[Rose-Hulman Undergraduate Mathematics Journal](https://scholar.rose-hulman.edu/rhumj)

Volume 20
Issue 1

Article 9

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Recommended Citation

Lepine, Dene (2019) "Monoidal Supercategories and Superadjunction," *Rose-Hulman Undergraduate Mathematics Journal*: Vol. 20 : Iss. 1 , Article 9.
Available at: https://scholar.rose-hulman.edu/rhumj/vol20/iss1/9

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Cover Page Footnote

Footnote: 2010 Mathematics Subject Classification. 18A40, 18A05. Acknowledgments: This document was written as an Undergraduate Honours Project at the University of Ottawa. The author would like to thank Professor Alistair Savage for his supervision and guidance throughout this project.

ℝose-Hulman Undergraduate Mathematics Journal
VOLUME 20, ISSUE 1, 2019

Monoidal Supercategories and Superadjunction

By *Dene Lepine*

Abstract. We define the notion of superadjunction in the context of supercategories. In particular, we give definitions in terms of counit-unit superadjunctions and hom-space superadjunctions, and prove that these two definitions are equivalent. These results generalize well-known statements in the non-super setting. In the super setting, they formalize some notions that have recently appeared in the literature. We conclude with a brief discussion of superadjunction in the language of string diagrams.

1 Introduction

Category theory is the study of all mathematical stuctures as a whole. In particular, category theorists study so-called 'categories'. These categories are characterized by their mathematical structures, or 'objects', and the maps, or 'morphisms', between said structures. Some examples include:

- the category of vector spaces which contains all vector spaces over some fixed field and the possible linear maps between these vector spaces,
- the category of groups and group homomorphisms between them, and
- the category of topological spaces and continuous maps.

With a bit of background in mathematics one can come up with countless other examples of categories.

The power of category theory can be observed while studying two objects which share some similarities but have such outstanding differences we cannot easily compare them. With category theory we can sieve through these differences and find the underlining commonalities between these objects.

In particular, category theorists compare categories by using 'functors'. Functors are maps between categories which preserve the structure of said categories. That is, they map objects to objects, map morphisms to morphisms, and preserve the composition of morphisms.

Mathematics Subject Classification. 18A40, 18A05
Keywords. supercategory, superadjunction, string diagrams

Once we have these functors we can explore how they might interact with each other, if at all. This brings us to *adjoint functors*. Like weeds, adjoint functors arise in seemingly unusual places. Unlike weeds, adjoint functors bring about alluring results in a far-reaching breadth of mathematics. First introduced in [4], adjoint functors have since been studied in a variety of settings. For basic definitions and background, we refer the reader to [6, Chap. 4, Sect. 1].

Adjoint functors abound in mathematics. Some examples include the following:

- The functor forming the free vector space on a set is left adjoint to the forgetful functor from vector spaces to sets.
- The functor forming the free group on a set is left adjoint to the forgetful functor from groups to sets.
- The functor which formally adjoins a multiplicative identity to a (non-unital) ring is left adjoint to the forgetful functor from unital rings to non-unital rings.
- If G is a finite group with subgroup H, then Frobenius reciprocity states that induction from the category of H-modules to the category of G-modules is both left and right adjoint to restriction in the other direction.
- The forgetful functor from topological spaces to sets is right adjoint to the functor endowing a set with the discrete topology, and is left adjoint to the functor endowing a set with the trivial topology.
- If X is an (R, S)-bimodule, for some rings R and S, then the functor that takes a right R-module, Y, to the R-tensor product $Y \otimes_R X$ is left adjoint to the functor which takes a right S-module, Z, to $\mathrm{Hom}_S(X, Z)$.
- The inclusion functor from compact Hausdorff spaces to topological spaces is right adjoint to Stone-Čech compactification.

The above examples illustrate the ubiquity of adjunction in mathematics. They can all be thought of as part of the "non-super" world. However, there has been a great deal of recent interest in "super mathematics". It is gradually becoming apparent that adjunction is a fundamental concept here as well. In particular, adjunction in the super setting plays a key role in the super analogues of rigid and pivotal categories that are fundamental in the field of categorification.

Super mathematics is originally motivated by physics and, in particular, superstring theory. For physicists the main goal of supermathematics is to study the behaviour of elementary particles called bosons and fermions. These particles interact within superspaces where bosons correspond to the even part of the vector space and fermions the odd part. Recently, super*categories* have appeared in the mathematical literature but lacked a coherent formalism, with various competing definitions. In [1] the authors formalized the notion of a supercategory and elaborated on related concepts.

The notion of superadjunction has also appeared implicitly in several papers. Examples include [2, (1.10), (1.18), (6.6)] which uses string diagrams (see Section 5) representing 2-morphisms in a 2-supercategory and [8, (6.7)] which makes use of the string diagrams representing morphisms in an additive $\Bbbk$-linear monoidal category. Other concrete examples will be described in Section 4. However, despite the existence of these examples, there does not seem to be a systematic treatment of the notion of superadjunction and superadjoint functors in the literature. The purpose of the current paper is to remedy this situation by formalizing the notion of adjunctions in this super setting, thereby filling this gap.

We begin in Section 2 by recalling the definitions of super vector spaces and supercategories, following [1]. In Section 3, which forms the core of the paper, we give two definitions of superadjuction: one in terms of a counit-unit adjunction, and the other in terms of a hom-space adjunction. We then show that these two definitions are, in fact, equivalent. In Section 4, we give a number of examples. In particular, we discuss induction and restriction functors for Frobenius superalgebras (see [7]). We conclude, in Section 5, with a brief discussion of adjunction in the formalism of string diagrams for monoidal supercategories.

Prerequisites

In general, this document should be accessible to readers with at least a background in undergraduate level abstract algebra and linear algebra. Readers would benefit from some experience in category theory but this is not required since all relevant definitions will be given.

2 Superspaces and Supercategories

In the following section we introduce the fundamental definitions needed to develop the definition of a *monoidal supercategory* and, in the next section, a *superadjunction* between superfunctors. For further information the reader may refer to [6, Chap. 1,2, & 4] for definitions and examples of the following outside of the super setting and to [1] for more on the standard definitions within the super setting. Throughout the following sections let $\Bbbk$ be some field of characteristic other than 2. We will also assume all tensor products are defined over $\Bbbk$, unless otherwise specified.

Definition 2.1 (Superspace)**.** A *superspace*, V, is a $\mathbb{Z}_2$-graded vector space over $\Bbbk$. That is, $\mathrm{V} = \mathrm{V}_{\bar{0}} \oplus \mathrm{V}_{\bar{1}}$. We say $v \in \mathrm{V}_\ell$, $\ell = \bar{0}, \bar{1}$, is *homogeneous* and define the *parity* of v to be $|v| = \ell$.

A linear map between superspaces V and W is said to be *even* (resp. *odd*) if it is parity preserving (resp. reversing). That is, T is even if $\mathrm{T}(\mathrm{V}_\ell) \subseteq \mathrm{W}_\ell$ and odd if $\mathrm{T}(\mathrm{V}_\ell) \subseteq \mathrm{W}_{\bar{1}-\ell}$ for $\ell = \bar{0}, \bar{1}$. Furthermore, this induces a super structure on the vector space Hom(V, W) with decomposition

$$\mathrm{Hom}(\mathrm{V},\mathrm{W}) = \mathrm{Hom}(\mathrm{V},\mathrm{W})_{\bar{0}} \oplus \mathrm{Hom}(\mathrm{V},\mathrm{W})_{\bar{1}},$$

where $\mathrm{Hom}(\mathrm{V},\mathrm{W})_{\bar{0}}$ is the vector space of all parity preserving linear maps and $\mathrm{Hom}(\mathrm{V},\mathrm{W})_{\bar{1}}$ is all parity reversing linear maps.

Let V and W be superspaces. We know that $\mathrm{V} \oplus \mathrm{W}$ is a vector space and, in fact, we can define a super structure on it by setting

$$(\mathrm{V} \oplus \mathrm{W})_\ell = \{(v, w) \in \mathrm{V} \oplus \mathrm{W} \mid v \in \mathrm{V}_\ell, w \in \mathrm{W}_\ell\},\ \ell = \bar{0}, \bar{1}.$$

Similarly, $\mathrm{V} \otimes \mathrm{W}$ is also a superspace with decomposition $(\mathrm{V} \otimes \mathrm{W})_{\bar{0}} = (\mathrm{V}_{\bar{0}} \otimes \mathrm{W}_{\bar{0}}) \oplus (\mathrm{V}_{\bar{1}} \otimes \mathrm{W}_{\bar{1}})$ and $(\mathrm{V} \otimes \mathrm{W})_{\bar{1}} = (\mathrm{V}_{\bar{1}} \otimes \mathrm{W}_{\bar{0}}) \oplus (\mathrm{V}_{\bar{0}} \otimes \mathrm{W}_{\bar{1}})$. In this setting, if f and g are homogeneous linear maps between superspaces, then their tensor product, denoted $f \otimes g$, is defined for all homogeneous $v \in \mathrm{V}$ and $w \in \mathrm{W}$ by

$$(f \otimes g)(v \otimes w) = (-1)^{|g||v|} f(v) \otimes g(w), \tag{1}$$

and then extended by linearity.

Definition 2.2 (Supercategory)**.** A *supercategory*, $\mathscr{C}$, consists of a class of objects, denoted $\mathrm{ob}(\mathscr{C})$, and a set of morphisms from a to b, denoted $\mathrm{Hom}_{\mathscr{C}}(a, b)$, for each pair $a, b \in \mathrm{ob}(\mathscr{C})$. Furthermore, each $\mathrm{Hom}_{\mathscr{C}}(a, b)$ is a superspace over $\Bbbk$ satisfying the following for any $a, b, c, d \in \mathrm{ob}(\mathscr{C})$:

- There is an even linear map, $\circ\colon \mathrm{Hom}(a, b) \otimes \mathrm{Hom}(b, c) \to \mathrm{Hom}(a, c)$, written as $g \circ f \in \mathrm{Hom}(a, c)$, where $f \in \mathrm{Hom}(a, b)$ and $g \in \mathrm{Hom}(b, c)$.
- $(h \circ g) \circ f = h \circ (g \circ f)$ for any $f \in \mathrm{Hom}(a, b)$, $g \in \mathrm{Hom}(b, c)$, and $h \in \mathrm{Hom}(c, d)$.
- For every $a \in \mathrm{ob}(\mathscr{C})$, there is an *identity morphism*, id_a, such that for every morphism $f\colon b \to a$ and $g\colon a \to c$ we have $\mathrm{id}_a \circ f = f$ and $g \circ \mathrm{id}_a = g$.

We denote the class of all morphisms between any two pairs of objects by $\mathrm{Hom}(\mathscr{C})$.

Example 2.3 (Category of Superspaces)**.** We define $\mathscr{SV}ec$ to be the category of all superspaces over $\Bbbk$ and all linear maps, including even, odd, and non-homogeneous maps. Similarly, we use $\mathscr{SV}ec_{fd}$ to denote the category of finite dimensional superspaces.

Take any $\mathrm{V}, \mathrm{U}, \mathrm{W} \in \mathrm{ob}(\mathscr{SV}ec)$, homogeneous $f \in \mathrm{Hom}_{\mathscr{SV}ec}(\mathrm{V}, \mathrm{U})$, and

$$g \in \mathrm{Hom}_{\mathscr{SV}ec}(\mathrm{U}, \mathrm{W}).$$

Observe that the standard composition of linear maps is such that

$$|g \circ f| = |g| + |f| = |f \otimes g|.$$

That is, $\circ$ is an even linear map. In particular, we have that $\mathscr{SV}ec$ is a supercategory.

Definition 2.4 (Superfunctor)**.** Let $\mathscr{C}$ and $\mathscr{D}$ be supercategories. A *covariant superfunctor* from $\mathscr{C}$ to $\mathscr{D}$ is a pair of maps $\mathrm{ob}(\mathscr{C}) \to \mathrm{ob}(\mathscr{D})$ and $\mathrm{Hom}(\mathscr{C}) \to \mathrm{Hom}(\mathscr{D})$. We denote both of these maps by F and write $\mathrm{F}\colon \mathscr{C} \to \mathscr{D}$.

Furthermore, we require that $\mathrm{F}(f) \in \mathrm{Hom}_{\mathscr{D}}(\mathrm{F}(\mathrm{X}), \mathrm{F}(\mathrm{Y}))$ for all $f \in \mathrm{Hom}_{\mathscr{C}}(\mathrm{X}, \mathrm{Y})$ and the following conditions hold:

- $\mathrm{F}(\mathrm{id}_{\mathrm{X}}) = \mathrm{id}_{\mathrm{F}(\mathrm{X})}$ for all $\mathrm{X} \in \mathrm{ob}(\mathbb{C})$,
- $\mathrm{F}\colon\ \mathrm{Hom}_{\mathscr{D}}(\mathrm{X}, \mathrm{Y}) \to \mathrm{Hom}_{\mathscr{D}}(\mathrm{F}(\mathrm{X}), \mathrm{F}(\mathrm{Y}))$ is an even map, and
- $\mathrm{F}(g \circ f) = \mathrm{F}(g) \circ \mathrm{F}(f)$ for $f \in \mathrm{Hom}_{\mathscr{C}}(\mathrm{X}, \mathrm{Y})$ and $g \in \mathrm{Hom}_{\mathscr{C}}(\mathrm{Y}, \mathrm{Z})$.

Throughout this paper when we say F is a superfunctor we mean that it is a covariant superfunctor.

Definition 2.5 (Supernatural Transformation)**.** Let $\mathscr{C}$ and $\mathscr{D}$ be supercategories and $\mathrm{F}, \mathrm{G}\colon \mathscr{C} \to \mathscr{D}$ superfunctors. A *supernatural transformation*, η, from F to G is a family of morphisms such that:

- To every $\mathrm{X} \in \mathrm{ob}(\mathscr{C})$ there is an associated morphism

  $$\eta_{\mathrm{X}} = \eta_{\mathrm{X},\bar{0}} + \eta_{\mathrm{X},\bar{1}} \in \mathrm{Hom}_{\mathscr{D}}(\mathrm{F}(\mathrm{X}), \mathrm{G}(\mathrm{X})),$$

  called the *component* of η at X.

- For every homogeneous $f \in \mathrm{Hom}_{\mathscr{C}}(\mathrm{X}, \mathrm{Y})$ we have the following commutative diagram for any $\ell \in \mathbb{Z}_2$.

$$\begin{tikzcd} \mathrm{F}(\mathrm{X}) \arrow[r, "\mathrm{F}(f)"] \arrow[d, "\eta_{\mathrm{X},\ell}"] & \mathrm{F}(\mathrm{Y}) \arrow[d, "\eta_{\mathrm{Y},\ell}"] \\ \mathrm{G}(\mathrm{X}) \arrow[r, "(-1)^{\ell|f|}\mathrm{G}(f)"] & \mathrm{G}(\mathrm{Y}) \end{tikzcd} \tag{2}$$

We say that $\eta_{\mathrm{X}}\colon \mathrm{F}(\mathrm{X}) \to \mathrm{G}(\mathrm{X})$ is *supernatural* in X and write $\eta\colon \mathrm{F} \Rightarrow \mathrm{G}$. Furthermore, a supernatural transformation η is said to be *even* if $\eta_{\mathrm{X}} = \eta_{\mathrm{X},\bar{0}}$ for all $\mathrm{X} \in \mathrm{ob}(\mathscr{C})$ and *odd* if $\eta_{\mathrm{X}} = \eta_{\mathrm{X},\bar{1}}$ for all $\mathrm{X} \in \mathrm{ob}(\mathscr{C})$.

Definition 2.6 (Equivalence of Supercategories)**.** We say that two supercategories, $\mathscr{C}$ and $\mathscr{D}$, are *equivalent* if there are superfunctors $\mathrm{F}\colon \mathscr{C} \to \mathscr{D}$ and $\mathrm{G}\colon \mathscr{D} \to \mathscr{C}$ and even supernatural isomorphisms $\varphi\colon \mathrm{FG} \to \mathrm{id}_{\mathscr{D}}$ and $\psi\colon \mathrm{id}_{\mathscr{C}} \to \mathrm{GF}$.

Definition 2.7 (Contravariant superfunctor)**.** A *contravariant superfunctor* F from $\mathscr{C}$ to $\mathscr{D}$ is a mapping such that:

- $\mathrm{F}(\mathrm{X}) \in \mathrm{ob}(\mathscr{D})$ for every $\mathrm{X} \in \mathrm{ob}(\mathscr{C})$ and

- $F(f) \in \mathrm{Hom}_{\mathscr{D}}(F(Y), F(X))$ for all $f \in \mathrm{Hom}_{\mathscr{C}}(X, Y)$ and the following conditions hold:
  - $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ for all $X \in \mathrm{ob}(\mathscr{C})$,
  - $F\colon \mathrm{Hom}_{\mathscr{D}}(X, Y) \to \mathrm{Hom}_{\mathscr{D}}(F(Y), F(X))$ is an even map, and
  - $F(g \circ f) = (-1)^{|f||g|} F(f) \circ F(g)$ for any homogeneous $f \in \mathrm{Hom}_{\mathscr{C}}(X, Y)$ and $g \in \mathrm{Hom}_{\mathscr{C}}(Y, Z)$.

Definition 2.8 (Product Category of Supercategories)**.** The *product category* of two supercategories, $\mathscr{C}$ and $\mathscr{D}$, is the category $\mathscr{C} \times \mathscr{D}$ defined as follows:

- $\mathrm{ob}(\mathscr{C} \times \mathscr{D})$ is the class of all pairs (X, Y), where $X \in \mathrm{ob}(\mathscr{C})$ and $Y \in \mathrm{ob}(\mathscr{D})$.
- $\mathrm{Hom}_{\mathscr{C} \times \mathscr{D}}((X, Y), (X', Y'))$ is the set of all pairs (f, g), where $f \in \mathrm{Hom}_{\mathscr{C}}(X, X')$ and $g \in \mathrm{Hom}_{\mathscr{D}}(Y, Y')$. Homogeneous elements are (f, g), where both f and g are homogeneous, with parity $|f| + |g|$.
- Composition is defined, for any homogeneous $(f, g) \in \mathrm{Hom}_{\mathscr{C} \times \mathscr{D}}((X, Y), (X', Y'))$ and $(f', g') \in \mathrm{Hom}_{\mathscr{C} \times \mathscr{D}}((X', Y'), (X'', Y''))$, to be

$$(f', g') \circ (f, g) = (-1)^{|f||g'|}(f' \circ f, g' \circ g). \tag{3}$$

Definition 2.9 (Superbifunctor)**.** Let $\mathscr{C}$, $\mathscr{D}$, and $\mathscr{C}'$ be supercategories. A covariant functor $F\colon \mathscr{C} \times \mathscr{D} \to \mathscr{C}'$ is called a *superbifunctor*.

Let $F\colon \mathscr{C} \times \mathscr{D} \to \mathscr{C}'$ be some superbifunctor. Notice that for each $X \in \mathrm{ob}(\mathscr{C})$ we can define $F(X, -)\colon \mathscr{D} \to \mathscr{C}'$ by $F(X, -)(Y) = F(X, Y)$ for all $Y \in \mathrm{ob}(\mathscr{D})$ and $F(X, -) = F(\mathrm{id}_X, f)$, for all $f \in \mathrm{Hom}(\mathscr{D})$. Similarly, can define $F(-, Y)\colon \mathscr{C} \to \mathscr{C}'$ in an analogous manner for any $Y \in \mathrm{ob}(\mathscr{D})$. Given these definitions one can show that these define covariant superfunctors.

Definition 2.10 (Opposite Supercategory)**.** If $\mathscr{C}$ is a supercategory, we define the opposite category, $\mathscr{C}^{\mathrm{op}}$, of $\mathscr{C}$ to be the supercategory whose objects are $\mathrm{ob}(\mathscr{C}^{\mathrm{op}}) = \mathrm{ob}(\mathscr{C})$ and morphisms are

$$\mathrm{Hom}(\mathscr{C}^{\mathrm{op}}) = \{f\colon Y \to X \mid f \in \mathrm{Hom}_{\mathscr{C}}(X, Y)\}.$$

Composition, $\circ_{\mathrm{op}}$, in $\mathscr{C}^{\mathrm{op}}$ is defined for all homogeneous $f \in \mathrm{Hom}_{\mathscr{C}}(Y, X)$ and $g \in \mathrm{Hom}_{\mathscr{C}}(Z, Y)$ by $f \circ_{\mathrm{op}} g = (-1)^{|g||f|}(g \circ f)$, where $\circ$ is composition in $\mathscr{C}$.

Remark 2.11. Notice that a contravariant superfunctor from $\mathscr{C}$ to $\mathscr{D}$ is exactly a covariant superfunctor from $\mathscr{C}^{\mathrm{op}}$ to $\mathscr{D}$.

Example 2.12 (Hom Functor)**.** Let $\mathscr{C}$ be a supercategory and define

$$\mathrm{Hom}_{\mathscr{C}}(-, -)\colon \mathscr{C} \times \mathscr{C} \to \mathscr{SV}ec$$

by

- for any $(X,Y) \in \mathscr{C} \times \mathscr{C}$ then $\mathrm{Hom}_{\mathscr{C}}(-,-)(X,Y) = \mathrm{Hom}_{\mathscr{C}}(X,Y)$ and
- for homogeneous $(f,g) \in \mathrm{Hom}_{\mathscr{C}\times\mathscr{C}}((X,Y),(X',Y'))$ define
$$\mathrm{Hom}_{\mathscr{C}}(f,g)\colon \mathrm{Hom}_{\mathscr{C}}(X',Y) \to \mathrm{Hom}_{\mathscr{C}}(X,Y')$$
for all homogeneous $h \in \mathrm{Hom}_{\mathscr{C}}(X',Y)$ by
$$\mathrm{Hom}_{\mathscr{C}}(f,g)(h) = (-1)^{|f|(|h|+|g|)} g \circ h \circ f. \tag{4}$$

Notice that for homogeneous $(f,g) \in \mathrm{Hom}(\mathscr{C}\times\mathscr{C})$ we have $|\mathrm{Hom}_{\mathscr{C}\times\mathscr{C}}(f,g)| = |(f,g)|$. Furthermore, if we fix the first argument of Hom to be $X \in \mathrm{ob}(\mathscr{C})$ it can be easily shown that $\mathrm{Hom}_{\mathscr{C}}(X,-)\colon \mathscr{C} \to \mathscr{SV}ec$ is a covariant superfunctor. Similarly, if the second argument is fixed then it can be shown that $\mathrm{Hom}_{\mathscr{C}}(-,X)\colon \mathscr{C} \to \mathscr{SV}ec$ is a contravariant superfunctor. That is, for $f \in \mathrm{Hom}_{\mathscr{C}}(Y,Y')$ then we have $\mathrm{Hom}_{\mathscr{C}}(f,X)\colon \mathrm{Hom}_{\mathscr{C}}(Y',X) \to \mathrm{Hom}_{\mathscr{C}}(Y,X)$. Since $\mathrm{Hom}_{\mathscr{C}}(-,-)$ is contravariant in the second argument it is not a superbifunctor. As a consequence, we often instead write $\mathrm{Hom}_{\mathscr{C}}(-,-)\colon \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathscr{SV}ec$ so that $\mathrm{Hom}_{\mathscr{C}}(-,-)$ is covariant in both arguments and therefore a superbifunctor.

Definition 2.13 (Strict Monoidal Supercategory)**.** A *strict monoidal supercategory* is a supercategory, $\mathscr{C}$, with a superbifunctor $\otimes\colon \mathscr{C}\times\mathscr{C} \to \mathscr{C}$ such that

- there is some object $I \in \mathrm{ob}(\mathscr{C})$ such that $I \otimes X = X = X \otimes I$ for all $X \in \mathrm{ob}(\mathscr{C})$ and
- $(X \otimes Y) \otimes Z = X \otimes (Y \otimes Z)$ for all $X,Y,Z \in \mathrm{ob}(\mathscr{C})$.

Furthermore, it follows from the fact that $\otimes$ is a superbifunctor that the composition of morphisms in this category is such that
$$(f \otimes g) \circ (h \otimes k) = (-1)^{|g||h|}(f \circ h) \otimes (g \circ k), \tag{5}$$
for homogeneous $f,g,h,k \in \mathrm{Hom}(\mathscr{C})$. We call this the *super interchange law.*

One can also define *monoidal supercategories.* Monoidal supercategories generalize the notion of a strict monoidal supercategory. In this setting, the tensor product is associative up to isomorphism, there is an object that acts as a multiplicative unit (up to isomorphism) with respect to the tensor product, and one imposes some coherence conditions on these isomorphisms. For more details in the non super setting see [6, Ch. 7, §1].

Example 2.14 (Category of Superspaces)**.** Recall the category, $\mathscr{SV}ec$, of all superspaces over $\Bbbk$. We have already seen that $\mathscr{SV}ec$ is a supercategory. Notice that for any homogeneous linear maps $f\colon V \to W$, $h\colon U \to V$, $g\colon Y \to Z$ and $k\colon X \to Y$, where $U,V,W,X,Y,Z \in \mathrm{ob}(\mathscr{SV}ec)$, we have
$$(f \otimes g) \circ (h \otimes k) = (-1)^{|g||h|}(f \circ h) \otimes (g \otimes k).$$
In particular, $\otimes$ can be shown to be a superbifunctor. Indeed, $\mathscr{SV}ec$ is a monoidal supercategory where $\Bbbk$ is the unit object.

3 Superadjunction

In this section we define counit-unit adjunction and Hom-set adjunction in the super setting. In the non-super setting it is well-known that these definitions are equivalent, and we will show that this is also the case in the super setting. Throughout this section let $\mathscr{C}$ and $\mathscr{D}$ be supercategories.

Definition 3.1 (Counit-unit Superadjunction)**.** Let $F\colon \mathscr{D} \to \mathscr{C}$ and $G\colon \mathscr{C} \to \mathscr{D}$ be superfunctors and (ε, η) be a pair of supernatural transformations, where $\varepsilon\colon FG \to \mathrm{id}_{\mathscr{C}}$ and $\eta\colon \mathrm{id}_{\mathscr{D}} \to GF$. We say (ε, η) are an *even* (resp. *odd*) *counit-unit superadjunction* if both ε and η are even (resp. odd) transformations and, if σ denotes the parity of ε and η, the following diagrams commute:

$$\begin{tikzcd} F \arrow[r, "F\eta"] \arrow[rr, bend right, "\mathrm{id}"'] & FGF \arrow[r, "\varepsilon F"] & F \end{tikzcd} \qquad \begin{tikzcd} G \arrow[r, "\eta G"] \arrow[rr, bend right, "(-1)^{\sigma}\mathrm{id}"'] & GFG \arrow[r, "G\varepsilon"] & G \end{tikzcd}$$

That is, for all $X \in \mathrm{ob}(\mathscr{C})$ and $Y \in \mathrm{ob}(\mathscr{D})$,

$$\mathrm{id}_{FY} = \varepsilon_{FY} \circ F(\eta_Y) \text{ and } (-1)^{\sigma}\mathrm{id}_{GX} = G(\varepsilon_X) \circ \eta_{GX}. \tag{6}$$

Here we say that F is *even* (resp. *odd*) *left adjoint* to G, G is *even* (resp. *odd*) *right adjoint* to F, and write $F \overset{\sigma}{\dashv} G$, where σ is the parity of the adjunction.

Definition 3.2 (Hom-space Superadjunction)**.** Let $F\colon \mathscr{D} \to \mathscr{C}$ and $G\colon \mathscr{C} \to \mathscr{D}$ be superfunctors and

$$\Phi\colon\ \mathrm{Hom}_{\mathscr{C}}(F-,-) \to \mathrm{Hom}_{\mathscr{D}}(-,G-)$$

be a supernatural isomorphism. We say Φ is an *even* (resp. *odd*) *hom-space superadjunction* between F and G if Φ is even (resp. odd), with parity denoted by σ, and the following diagram commutes for all homogeneous $f \in \mathrm{Hom}_{\mathscr{D}}(B, Y)$ and $g \in \mathrm{Hom}_{\mathscr{C}}(X, A)$:

$$\begin{tikzcd} \mathrm{Hom}_{\mathscr{C}}(FY, X) \arrow[r, "\mathrm{Hom}_{\mathscr{C}}(Ff, g)"] \arrow[d, "\Phi_{Y,X}"] & \mathrm{Hom}_{\mathscr{C}}(FB, A) \arrow[d, "\Phi_{B,A}"] \\ \mathrm{Hom}_{\mathscr{D}}(Y, GX) \arrow[r, "(-1)^{\sigma|f|}\mathrm{Hom}_{\mathscr{D}}(f, Gg)"] & \mathrm{Hom}_{\mathscr{D}}(B, GA) \end{tikzcd} \tag{7}$$

As we will see, hom-space superadjunction and counit-unit superadjunction are equivalent. Therefore, in either case we can say that F is *even* (resp. *odd*) *left adjoint* to G, G is *even* (resp. *odd*) *right adjoint* to F, and write $F \overset{\sigma}{\dashv} G$, where σ is the parity of the adjunction.

Proposition 3.3. *Let* $F: \mathscr{D} \to \mathscr{C}$ *and* $G: \mathscr{C} \to \mathscr{D}$ *be superfunctors between supercategories. Then there is an even (resp. odd) counit-unit adjunction from* F *to* G *if and only if there is an even (resp. odd) hom-space adjunction from* F *to* G.

Proof. Suppose that (ε, η) is a counit-unit adjunction from F to G of parity σ. We define $\Phi_{Y,X}: \mathrm{Hom}_{\mathscr{C}}(FY, X) \to \mathrm{Hom}_{\mathscr{D}}(Y, GX)$ and $\Psi_{Y,X}: \mathrm{Hom}_{\mathscr{D}}(Y, GX) \to \mathrm{Hom}_{\mathscr{C}}(FY, X)$, for any $X \in \mathrm{ob}(\mathscr{C})$ and $Y \in \mathrm{ob}(\mathscr{D})$, by

$$\Phi_{Y,X}(f) = G(f) \circ \eta_Y \text{ and } \Psi_{Y,X}(g) = (-1)^{\sigma(|g|+1)} \varepsilon_X \circ F(g),$$

where $f: FY \to X$ and $g: Y \to GX$ are homogeneous. Observe that Φ and Ψ have parity σ. Furthermore, notice the following for any homogeneous $f: FY \to X$:

$$\begin{aligned}
\Psi_{Y,X}\Phi_{Y,X}(f) &= (-1)^{\sigma(|\Phi_{Y,X}(f)|+1)} \varepsilon_X \circ F(G(f) \circ \eta_Y) \\
&= (-1)^{\sigma|f|} \varepsilon_X \circ FG(f) \circ F(\eta_Y) && \text{(Since F is a functor)} \\
&= f \circ \varepsilon_{FY} \circ F(\eta_Y) && \text{(By naturality of } \varepsilon) \\
&= f \circ \mathrm{id}_{FY} && \text{(By (6))} \\
&= f
\end{aligned}$$

and

$$\begin{aligned}
\Phi_{Y,X}\Psi_{Y,X}(g) &= (-1)^{\sigma(|g|+1)} G(\varepsilon_X \circ F(g)) \circ \eta_Y \\
&= (-1)^{\sigma(|g|+1)} G(\varepsilon_X) \circ GF(g) \circ \eta_Y && \text{(Since G is a functor)} \\
&= (-1)^{\sigma} G(\varepsilon_X) \circ \eta_{GX} \circ g && \text{(By naturality of } \eta) \\
&= \mathrm{id}_{GX} \circ g && \text{(By (6))} \\
&= g.
\end{aligned}$$

By extending linearly we have that $\Psi_{Y,X} = \Phi_{Y,X}^{-1}$ and therefore $\Phi_{Y,X}$ is a bijection.

We need to show that (7) commutes for $\Phi_{Y,X}$ and $\Psi_{Y,X}$, where $Y \in \mathscr{D}$ and $X \in \mathscr{C}$. Observe that for any $g \in \mathrm{Hom}_{\mathscr{C}}(X, A)$ and $f \in \mathrm{Hom}_{\mathscr{C}}(FY, X)$ we have the following

$$\begin{aligned}
\Phi_{Y,A} \circ \mathrm{Hom}_{\mathscr{C}}(FY, g)(f) &= \Phi_{Y,A}(g \circ f) && \text{(By (4) on } \mathrm{Hom}_{\mathscr{C}}(FY, g)) \\
&= G(g \circ f) \circ \eta_Y \\
&= Gg \circ Gf \circ \eta_Y && \text{(Since G is a functor)} \\
&= \mathrm{Hom}_{\mathscr{D}}(Y, Gg)(Gf \circ \eta_y) && \text{(By (4) on } \mathrm{Hom}_{\mathscr{D}}(Y, Gg)) \\
&= \mathrm{Hom}_{\mathscr{D}}(Y, Gg) \circ \Phi_{Y,X}(f).
\end{aligned}$$

Furthermore, fix any $X \in \mathrm{ob}(\mathscr{C})$ and notice that we have the following for homogeneous $f \in \mathrm{Hom}_{\mathscr{D}}(B, Y)$ and $g \in \mathrm{Hom}_{\mathscr{C}}(FY, X)$:

$$\Phi_{Y,X} \circ \mathrm{Hom}_{\mathscr{C}}(Ff, X)(g) = \Phi_{Y,X}(g \circ Ff) \qquad \text{(By (4) on } \mathrm{Hom}_{\mathscr{C}}(Ff, X))$$

$$
\begin{aligned}
&= G(g \circ Ff) \circ \eta_Y \\
&= Gg \circ GFf \circ \eta_Y && \text{(Since G is a functor)} \\
&= (-1)^{\sigma|f|} Gg \circ \eta_B \circ f && \text{(By (2))} \\
&= (-1)^{\sigma|f|} \Phi_{B,X}(g) \circ f \\
&= (-1)^{\sigma|f|} \operatorname{Hom}_{\mathscr{D}}(f, GX) \circ \Phi_{B,X}(g). && \text{(By (4) on } \operatorname{Hom}_{\mathscr{D}}(g, GX))
\end{aligned}
$$

Thus, the diagram (7) commutes. Moreover, since (7) commutes for Φ, it is a similar exercise to show that (7) also commutes for Ψ.

Therefore, Φ is a hom-space superadjunction from F to G or parity σ.

Now suppose that $\Phi\colon \operatorname{Hom}_{\mathscr{C}}(F-,-) \to \operatorname{Hom}_{\mathscr{D}}(-,G-)$ is a hom-space superadjunction from F to G, with parity σ. For all $X \in \operatorname{ob}(\mathscr{C})$ and $Y \in \operatorname{ob}(\mathscr{D})$ we claim that

$$\varepsilon_X = (-1)^{\sigma} \Phi^{-1}_{GX,X}(\mathrm{id}_{GX}) \text{ and } \eta_Y = \Phi_{Y,FY}(\mathrm{id}_{FY})$$

define a counit-unit superadjunction from F to G. Since the identity is always an even homomorphism we have that $|\eta| = |\varepsilon| = \sigma$. Furthermore, we know that Φ makes (7) commute and therefore if we set $X = FY$, $B = Y$, $A = X$, and $f = \mathrm{id}_Y$ then for any $g\colon FY \to X$ we have the following:

$$
\begin{aligned}
\Phi_{Y,X} \circ \operatorname{Hom}_{\mathscr{C}}(F\mathrm{id}_Y, g)(\mathrm{id}_{FY}) &= (-1)^{\sigma|\mathrm{id}_Y|} \operatorname{Hom}_{\mathscr{D}}(\mathrm{id}_Y, Gg) \circ \Phi_{Y,FY}(\mathrm{id}_{FY}) \\
\Phi_{Y,X}(g \circ \mathrm{id}_{FY} \circ F\mathrm{id}_Y) &= Gg \circ \Phi_{Y,FY}(\mathrm{id}_{FY}) \circ \mathrm{id}_Y && \text{(By (4))} \\
\Phi_{Y,X}(g) &= Gg \circ \Phi_{Y,FY}(\mathrm{id}_{FY}).
\end{aligned}
$$

Similarly, for any $f\colon Y \to GX$ we have the following:

$$
\begin{aligned}
(-1)^{\sigma|f|} \operatorname{Hom}_{\mathscr{C}}(Ff, \mathrm{id}_X) \circ \Phi^{-1}_{GX,X}(\mathrm{id}_{GX}) &= \Phi^{-1}_{Y,X} \circ \operatorname{Hom}_{\mathscr{D}}(f, G\mathrm{id}_X)(\mathrm{id}_{GX}) \\
(-1)^{\sigma|f|} \mathrm{id}_X \circ \Phi^{-1}_{GX,X}(\mathrm{id}_{GX}) \circ Ff &= \Phi^{-1}_{Y,X}(G\mathrm{id}_X \circ \mathrm{id}_{GX} \circ f) && \text{(By (4))} \\
(-1)^{\sigma|f|} \Phi^{-1}_{GX,X}(\mathrm{id}_{GX}) \circ Ff &= \Phi^{-1}_{Y,X}(f).
\end{aligned}
$$

That is, for any $g\colon Y \to GX$ and $f\colon FY \to X$ we have the following two equations:

$$\Phi_{Y,X}(f) = Gf \circ \eta_Y \tag{8}$$

$$\Phi^{-1}_{Y,X}(g) = (-1)^{\sigma(1+|g|)} \varepsilon_X \circ Fg \tag{9}$$

Knowing this and the fact that both $|\eta| = \sigma$ and $\sigma^2 = \sigma$, we consider the following:

$$
\begin{aligned}
\varepsilon_{FY} \circ F(\eta_Y) &= (-1)^{\sigma(|\eta|+1)} \varepsilon_{FY} \circ F(\eta_Y) \\
&= \Phi^{-1}_{Y,X}(\eta_Y) \\
&= \Phi^{-1}_{Y,FY}(\Phi_{Y,FY}(\mathrm{id}_{FY})) && \text{(By (9))}
\end{aligned}
$$

$$= \mathrm{id}_{\mathrm{FY}}$$

and

$$\begin{aligned}
\mathrm{G}(\varepsilon_{\mathrm{X}}) \circ \eta_{\mathrm{GX}} &= \Phi_{\mathrm{Y,X}}(\eta_{\mathrm{X}}) \\
&= (-1)^{\sigma} \Phi_{\mathrm{GX,X}}(\Phi^{-1}_{\mathrm{GX,X}}(\mathrm{id}_{\mathrm{GX}})) && \text{(By (8))} \\
&= (-1)^{\sigma} \mathrm{id}_{\mathrm{GX}}.
\end{aligned}$$

Thus, the proposed ε and η satisfy (6).

Lastly, we show that both η and ε are supernatural transformations. Take any homogeneous $g \in \mathrm{Hom}_{\mathscr{C}}(\mathrm{X}, \mathrm{Y})$ and consider the following:

$$\begin{aligned}
\varepsilon_{\mathrm{Y}} \circ \mathrm{FG}(g) &= (-1)^{\sigma} \Phi^{-1}_{\mathrm{GY,Y}}(\mathrm{id}_{\mathrm{GY}}) \circ \mathrm{FG}(g) \\
&= (-1)^{\sigma} \mathrm{Hom}_{\mathscr{C}}(\mathrm{FG}g, \mathrm{Y}) \circ \Phi^{-1}_{\mathrm{GY,Y}}(\mathrm{id}_{\mathrm{GY}}) && \text{(By (4))} \\
&= (-1)^{\sigma(1+|g|)} \Phi^{-1}_{\mathrm{GX,Y}} \circ \mathrm{Hom}_{\mathscr{D}}(\mathrm{G}g, \mathrm{GY})(\mathrm{id}_{\mathrm{GY}}) && \text{(By (7))} \\
&= (-1)^{\sigma(1+|g|)} \Phi^{-1}_{\mathrm{GX,Y}}(\mathrm{G}g) \\
&= (-1)^{\sigma(1+|g|)} \Phi^{-1}_{\mathrm{GX,Y}} \circ \mathrm{Hom}_{\mathscr{D}}(\mathrm{GX}, \mathrm{G}g)(\mathrm{id}_{\mathrm{GX}}) && \text{(By (4))} \\
&= (-1)^{\sigma(1+|g|)} \mathrm{Hom}_{\mathscr{D}}(\mathrm{FGX}, g) \circ \Phi^{-1}_{\mathrm{GX,X}}(\mathrm{id}_{\mathrm{GX}}) && \text{(By (7))} \\
&= (-1)^{\sigma(1+|g|)} g \circ \Phi^{-1}_{\mathrm{GX,X}}(\mathrm{id}_{\mathrm{GX}}) = (-1)^{\sigma|g|} g \circ \varepsilon_{\mathrm{X}}. && \text{(By (4))}
\end{aligned}$$

By an analogous argument, for any homogeneous $f \in \mathrm{Hom}_{\mathscr{D}}(\mathrm{X}, \mathrm{Y})$ we have that:

$$\mathrm{GF}(f) \circ \eta_{\mathrm{X}} = (-1)^{\sigma|f|} \eta_{\mathrm{Y}} \circ f.$$

Thus, both η and ε are supernatural transformations. Moreover, η and ε are, in fact, a counit-unit superadjunction from F to G of parity σ. □

Given this equivalence of counit-unit superadjunction and hom-space superadjunction, the general notation $\mathrm{F} \overset{\sigma}{\dashv} \mathrm{G}$ is, in fact, unambiguous.

The counit-unit formulation of superadjunction extends to the setting of monoidal supercategories in the following way. Suppose $\mathscr{M}$ is a monoidal supercategory and F is some object in $\mathscr{M}$. We can think of F as a superfunctor by defining $\mathrm{F} \otimes - : \mathscr{M} \to \mathscr{M}$ for any $\mathrm{G}, \mathrm{M}, \mathrm{N} \in \mathrm{ob}(\mathscr{M})$ and $f \in \mathrm{Hom}_{\mathscr{M}}(\mathrm{M}, \mathrm{N})$ by

$$(\mathrm{F} \otimes -)(\mathrm{G}) = \mathrm{F} \otimes \mathrm{G} \text{ and } (\mathrm{F} \otimes -)(f) = \mathrm{id}_{\mathrm{F}} \otimes f.$$

Indeed, it is straightforward to check that $\mathrm{F} \otimes -$ is a superfunctor. Moreover, by thinking of $\mathrm{F}, \mathrm{G} \in \mathrm{ob}(\mathscr{M})$ as superfunctors we say that F is even (resp. odd) left adjoint to G if there exist even (resp. odd) morphisms $\varepsilon : \mathrm{F} \otimes \mathrm{G} \to \mathrm{I}$ and $\eta : \mathrm{I} \to \mathrm{G} \otimes \mathrm{F}$ such that the diagrams in Definition 3.1 commute.

4 Examples

Now that we have formulated the definition of a superadjunction between two functors it is important to explore some examples. Here we give the basic definitions needed and then go through examples in detail. For other examples see [8, (6.7)], [1] and [2, (1.10),(1.18)].

Definition 4.1 (Superalgebra)**.** An *associative superalgebra*, $\mathscr{A}$, is a superspace equipped with a multiplication map $\cdot\colon \mathscr{A}\times\mathscr{A}\to\mathscr{A}$ such that for all $\ell, k\in\mathbb{Z}_2$,

$$\mathscr{A}_\ell\cdot\mathscr{A}_k\subseteq\mathscr{A}_{\ell+k}.$$

We say that $\mathscr{A}$ is *unital* if there exists a multiplicative unit in $\mathscr{A}$.

Definition 4.2 ($\mathscr{A}$-Supermodule)**.** Let $\mathscr{A}$ be a unital superalgebra over $\Bbbk$. A superspace, M, over $\Bbbk$ is a *left $\mathscr{A}$-supermodule* if it is equipped with $\cdot\colon \mathscr{A}\times \mathrm{M}\to\mathrm{M}$ such that:

- $\mathscr{A}_i\cdot\mathrm{M}_j\subseteq\mathrm{M}_{i+j}$ for each $i,j\in\mathbb{Z}_2$,
- $1\cdot m=m$ for all $m\in\mathrm{M}$, and
- $(ab)\cdot m=a\cdot(b\cdot m)$ for all $a,b\in\mathscr{A}$ and $m\in\mathrm{M}$.

Similarly, a supervector space, M, over $\Bbbk$ is a *right $\mathscr{A}$-supermodule* if it is equipped with $\cdot\colon \mathrm{M}\times\mathscr{A}\to\mathrm{M}$ such that:

- $\mathrm{M}_j\cdot\mathscr{A}_i\subseteq\mathrm{M}_{i+j}$ for each $i,j\in\mathbb{Z}_2$,
- $m\cdot 1=m$ for all $m\in\mathrm{M}$, and
- $m\cdot(ab)=(m\cdot a)\cdot b$ for all $a,b\in\mathscr{A}$ and $m\in\mathrm{M}$.

We write ${}_{\mathscr{A}}\mathrm{M}$ or $\mathrm{M}_{\mathscr{A}}$ to emphasize that M is a left $\mathscr{A}$-supermodule or right $\mathscr{A}$-supermodule, respectively. Furthermore, we will often write $am:=a\cdot m$ and $ma:=m\cdot a$.

Fix a unital superalgebra $\mathscr{A}$. The category of all left $\mathscr{A}$-supermodules, denoted by $\mathscr{A}$-SMod, has all left supermodules as objects and for all $\mathrm{M},\mathrm{N}\in\mathrm{ob}(\mathscr{A}\text{-SMod})$ we have $\mathrm{Hom}_{\mathscr{A}}(\mathrm{M},\mathrm{N})=\mathrm{Hom}_{\mathscr{A}}(\mathrm{M},\mathrm{N})_{\bar 0}\oplus\mathrm{Hom}_{\mathscr{A}}(\mathrm{M},\mathrm{N})_{\bar 1}$, where

$$\mathrm{Hom}_{\mathscr{A}}(\mathrm{M},\mathrm{N})_\sigma=\{\mathrm{T}\in\mathrm{Hom}_{\Bbbk}(\mathrm{M},\mathrm{N})_\sigma\mid \mathrm{T}a=(-1)^{\sigma|a|}a\mathrm{T}\text{ for all }a\in\mathscr{A}_{\bar 0}\cup\mathscr{A}_{\bar 1}\}.\tag{10}$$

Definition 4.3 (Superbimodule)**.** Let $\mathscr{A}$ and $\mathscr{B}$ be superalgebras. We say a superspace M is a $(\mathscr{A},\mathscr{B})$*-superbimodule* if M is a left $\mathscr{A}$-module and a right $\mathscr{B}$-module such that for all $a\in\mathscr{A}$, $b\in\mathscr{B}$ and $m\in\mathrm{M}$

$$(am)b=a(mb).$$

To emphasize that M is a superbimodule, we will write ${}_{\mathscr{A}}\mathrm{M}_{\mathscr{B}}$.

Remark 4.4. We recall the definition of tensor product, $\otimes_{\mathscr{A}}$, for an $\mathscr{A}$-superalgebra. Let $M_{\mathscr{A}}$ and ${}_{\mathscr{A}}N$ be right and left $\mathscr{A}$-supermodules, respectively. Set $\mathscr{F}(M_{\mathscr{A}} \times {}_{\mathscr{A}}N)$ to be the is the free vector space on the set $M_{\mathscr{A}} \times {}_{\mathscr{A}}N$ and Z is the subspace

$$Z = \operatorname{span}\left\{ \begin{matrix} (m_1+m_2,n)-(m_1,n)-(m_2,n) \\ (m,n_1+n_2)-(m,n_1)-(m,n_2) \\ (ma,n)-(m,an) \end{matrix} \;\middle|\; m,m_1,m_2 \in M_{\mathscr{A}},\ n,n_1,n_2 \in {}_{\mathscr{A}}N,\ a \in \mathscr{A} \right\}.$$

We define $M_{\mathscr{A}} \otimes_{\mathscr{A}} {}_{\mathscr{A}}N = \mathscr{F}(M_{\mathscr{A}} \times {}_{\mathscr{A}}N)/Z$. Note that we will omit the subscript on $\otimes_{\mathscr{A}}$ and write $M_{\mathscr{A}} \otimes {}_{\mathscr{A}}N$ when it is clear that we are taking the $\mathscr{A}$-tensor product. Furthermore, we write $m \otimes n := (m,n)+Z$. Notice that if $\mathscr{A} = \Bbbk$, then we get the standard $\Bbbk$-bilinear tensor product.

Now let $\mathscr{A}$ and $\mathscr{B}$ be superalgebras and fix ${}_{\mathscr{A}}M_{\mathscr{B}}$, some $(\mathscr{A},\mathscr{B})$-superbimodule. The functor $M\colon \mathscr{B}\text{-SMod} \to \mathscr{A}\text{-SMod}$ is defined for all ${}_{\mathscr{B}}N \in \mathrm{ob}(\mathscr{B}\text{-SMod})$ by

$$\begin{aligned} M({}_{\mathscr{B}}N) &= {}_{\mathscr{A}}M_{\mathscr{B}} \otimes {}_{\mathscr{B}}N \text{ where } \mathscr{A} \text{ acts by} \\ a\cdot(m\otimes n) &= am\otimes n \end{aligned} \tag{11}$$

for any $a \in \mathscr{A}$, $m \in {}_{\mathscr{A}}M_{\mathscr{B}}$, and $n \in {}_{\mathscr{B}}N$ and extended by linearity. Furthermore, for homogeneous $T \in \mathrm{Hom}_{\mathscr{B}}({}_{\mathscr{B}}N, {}_{\mathscr{B}}V)$, $m \in {}_{\mathscr{A}}M_{\mathscr{B}}$, and $n \in {}_{\mathscr{B}}N$

$$M(T)(m\otimes n) = (-1)^{|m||T|} m \otimes T(n), \tag{12}$$

then extend by linearity.

Example 4.5 (Superadjunction of Res and Ind)**.** Let $\mathscr{A}$ be a superalgebra over $\Bbbk$. Throughout the following we view $\mathscr{A}$ as $(\Bbbk,\mathscr{A})$-bimodule by letting $\Bbbk$ act by left scalar multiplication and $\mathscr{A}$ act by right multiplication. Similarly, $\mathscr{A}$ is a $(\mathscr{A},\Bbbk)$-bimodule where $\mathscr{A}$ acts by left multiplication and $\Bbbk$ by right scalar multiplication.

Now define two superfunctors,

$$\mathrm{Res}\colon \mathscr{A}\text{-SMod} \to \Bbbk\text{-SMod} \qquad \text{and} \qquad \mathrm{Ind}\colon \Bbbk\text{-SMod} \to \mathscr{A}\text{-SMod}.$$

Both Res and Ind are special cases of (11) and (12). In particular, Res is defined for any ${}_{\mathscr{A}}M \in \mathrm{ob}(\mathscr{A}\text{-SMod})$ to be

$$\mathrm{Res}({}_{\mathscr{A}}M) = {}_{\Bbbk}\mathscr{A}_{\mathscr{A}} \otimes {}_{\mathscr{A}}M$$

and Ind is defined for any ${}_{\Bbbk}U \in \Bbbk\text{-SMod}$, to be

$$\mathrm{Ind}({}_{\Bbbk}U) = {}_{\mathscr{A}}\mathscr{A}_{\Bbbk} \otimes {}_{\Bbbk}U.$$

We claim that $\mathrm{Ind} \overset{\bar{0}}{\dashv} \mathrm{Res}$. Consider $\varepsilon\colon \mathrm{Ind}\,\mathrm{Res} \to \mathrm{id}_{\mathscr{A}\text{-SMod}}$ and $\eta\colon \mathrm{id}_{\Bbbk\text{-SMod}} \to \mathrm{Res}\,\mathrm{Ind}$ defined for any $m \in {}_{\mathscr{A}}\mathrm{M}$, $u \in {}_{\Bbbk}\mathrm{U}$, and $a, b \in \mathscr{A}$ by

$$\varepsilon_{{}_{\mathscr{A}}\mathrm{M}}(a \otimes b \otimes m) = abm \text{ and } \eta_{{}_{\Bbbk}\mathrm{U}}(u) = 1 \otimes 1 \otimes u, \tag{13}$$

then extend linearly. Notice that both ε and η are even maps. First we will show naturality of ε and η. Take any homogeneous $\mathrm{T} \in \mathrm{Hom}_{\mathscr{A}}({}_{\mathscr{A}}\mathrm{M}, {}_{\mathscr{A}}\mathrm{N})$, $a, b \in \mathscr{A}$, $m \in {}_{\mathscr{A}}\mathrm{M}$, and observe that

$$\begin{aligned}
\varepsilon_{{}_{\mathscr{A}}\mathrm{N}} \circ \mathrm{Ind}\,\mathrm{Res}(\mathrm{T})(a \otimes b \otimes m) &= \varepsilon_{{}_{\mathscr{A}}\mathrm{N}}((-1)^{|\mathrm{T}|(|a|+|b|)} a \otimes b \otimes \mathrm{T}m) && \text{(By (12))}\\
&= (-1)^{|\mathrm{T}|(|a|+|b|)} ab\mathrm{T}m && \text{(By (13))}\\
&= \mathrm{T}(abm) && \text{(By (10))}\\
&= \mathrm{id}_{\mathscr{A}\text{-SMod}}(\mathrm{T}) \circ \varepsilon_{{}_{\mathscr{A}}\mathrm{M}}(a \otimes b \otimes m), && \text{(By (13))}
\end{aligned}$$

then we extend by linearity. Thus, we conclude that $\varepsilon_{{}_{\mathscr{A}}\mathrm{N}} \circ \mathrm{Ind}\,\mathrm{Res}(\mathrm{T}) = \mathrm{id}_{\mathscr{A}\text{-SMod}}(\mathrm{T}) \circ \varepsilon_{{}_{\mathscr{A}}\mathrm{M}}$ for any $\mathrm{T} \in \mathrm{Hom}_{\mathscr{A}}({}_{\mathscr{A}}\mathrm{M}, {}_{\mathscr{A}}\mathrm{N})$. Similarly, take homogeneous $\mathrm{T} \in \mathrm{Hom}_{\Bbbk}({}_{\Bbbk}\mathrm{U}, {}_{\Bbbk}\mathrm{V})$, $u \in {}_{\Bbbk}\mathrm{U}$ and consider

$$\begin{aligned}
\eta_{{}_{\Bbbk}\mathrm{V}} \circ \mathrm{id}_{\Bbbk\text{-SMod}}(\mathrm{T})(u) &= \eta_{{}_{\Bbbk}\mathrm{V}}(\mathrm{T}u)\\
&= 1 \otimes 1 \otimes \mathrm{T}u && \text{(By (13))}\\
&= \mathrm{Res}\,\mathrm{Ind}(\mathrm{T})((-1)^{|\mathrm{T}|(|1|+|1|)} 1 \otimes 1 \otimes u) && \text{(By (12))}\\
&= \mathrm{Res}\,\mathrm{Ind}(\mathrm{T})(1 \otimes 1 \otimes u) && \text{(Since } |1| = \bar{0})\\
&= \mathrm{Res}\,\mathrm{Ind}(\mathrm{T}) \circ \eta_{{}_{\Bbbk}\mathrm{U}}(u). && \text{(By (13))}
\end{aligned}$$

Therefore, we have that both ε and η are even supernatural transformations.

To verify equations (6) for our ε and η, consider the following for any homogeneous $a \in \mathscr{A}$, $u \in {}_{\Bbbk}\mathrm{U}$, and $m \in {}_{\mathscr{A}}\mathrm{M}$:

$$\begin{aligned}
\varepsilon_{\mathrm{Ind}_{\Bbbk}\mathrm{U}} \circ \mathrm{Ind}(\eta_{{}_{\Bbbk}\mathrm{U}})(a \otimes u) &= \varepsilon_{\mathrm{Ind}_{\Bbbk}\mathrm{U}}((-1)^{|a||\eta|} a \otimes 1 \otimes 1 \otimes u)\\
&= \varepsilon_{\mathrm{Ind}_{\Bbbk}\mathrm{U}}(a \otimes 1 \otimes 1 \otimes u) && \text{(Since } |\eta| = \bar{0})\\
&= a \otimes u
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{Res}(\varepsilon_{{}_{\mathscr{A}}\mathrm{M}}) \circ \eta_{\mathrm{Res}_{\mathscr{A}}\mathrm{M}}(a \otimes m) &= \mathrm{Res}(\varepsilon_{{}_{\mathscr{A}}\mathrm{M}})(1 \otimes 1 \otimes a \otimes m)\\
&= (-1)^{|1||\varepsilon|} 1 \otimes am\\
&= a \otimes m, && \text{(By the definition of } \otimes \text{ and } |\varepsilon| = \bar{0})
\end{aligned}$$

we extend both of these arguments by linearity. Thus, we can conclude that (6) are satisfied and $\mathrm{Ind} \overset{\bar{0}}{\dashv} \mathrm{Res}$. Indeed, this is analogous to the well-known fact that Ind is left adjoint to Res in the non-super setting.

Our next example is the super analog to the well known result that if $\mathscr{A}$ is a Frobenius algebra (to be discussed) then Res $\dashv$ Ind. Moreover, we will show that in the super setting both Res $\overset{\bar{0}}{\dashv}$ Ind and Res $\overset{\bar{1}}{\dashv}$ Ind are possible.

Definition 4.6 (Frobenius Superalgebra)**.** A *Frobenius superalgebra* is a finite dimensional, unital associative superalgebra, $\mathscr{A}$, over $\mathbb{k}$ together with a linear map $\mathrm{tr}\colon \mathscr{A} \to \mathbb{k}$, of parity σ, such that ker(tr) contains no non-zero left ideals of $\mathscr{A}$. We say that tr is the *trace map* of $\mathscr{A}$.

Example 4.7 (Clifford Superalgebra)**.** Define the *Clifford superalgebra* to be the superalgebra $\mathrm{C}\ell = \langle c \mid c^2 = 1\rangle$ with superdecomposion $\mathrm{C}\ell = \mathbb{k} \oplus \mathbb{k}c$, where c is odd. Moreover, $\mathrm{C}\ell$ can be made into a Frobenius superalgebra by defining $\mathrm{tr}\colon \mathrm{C}\ell \to \mathbb{k}$. Notice there are the following two choices, up to scalar multiplication, for defining tr:

$$\begin{gathered}\mathrm{tr}_{\mathrm{even}}(1) = 1 \text{ and } \mathrm{tr}_{\mathrm{even}}(c) = 0 \text{ or}\\ \mathrm{tr}_{\mathrm{odd}}(1) = 0 \text{ and } \mathrm{tr}_{\mathrm{odd}}(c) = 1,\end{gathered}$$

then extend both linearly.

We say that a trace map, $\mathrm{tr}\colon \mathscr{A} \to \mathbb{C}$, is *supersymmetric* if $\mathrm{tr}(ab) = (-1)^{|a||b|}\mathrm{tr}(ba)$ for all homogeneous $a, b \in \mathscr{A}$ and is *symmetric* if $\mathrm{tr}(ab) = \mathrm{tr}(ba)$ for all $a, b \in \mathscr{A}$.

Notice that $\mathrm{tr}_{\mathrm{even}}(xy) = \mathrm{tr}_{\mathrm{even}}(yx)$, for all $x, y \in \mathrm{C}\ell$. When $x = y \in \{1, c\}$ this is obvious, and we are left with the final case

$$\mathrm{tr}_{\mathrm{even}}(1c) = \mathrm{tr}_{\mathrm{even}}(c) = 0 = \mathrm{tr}_{\mathrm{even}}(c) = \mathrm{tr}_{\mathrm{even}}(c1).$$

That is, $\mathrm{tr}_{\mathrm{even}}$ is symmetric. Furthermore, observe that $\mathrm{tr}_{\mathrm{even}}$ is not supersymmetric since

$$\mathrm{tr}_{\mathrm{even}}(cc) = \mathrm{tr}_{\mathrm{even}}(1) = 1 \neq -1 = (-1)^{|c||c|}\mathrm{tr}_{\mathrm{even}}(cc).$$

Similarly, for $\mathrm{tr}_{\mathrm{odd}}$ we have that

$$\begin{aligned}
&\mathrm{tr}_{\mathrm{odd}}(1^2) = \mathrm{tr}_{\mathrm{odd}}(1) = 0 = (-1)^{|1||1|}\mathrm{tr}_{\mathrm{odd}}(1^2), && \text{(Since } |1| = 0)\\
&\mathrm{tr}_{\mathrm{odd}}(1c) = (-1)^{|1||c|}\mathrm{tr}_{\mathrm{odd}}(c1) = \mathrm{tr}_{\mathrm{odd}}(c1), \text{ and} && \text{(Since } |1| = 0)\\
&\mathrm{tr}_{\mathrm{odd}}(cc) = \mathrm{tr}_{\mathrm{odd}}(1) = 0 = (-1)^{|c||c|}\mathrm{tr}_{\mathrm{odd}}(cc). && \text{(Since } c^2 = 1)
\end{aligned}$$

Thus, $\mathrm{tr}_{\mathrm{odd}}$ is not only symmetric but also supersymmetric.

Example 4.8. With the Clifford superalgebra we have a choice of the parity of the trace map. In general, that is not always the case. An example of a superalgebra without such a choice is $\mathbb{k}[x]/(x^k)$, where $k > 1$ and x is odd. We equip $\mathbb{k}[x]/(x^k)$ with the following decomposition:

$$\left(\mathbb{k}[x]/(x^k)\right)_0 = \mathbb{k} \oplus \mathbb{k}x^2 \oplus \cdots \oplus \mathbb{k}x^\ell \text{ and}$$

$$\left(\Bbbk[x]/(x^k)\right)_1 = \Bbbk x \oplus \Bbbk x^3 \oplus \cdots \oplus \Bbbk x^{\ell'},$$

where

$$\ell = \max\{p \in \mathbb{N} \mid p < k \text{ and } p = 2q \text{ for some } a \in \mathbb{N}\}$$
$$\ell' = \max\{p \in \mathbb{N} \mid p < k \text{ and } p = 2q+1 \text{ for some } q \in \mathbb{N}\}.$$

Notice that the ideal generated by x^{k-1} is proper and since $\ker(\mathrm{tr})$ must not contain non-zero left ideals we must have that $\mathrm{tr}(x^{k-1}) \neq 0$. With this restriction it immediately follows that $|\mathrm{tr}| \equiv k-1 \mod 2$.

More can be found on the trace map in the super setting in [7]. Of particular interest is Proposition 5.5 of [7], where the authors show that if tr_1 and tr_2 are two trace maps of a Frobenius superalgebra, $\mathscr{A}$, then there is some invertible $a \in \mathscr{A}$ such that $\mathrm{tr}_1(x) = \mathrm{tr}_2(ax)$ for all $x \in \mathscr{A}$. Using this it follows that if $\mathscr{A}$ is a Frobenius superalgebra with even trace map $\mathrm{tr}_{\mathrm{even}}$ then there is an odd trace map $\mathrm{tr}_{\mathrm{odd}}$ if and only if there is some invertible $a \in \mathscr{A}_1$. In particular, $\mathrm{tr}_{\mathrm{odd}}(x) = \mathrm{tr}_{\mathrm{even}}(ax)$ for all $x \in \mathscr{A}$. Similarly, if we have $\mathrm{tr}_{\mathrm{odd}}$ then there is $\mathrm{tr}_{\mathrm{even}}$ if and only if there is some invertible $a \in \mathscr{A}_1$.

Example 4.9 (Matrix Superalgebra)**.** Consider the vector space, $\mathrm{Mat}_{2n\times 2n}(\Bbbk)$, of $2n \times 2n$ matrices over $\Bbbk$. We say that $\mathrm{A} \in \mathrm{Mat}_{2n\times 2n}(\Bbbk)$ has even parity if it is block diagonal. That is, there are some $\mathrm{X}, \mathrm{Y} \in \mathrm{Mat}_{n\times n}(\Bbbk)$ such that

$$\mathrm{A} = \left[\begin{array}{c|c} \mathrm{X} & 0 \\ \hline 0 & \mathrm{Y} \end{array}\right],$$

where 0 is the $n \times n$ zero matrix. Analogously, we say that $\mathrm{B} \in \mathrm{Mat}_{2n\times 2n}(\Bbbk)$ has odd parity if it is block skew diagonal. That is, we can write

$$\mathrm{B} = \left[\begin{array}{c|c} 0 & \mathrm{X} \\ \hline \mathrm{Y} & 0 \end{array}\right],$$

for some $\mathrm{X}, \mathrm{Y} \in \mathrm{Mat}_{n\times n}(\Bbbk)$. It is straightforward to check that this decomposition gives rise to a superalgebra structure on $\mathrm{Mat}_{2n\times 2n}(\Bbbk)$.

When equipped with this superdecomposition we say that $\mathrm{Mat}_{2n\times 2n}(\Bbbk)$ is the superalgebra of $n|n$ supermatrices over $\Bbbk$ and we denote it by $\mathrm{Mat}_{n|n}(\Bbbk)$. Furthermore, we can equip $\mathrm{Mat}_{n|n}(\Bbbk)$ with the standard trace $\mathrm{tr}(\mathrm{X}) = \sum_{j=1}^{2n} x_{j,j}$, where the entries of $\mathrm{X} \in \mathrm{Mat}_{n|n}(\Bbbk)$ are $x_{j,k} \in \Bbbk$ for $1 \le j, k \le 2n$. We know that the standard trace is non-degenerate and therefore has no left ideals in $\ker(\mathrm{tr})$. Moreover, it is clear $\mathrm{tr}\left(\mathrm{Mat}_{n|n}(\Bbbk)_1\right) = 0$ and therefore $|\mathrm{tr}| = \bar{0}$. Now consider $\mathrm{tr}_{\mathrm{A}} \colon \mathrm{Mat}_{n|n}(\Bbbk) \to \Bbbk$ defined for all $\mathrm{X} \in \mathrm{Mat}_{n|n}(\Bbbk)$ by

$$\mathrm{tr}_{\mathrm{A}}(\mathrm{X}) = \mathrm{tr}(\mathrm{AX}),$$

where

$$\mathrm{A} = \left[\begin{array}{c|c} 0 & \mathrm{id}_{n\times n} \\ \hline \mathrm{id}_{n\times n} & 0 \end{array}\right].$$

Notice that A is invertible and $A \in \mathscr{A}_1$. Therefore tr_A is also a trace map such that $|\mathrm{tr}_A| = 1$.

Moreover, the standard trace is not the only even trace map. We can consider the map tr_B defined analogously as above with

$$B = \begin{bmatrix} \mathrm{id}_{n\times n} & 0 \\ 0 & -\mathrm{id}_{n\times n} \end{bmatrix}.$$

Since $B^2 = \mathrm{id}_{n|n}$ and $B \in \mathscr{A}_0$ we have that tr_B is also an even trace map. This trace is typically called the supertrace.

Example 4.10. Let $\mathscr{A}$ be a Frobenius superaglebra with $\sigma = |\mathrm{tr}|$. We claim that $\mathrm{Res} \overset{\sigma}{\dashv} \mathrm{Ind}$.

Let B be some basis of $\mathscr{A}$ consisting of only homogeneous elements and $B^\vee = \{b^\vee \mid b \in B\}$ be a left dual basis with respect to tr. That is, for all $a, b \in B$

$$\mathrm{tr}(a^\vee b) = \delta_{a,b}.$$

Notice that for all $c \in \mathscr{A}$ we have the following

$$\sum_{b\in B} \mathrm{tr}(b^\vee c)b = c = \sum_{b\in B} \mathrm{tr}(cb)b^\vee. \tag{14}$$

Also, for all $b \in B$, we have $|b^\vee| = |b| + \sigma$.

Define $\varepsilon_{\Bbbk M} \colon \mathrm{Res}\,\mathrm{Ind}_{\Bbbk} M \to \mathrm{id}({}_{\Bbbk}M)$ and $\eta_{\mathscr{A}N} \colon \mathrm{id}_{\mathscr{A}N} \to \mathrm{Ind}\,\mathrm{Res}_{\mathscr{A}} N$ for all $a_1, a_2 \in \mathscr{A}$, $m \in {}_{\Bbbk}M$, and $n \in {}_{\mathscr{A}}N$ by

$$\varepsilon_{\Bbbk M}(a_1 \otimes a_2 \otimes m) = \mathrm{tr}(a_1 a_2)m$$

and

$$\eta_{\mathscr{A}N}(n) = \sum_{b\in B} (-1)^{\sigma|b^\vee|} b \otimes b^\vee \otimes n,$$

then extend both by linearity. Note that $|\varepsilon_{\Bbbk M}| = |\mathrm{tr} \otimes \mathrm{id}_{\Bbbk M}| = \sigma$ and $|\eta_{\mathscr{A}N}(n)| = |b| + |b^\vee| + |n| = \sigma + |n|$. Therefore, both ε and η are of parity σ.

We first check that these are, in fact, supernatural transformations. Take any homogeneous $T \in \mathrm{Hom}_{\Bbbk}({}_{\Bbbk}M, {}_{\Bbbk}N)$, $a_1, a_2 \in \mathscr{A}$, $m \in {}_{\Bbbk}M$, and consider

$$\begin{aligned}
\varepsilon_{\Bbbk N} \circ \mathrm{Res}\,\mathrm{Ind}(T)(a_1 \otimes a_2 \otimes m) &= \varepsilon_{\Bbbk N}\left((-1)^{|T|(|a_1|+|a_2|)} a_1 \otimes a_2 \otimes Tm\right) \\
&= (-1)^{|T|(|a_1|+|a_2|)}\mathrm{tr}(a_1 a_2)Tm \\
&= (-1)^{|T|(|a_1|+|a_2|)} T\left(\mathrm{tr}(a_1 a_2)m\right) && \text{(By (10))} \\
&= (-1)^{|T|\sigma} T\left(\mathrm{tr}(a_1 a_2)m\right) \\
& && \text{(If } \mathrm{tr}(a_1 a_2) \neq 0 \text{ then } |a_1| + |a_2| = \sigma) \\
&= (-1)^{|T|\sigma} T \circ \varepsilon_{\Bbbk M}(a_1 \otimes a_2 \otimes m).
\end{aligned}$$

Extending the above by linearity we conclude that ε is a supernatural transformation. Similarly, take homogeneous $\mathrm{T} \in \mathrm{Hom}_{\mathscr{A}}({}_{\mathscr{A}}\mathrm{M}, {}_{\mathscr{A}}\mathrm{N})$, $m \in {}_{\mathscr{A}}\mathrm{M}$, and consider

$$\begin{aligned}
\mathrm{Ind}\,\mathrm{Res}(\mathrm{T}) \circ \eta_{\Bbbk\mathrm{M}}(m) &= \mathrm{Ind}\,\mathrm{Res}(\mathrm{T})\left(\sum_{b\in\mathscr{B}} (-1)^{\sigma|b^\vee|} b \otimes b^\vee \otimes m\right) \\
&= \sum_{b\in\mathrm{B}} (-1)^{\sigma|b^\vee|+|\mathrm{T}||b|+|\mathrm{T}||b^\vee|} b \otimes b^\vee \otimes \mathrm{T}m \\
&= (-1)^{\sigma|\mathrm{T}|} \sum_{b\in\mathrm{B}} (-1)^{\sigma|b^\vee|} b \otimes b^\vee \otimes \mathrm{T}m && \text{(Since } |b|+|b^\vee| = \sigma) \\
&= (-1)^{\sigma|\mathrm{T}|} \eta_{\Bbbk\mathrm{N}} \circ \mathrm{T}(m).
\end{aligned}$$

Thus, we have that η is a supernatural transformation.

Now to check that both satisfy the equations (6), take any $a \in \mathscr{A}$, $m \in {}_{\Bbbk}\mathrm{M}$ and consider the following:

$$\begin{aligned}
\mathrm{Res}(\varepsilon_{\Bbbk\mathrm{M}}) \circ \eta_{\mathrm{Res}_{\Bbbk}\mathrm{M}}(a\otimes m) &= \mathrm{Res}(\varepsilon_{\Bbbk\mathrm{M}})\left(\sum_{b\in\mathscr{B}} (-1)^{\sigma|b^\vee|} b \otimes b^\vee \otimes a \otimes m\right) \\
&= \sum_{b\in\mathrm{B}} (-1)^{\sigma|b^\vee|+\sigma|b|} b \otimes \mathrm{tr}(b^\vee a) m && \text{(By (12))} \\
&= \sum_{b\in\mathrm{B}} (-1)^{\sigma|b|+\sigma^2+\sigma|b|} \mathrm{tr}(b^\vee a) b \otimes m && \text{(Since } |b^\vee| = |b|+\sigma) \\
&= \sum_{b\in\mathrm{B}} (-1)^{\sigma} \mathrm{tr}(b^\vee a) b \otimes m && \text{(Since } \sigma^2 = \sigma) \\
&= (-1)^{\sigma}\left(\sum_{b\in\mathrm{B}} \mathrm{tr}(b^\vee a) b\right) \otimes m \\
&= (-1)^{\sigma} a \otimes m, && \text{(By (14))}
\end{aligned}$$

then extend by linearity. Thus, we have that $\mathrm{Res}(\varepsilon_{\Bbbk\mathrm{M}}) \circ \eta_{\mathrm{Res}_{\Bbbk}\mathrm{M}} = (-1)^{\sigma}\mathrm{id}_{\mathrm{Ind}_{\Bbbk}\mathrm{M}}$. Similarly, take any homogeneous $a \in \mathscr{A}$, $n \in {}_{\mathscr{A}}\mathrm{N}$, and consider

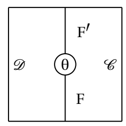
$$\begin{aligned}
\varepsilon_{\mathrm{Res}_{\mathscr{A}}\mathrm{N}} \circ \mathrm{Res}(\eta_{\mathscr{A}\mathrm{N}})(a\otimes n) &= \varepsilon_{\mathrm{Res}_{\mathscr{A}}\mathrm{N}}\left(\sum_{b\in\mathrm{B}} (-1)^{\sigma(|a|+|b^\vee|)} a \otimes b \otimes b^\vee \otimes n\right) \\
&= \sum_{b\in\mathrm{B}} (-1)^{\sigma(|a|+|b^\vee|)} \mathrm{tr}(ab) b^\vee \otimes n \\
&= \left(\sum_{b\in\mathrm{B}} \mathrm{tr}(ab) b^\vee\right) \otimes n && \text{(If } \mathrm{tr}(ab) \neq 0 \text{ then } |a| = |b^\vee|) \\
&= a \otimes n, && \text{(By (14))}
\end{aligned}$$

then extend by linearity. Therefore, we have that $\varepsilon_{\mathrm{Res}_{\mathscr{A}}\mathrm{N}} \circ \mathrm{Res}(\eta_{\mathscr{A}\mathrm{N}}) = \mathrm{id}_{\mathrm{Res}_{\mathscr{A}}\mathrm{N}}$. That is, (6) is satisfied and $\mathrm{Res} \overset{\sigma}{\dashv} \mathrm{Ind}$.

5 String Diagrams

String diagrams are an invaluable visualization and computational tool in the theory of monoidal categories and 2-categories. We conclude the paper with a brief discussion of how the notion of superadjunction translates into string diagrams. We refer the interested reader to [5] for more details on the appearance of string diagrams in categorification in the non-super setting, and to [1] for the super setting.

Suppose that $\mathscr{D}, \mathscr{C}$, and $\mathscr{E}$ are monoidal supercategories. Then we read the following diagram from right to left and bottom to top.

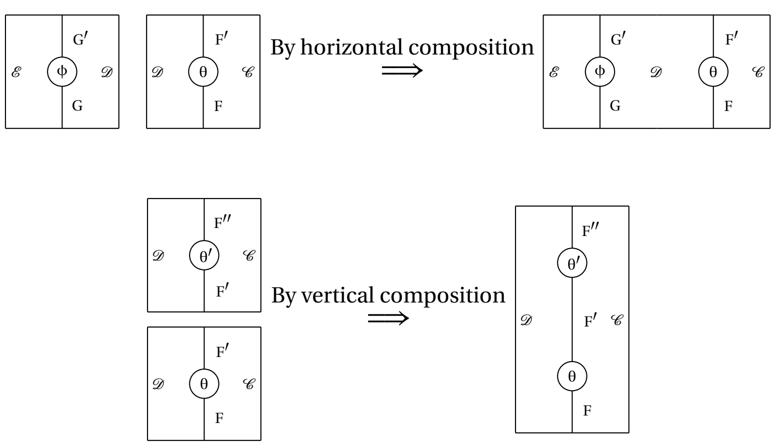


We interpret this diagram as follows:

- the area containing each of $\mathscr{C}$ and $\mathscr{D}$ represent each of these categories, respectively,
- $F, F' : \mathscr{C} \to \mathscr{D}$ are superfunctors and the strands labelled F and F′ represent them, and
- $\theta : F' \to F$ is a supernatural transformation and the node labelled by θ represents it.

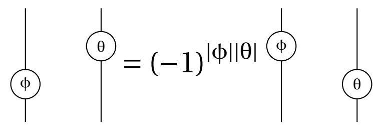
Often times we will suppress most, if not all, of the labels we have above. For the time being we will maintain them as we outline some of the tools we have while working with these diagrams.

Given two diagrams we can, if compatible, compose them either vertically or horizontally. That is, we have the following diagrams for superfunctors, $F, F', F'' : \mathscr{C} \to \mathscr{D}$ and $G, G' : \mathscr{D} \to \mathscr{E}$, and supernatural transformations, $\theta : F \to F'$, $\theta' : F' \to F''$ and $\phi : G \to G'$.

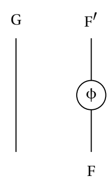


Here the horizontal composition corresponds to composing functors and the vertical composition to composing natural transformations.

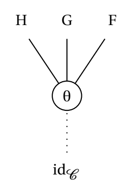
Since there are two types of composition the natural question to ask is 'do they commute?'. This leads us to the *super interchange law* as motivated by (5). For the following we transition to the standard notation and will omit the functor and category labels as well as the border lines but retain the labels for the supernatural transformations, $\phi\colon G \to G'$ and $\theta\colon F \to F'$. The super interchange law states the following:



Some more elaborate string diagrams can be constructed. For example, let $\theta\colon \mathrm{id}_{\mathscr{C}} \to FGH$ and $\phi\colon F \to F'$ be supernatural transformations and consider the following diagrams:

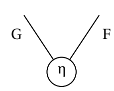


and

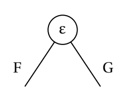


In the right diagram we use a dotted line to represent the identity but this line is often omitted.

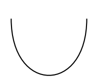
We are particularly interested in superadjoint functors and would like translate the relations, (6), to these string diagrams. For counit and unit supertransformations, we have $\eta\colon \mathrm{id}_{\mathscr{C}} \to GF$ and $\varepsilon\colon FG \to \mathrm{id}_{\mathscr{D}}$ which admit the following string diagrams with the dotted line representing the identity omitted:



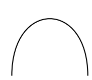
and



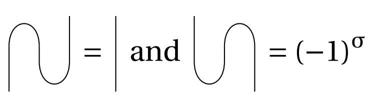
Moreover, we will be using these transformations almost exclusively, so we will omit the labelling and 'smooth' the structure of the diagrams to get the following diagrams for ε and η, respectively.



and



Using this we can finally write the equations (6) diagrammatically to get the following relations.

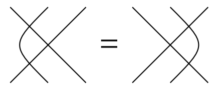


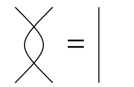
(15)

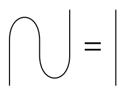
Observe the similarities between these diagrams and the ones found in [8, (6.7)].

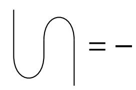
We conclude with the example of the *odd Brauer supercategory* from [1] which follows the string calculus from [3]. The odd Brauer supercategory involves relations that correspond to (15) in the case that $\sigma = \bar{1}$.

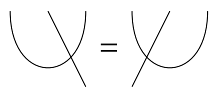
Definition 5.1 (Odd Brauer Supercategory)**.** The *odd Brauer supercategory* is the strict monoidal supercategory $\mathscr{SB}$ generated by the object $\cdot$, an even morphism $\times\colon \cdot \otimes \cdot \to \cdot \otimes \cdot$, and two odd morphisms $\cup\colon \mathrm{I} \to \cdot \otimes \cdot$ and $\cap\colon \cdot \otimes \cdot \to \mathrm{I}$. Moreover, these morphisms are such that the following relations hold:

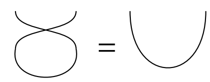

(16)


(17)


(18)


(19)


(20)


(21)

Relations (18) and (19) can be interpreted as the statement that the object $\cdot$ is odd adjoint to itself.

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