for this work stems from questions Pr. Athanassios Economou and his research group at the Shape Computation Lab at Georgia Tech have been pursuing in shape matching and visual recognition in computer-aided design (CAD) systems. The two-part representation of shape in terms of line segments and underlying lines is the very same model used in the Shape Machine [5], a shape grammar interpreter currently developed at the Shape Computation Lab. The enumeration discussed in this work is envisioned as a complement to the work in the lab and as a foundation for a formal and well defined standard of reference of the simplest possible arrangements of shapes made up of lines in the plane. Aspects of the work have been presented at the Shape Machine Symposium [6] and the Shape Atlas Exhibition, an exhibition hosted by the School for Architecture at Georgia Tech.

2 Line Segments on a Line With k Intersections

Now it would be helpful to understand how many possible distinct line segments can be placed on a given line.

Theorem 2.1. *There are $2k^2+2k+1$ distinct ways to place one line segment on a line with k intersections.*

Proof. Consider a line with k points of intersection. On a line, k cuts will result in $k+1$ zones between the cuts.

Now we can split all possible line segments into two categories, ones that span multiple intersections and zones, and those that are contained within a single zone.

The number of line segments that span multiple intersections and zones are $\binom{2k+1}{2}$, as for any two different intersections or zones, there is only one line segment starting in one and ending in the other, for a total of

$$\frac{(2k+1)(2k)}{2} = \frac{4k^2+2k}{2} = 2k^2+k$$

possible line segments.

Now the number of line segments that are contained within a single zone (as a line segment that is contained in only an intersection is a single point, and thus not a valid line segment) are equal to the number of non-zero length zones, which as previously mentioned is $k+1$.

Combining the two classes of line segments, we reach $2k^2+2k+1$ unique classes of line segments, completing the proof. □

What if we relax the restriction that there must be exactly one line segment per line? How many ways can you put two segments on a line with k intersections? What about three segments? We have to be careful not to choose endpoints such that the segments intersect each other, as this would yield a single line segment. But at the same time, we could potentially have all line segments contained in the same zone without any two line segments intersecting.

Theorem 2.2. *Let*

$$\mathrm{F}_i(x) := \sum_{k\geq 0} c_{i,k}x^k$$

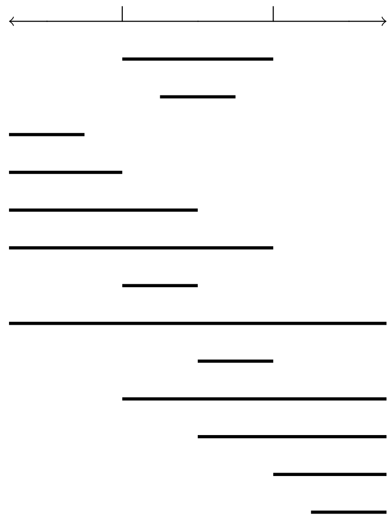


Figure 3: Example: There are 13 ways to place a line segment on a line with 2 intersections.

where $c_{i,k}$ denotes the number of ways to place i line segments on a line with k distinguished points.

This generating function has the closed form

$$\mathrm{F}_i(x) = \frac{(1+x)^{2i}}{(1-x)^{2i+1}}.$$

Proof. Imagine we fix the i line segments on the line, and then place the k distinguished points on the oriented line around them. Each line segment has two endpoints for a total of $2i$ endpoints, and we want to know where they fall relative to the distinguished points. At most one of the distinguished points can coincide with each endpoint, so we can consider each of the $2i$ different $(1+x)$ terms as corresponding to a specific endpoint and whether or not we choose to place a distinguished point there. The rest of the oriented line is divided into $2i+1$ distinct sections by these endpoints, and we can place any number of distinguished points in any of the sections. This corresponds to there being $2i+1$ different $(1-x)^{-1}$ terms, each of which tracks the (arbitrary) number of distinguished points in their corresponding section. (Recall that $(1-x)^{-1} = 1+x+x^2+\ldots$.) These choices uniquely determine the number of ways to place i line segments on an oriented line with k distinguished points, and thus is described by the above generating function.

□

This theorem also gives us a formula to calculate $c_{i,k}$ directly by summing up at most $2i+1$ products of binomial coefficients.

Corollary 2.3.

$$c_{i,k} = \sum_{j\geq 0}^{2i} \binom{2i}{j}\binom{k+2i-j}{2i}$$

Proof. Let us take the generating function from Theorem 2.2. We have that

$$\mathrm{F}_i(x) = \frac{(1+x)^{2i}}{(1-x)^{2i+1}}.$$

Expanding out the polynomial in the numerator, we get

$$\mathrm{F}_i(x) = \sum_{j\geq 0}^{2i} \binom{2i}{j} \frac{x^j}{(1-x)^{2i+1}}.$$

Now we use the identity that $\frac{1}{(1-x)^k} = \sum_{n\geq 0} \binom{n+k-1}{k-1} x^n$, and we get that

$$\begin{aligned}
\mathrm{F}_i(x) &= \sum_{j\geq 0}^{2i} \binom{2i}{j} \left(x^j \sum_{k\geq 0} \binom{k+2i}{2i} x^k \right) \\
&= \sum_{j\geq 0}^{2i} \binom{2i}{j} \left(\sum_{k\geq 0} \binom{k+2i-j}{2i} x^k \right) \\
&= \sum_{k\geq 0} \left(\sum_{j\geq 0}^{2i} \binom{2i}{j} \binom{k+2i-j}{2i} \right) x^k
\end{aligned}$$

and we simply extract the coefficient corresponding to x^k to get

$$c_{i,j} = \sum_{j\geq 0}^{2i} \binom{2i}{j}\binom{k+2i-j}{2i}$$

as desired.

□

We came upon this generating function by noticing that the sequence of ways to place one line segment on a line with k distinguished points appeared to be the same as k-th crystal ball number on a cubic lattice in $\mathbb{R}^2$ [3]. The k-th crystal ball number on a cubic lattice in $\mathbb{R}^n$ is defined as the number of elements of $\mathbb{Z}^n$ within k taxicab (or L_1) distance of the origin. We saw that the number of ways to place i line segments on a line with k distingushed points seemed to match the k-th crystal ball number on a cubic lattice in $\mathbb{R}^{2i}$ [2].

Proposition 2.4. *There is a bijection between the number of ways to place i line segments on an oriented line with k distinguished points and the k-th crystal ball number on a cubic lattice in $\mathbb{R}^{2i}$.*

Proof. We claim that this bijection follows from the fact that the two sequences have the same generating function

$$\mathrm{F}_i(x) = \frac{(1+x)^{2i}}{(1-x)^{2i+1}}.$$

We have already shown that this is the generating function for the number of ways to place i line segments on a line with k distinguished points, so it remains to show that this generating function describes the crystal ball numbers.

Now we note that the k-th crystal ball sequence on $\mathbb{R}^{2i}$ describes the number of $2i$-tuples of integers such that the total magnitude of all the integers in the tuple sum to less than or equal to k. (In this notation, the sequence starts at $k = 0$, valued at 1 for all i.) We claim that the generating function

$$\mathrm{G}(x) = \frac{(1+x)^y}{(1-x)^y}$$

is the generating function for the number of y-tuples of integers whose total magnitudes sum to exactly k, for the k-th term in the sequence. If this is true then it is clear that the claimed generating function is correct for the sequence, since the extra $(1-x)^{-1}$ serves as a slack term.

We can separate the generating function G into y terms of the form $(1+x)/(1-x)$. Each of these terms corresponds to a different element of the y-tuple. Recall that

$$(1-x)^{-1} = 1 + x + x^2 + \ldots.$$

Thus, $(1+x)(1-x)^{-1} = 1+2x+2x^2+2x^3+\ldots$. This expresses that there is one way to choose 0, and two ways to choose any $|a| \neq 0$ value for a particular integer in a y-tuple, $|a|$ and $-|a|$. The coefficient of x^b in $\mathrm{G}(x)$ will be the number of ways that y integers can have total magnitude equal to b. Thus, G is a generating function for the number of elements of $\mathbb{Z}^y$ exactly k away from the origin in $\mathbb{R}^y$ (using L_1 distance), and so $\mathrm{G}(x)(1-x)^{-1}$ is the generating function for the number of elements of $\mathbb{Z}^y$ **at most** k away from the origin in $\mathbb{R}^y$. So $\frac{(1+x)^{2i}}{(1-x)^{2i+1}}$ is the generating function for the number of elements of $\mathbb{Z}^{2i}$ at most k away from the origin in $\mathbb{R}^{2i}$, which is exactly the definition of the k-th crystal ball number on a cubic lattice in $\mathbb{R}^{2i}$. □

3 Total number of shapes on a given line arrangement

Now that we can count the number of distinct line segments per line, the way we count the total number of shapes (without considering isomorphism under symmetry) is simple. For a line arrangement with n lines having $k_1, k_2, \ldots, k_n$ intersections, recall that we calculated the number of ways to place a single line segment on a line with k_a intersections to be $2k_a^2 + 2k_a + 1$ in Theorem 2.1. The total number of shapes using exactly one line segment per line is then

$$\prod_{a=1}^{n}\left(2k_a^2+2k_a+1\right)$$

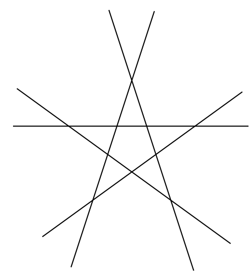


Figure 4: A five-pointed star

For example, consider Figure 3. Each of the five lines in this arrangement has 4 distinguished points, which means each line can have $2*(4)^2+2*4+1=41$ distinct line segments when we only allow one segment per line. So the number of shapes that can be generated on this line arrangement, disregarding symmetry, is $41^5 = 115,856,201$. However, this arrangement is highly symmetric, so we will see that many of these potential shapes are in fact not distinct.

Let us now turn our attention to a more generalized shape, where we can have more than one line segment per line. For a given construction line arrangement with n lines, with $i_1, i_2, \ldots, i_n$ line segments on lines $1, 2, \ldots n$, we can calculate the number of shapes (again disregarding isomorphism under symmetry) by finding the number of ways to place i_a line segments on a line with k_a intersections using the generating function described in Theorem 2.2, and then taking the product of all of these ways for each a from 1 to n.

4 Shapes Under Symmetry

Two shapes are equivalent if there is a bijection from intersection points to intersection points in such a way that underlying lines map to lines.

This then forms a symmetry group of these operations from any given shape to itself. For the 5-pointed star example of Figure 3, the operations are the symmetry group of the regular pentagon. This consists of rotations by multiples of 72° (5 including the identity), and reflections across an axis that contain both a point of the pentagon and the midpoint of the opposite edge (5 points).

Burnside's lemma says that for a set X and a symmetry group G acting on X, the total number of orbits is described by

$$|X/G| = \frac{1}{|G|} \sum_{g \in G} |X^g|,$$

where X^g is the set of elements of X that are fixed by the group element g.

If we define our set X to be the set of shapes in our given configuration, and the group to be the symmetry group that acts on the line arrangement, then direct application of Burnside's lemma gives us the number of distinct shapes.

Let us calculate the total number of distinct one-segment-per-line shapes for the 5-pointed star as an example. We noted earlier that it has the symmetry group of the pentagon. Notice that the fixed shapes of the five reflections require one line segment (of the line perpendicular to the axis of reflection) to remain fixed when reflected across the middle point of the line, meaning there are only five valid line segments that remain fixed, namely the ones that are symmetric across that axis of reflection. The other four lines are uniquely determined by a choice of two of the lines, since reflection will superimpose two of the lines on the other two, resulting in 41^2 choices. Thus for each reflection, there are $5 * 41^2$ shapes that are fixed by that reflection. For the rotations, it is enough to notice that each line must be a rotational copy of each other line, which means a choice of a line segment for one line determines the choice for all line segments, so there are 41 shapes fixed by each non-identity rotation. And the identity map contains 41^5 elements that are obviously fixed.

So in conclusion, there are $\frac{1}{10}(5 * 5 * 41^2 + 4 * 41 + 41^5) = 11{,}589{,}839$ distinct shapes under the line arrangement isomorphic to the 5-pointed star.

In general, computing this number will require finding the symmetry group that acts on the line arrangement, and then calculating the fixed shapes under each of the elements of the symmetry group and applying Burnside's lemma.

Note that calculating the fixed shapes for reflections will often require calculating the number of ways to place line segments on a line such that it is fixed under reflection.

Theorem 4.1. *The number of ways to place i line segments on a line with k intersection points, such that the resulting placement is fixed under reflection across that line, is the $\lfloor k/2 \rfloor$-th degree coefficient of the generating function*

$$H_i(x) = \frac{(1+x)^i}{(1-x)^{i+1}}$$

Proof. For a choice of line segments on a line to remain fixed under reflection, exactly half of the endpoints of the line segments will be on one side of the line, and must match up with the other half on reflection. So, if we will place i line segments (i.e. $2i$ line segment endpoints) on a line with k intersections, placing i line segment endpoints on one half of the line will fix the placement of the other segment endpoints on the other half of the line.

Note that no line segment endpoint can fall on the point of reflection of the line, since that would require two line segments ending on the same point.

So, it remains to find the number of ways to place i points, corresponding to half of the line segment endpoints, on half of the line. Since the line has k distinguished points, half of them (rounded down) fall on one half of the line. So we must find a way to place i points on a line with $\lfloor k/2 \rfloor$ distinguished points.

Thus, by the same logic as in Theorem 2.2, this formula has the generating function

$$\mathrm{H}_i(x) = \frac{(1+x)^i}{(1-x)^{i+1}}.$$

□

5 Conclusion

While we have shown how to enumerate shapes given a single line segment per line and how to enumerate multiple line segments per line, we have not shown how to enumerate shapes given multiple line segments per line. Another potential area of interest is in curves: either relaxing the definitions into pseudoline arrangements, or fully allowing arbitrary self-intersection.

Appendix: Enumerating Shapes with Four and Five Lines

The following section contains a complete catalog of all four-line and five-line affine line arrangements, as well as a list of the number of shapes for each arrangement both with and without symmetry. This table can also be found at
http://people.math.gatech.edu/ jyu67/shapes/webpage/table.html.
These line arrangements were generated from the rank 3, cardinality 5 and 6 catalog of hyperplane arrangement chirotopes in [1]. This is in general a difficult problem to do, since by Mnëv's universality theorem [4] the realization space of an arbitrary oriented matroid is stably equivalent to solving a system of polynomial equations and inequalities. These were done by hand, which was feasible given how small the matroids were. These arrangements are rank 3 in projective space, so one line was chosen "at infinity" in order to generate the affine arrangement. Also attached to each arrangement is the corresponding Rev-Lex Index of the line arrangement, which is a notation borrowed from [1]. The Rev-Lex Index of an arrangement is the largest chirotope in its corresponding isomorphism class, with chirotopes represented as strings of signs whose related bases are ordered in *rev*erse *lex*icographical order, and the largest chirotope is the lexicographically maximal one, with $0 > + > -$.

Figure 5: 4 line segments

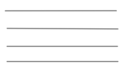
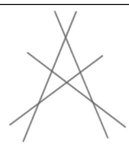
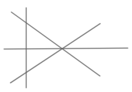
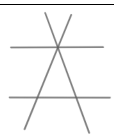
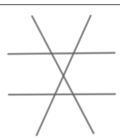
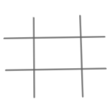
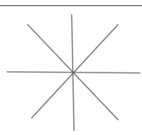
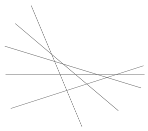
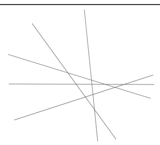
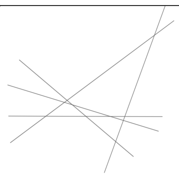
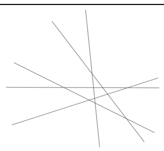
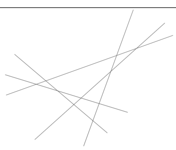
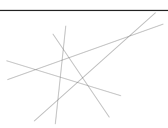
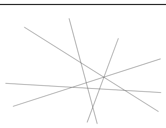
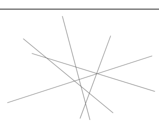
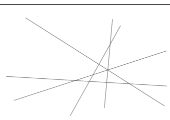
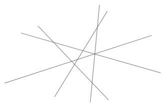
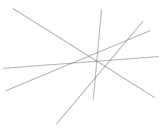
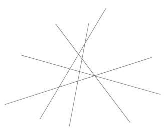
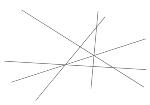
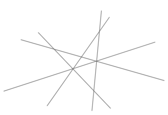
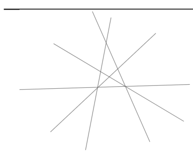
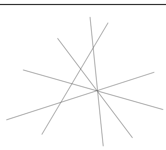
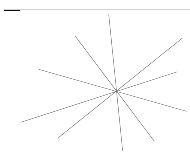
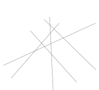
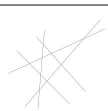
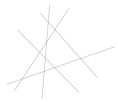
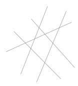
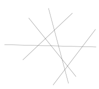
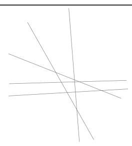
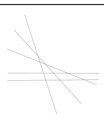
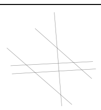
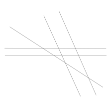
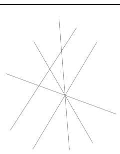
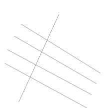
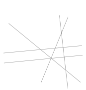
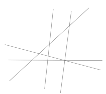
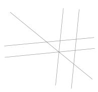
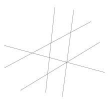
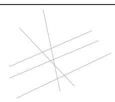
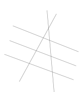
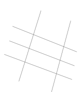
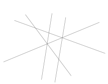
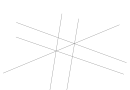
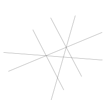
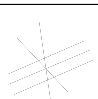
Images	**Symmetries**	**Before Symmetry**	**After Symmetry**	**Rev-Lex Index**
	4	1	1	0000000000
	2	390625	195625	++++++++++
	2	54925	27716	0+++++++++
	2	105625	52925	++++0+++++
	2	10985	5512	0++++++0++
	4	105625	26573	++++0+−+−−
	4	3125	810	0+++000+++
	8	28561	3666	++++0++++0
	16	625	52	0000++++++

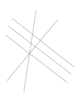
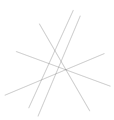
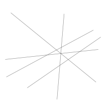
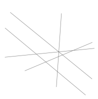
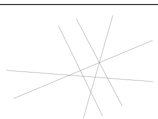
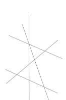
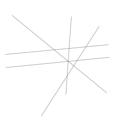
Figure 6: 5 line segments

Images	Symmetries	Before Symmetry	After Symmetry	Rev-Lex Index
	2	115856201	57932303	++++++++++++++++++++
	1	115856201	115856201	+++++++++++++++++++−
	2	115856201	57932303	++++++++++++++++−+++
	2	115856201	57932303	++++++++++++++++++−−
	2	115856201	57932303	++++++++++++++++−−++
	10	115856201	11589839	+++++++++++++−++−−++
	2	26265625	13134350	0++++++++++++++−+−−−
	1	26265625	26265625	0++++++++++++++−++−−
	1	26265625	26265625	0++++++++++++++++−−−

Images	Symmetries	Before Symmetry	After Symmetry	Rev-Lex Index
	1	26265625	26265625	0+++++++++++++++++−−
	2	26265625	13134350	0+++++++++++++++−−−+
	1	26265625	26265625	0+++++++++++++++++++
	2	5078126	2543125	0++++++0+++++++++−−−
	2	5078125	2540000	0++++++0++++++++++−−
	2	5078125	2540000	0++++++0+++++++++++−
	2	1171001	585923	0000++++++++++++++++
	20	3125	195	0000000000++++++++++
	1	43075625	43075625	+++++++++++0−−−−−−−+
	2	43075625	21540375	+++++++++++0+−++−−++

Images	Symmetries	Before Symmetry	After Symmetry	Rev-Lex Index
	1	43075625	43075625	+++++++++++0+−++−+++
	2	16015625	8009375	+++++++++++0+−0+−−++
	2	43075625	21540375	++++++++++0+−+−−+−−−
	1	43075625	43075625	++++++++++0++++++++−
	1	43075625	43075625	++++++++++0+++++++++
	2	16015625	8009375	++++++++++0+++++++0−
	2	16015625	8009375	++++++++++0++++0++−−
	2	274625	137566	0000++++++++++++0+++
	4	25625	6479	0000++++++000000++++
	1	8328125	8328125	0++++++++++++0++−−++

Images	Symmetries	Before Symmetry	After Symmetry	Rev-Lex Index
	1	8328125	8328125	0 + + + + + + + + + + + +0 + + − + + +
	2	2640625	1320800	0 + + + + + + + + + + + +0 + + − −0+
	1	2640625	2640625	0 + + + + + + + + + + + +0 + + − 0 + +
	2	3693157	1847132	0 + + + + + + + + + 000 + + − + + −−
	2	3693157	1847132	0 + + + + + + + + + 000 + + + + + ++
	4	1373125	344074	0 + + + + + + + + + 000 + + + + + +0
	1	1373125	1373125	0 + + + + + +0 + +$_{+}$ + + + +0 + + − −
	4	371293	93626	0 + + + + + +0 + + + + + + + 0 + 0 − −
	2	1373125	687050	0 + + + + + +0 + + + + + +0 − + − −−
	4	528125	132523	0 + + + + + +0 + +000 + + − + + −−

Images	Symmetries	Before Symmetry	After Symmetry	Rev-Lex Index
	2	528125	264175	$0++++++0++000++++++-$
	2	9765625	4885625	$0++++++++++++++++++0$
	4	9765625	2443450	$0+++++++++++++++---0$
	1	8328125	8328125	$0+++++++++++++0-----$
	1	8328125	8328125	$0+++++++++++++0-+---$
	1	8328125	8328125	$0++++++++++++0------$
	1	8328125	8328125	$0++++++++++++0++---+$

References

[1] Lukas Finschi, *Homepage of Oriented Matroids*, http://www.om.math.ethz.ch/, 2010.

[2] Simon Plouffe, *Approximations of generating functions and a few conjectures*, 1992. arXiv:0911.4975

[3] Neil James Alexander Sloane. *The On-Line Encyclopedia of Integer Sequences*, http://oeis.org. Sequences A001844, A001846.

[4] Nikolai Mnëv, *The universality theorems on the classification problem of configuration varieties and convex polytopes varities. Topology and geometry.* —Rohlin Seminar, 527–543, Lecture Notes in Math., 1346, Springer, Berlin, 1988.

[5] Shape Machine. https://shape.design.gatech.edu/Machine/index2.html

[6] Shape Machine Symposium. https://shape.design.gatech.edu/Symposium/index.html

May Cai
Georgia Institute of Technology
mayhcai@gmail.com

Nicholas Liao
Georgia Institute of Technology
nliao7@gatech.edu