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Graphs, Random Walks, and the Tower of Hanoi

By *Stephanie Egler*

Abstract. The Tower of Hanoi puzzle with its disks and poles is familiar to students in mathematics and computing. Typically used as a classroom example of the important phenomenon of recursion, the puzzle has also been intensively studied its own right, using graph theory, probability, and other tools. The subject of this paper is "Hanoi graphs", that is, graphs that portray all the possible arrangements of the puzzle, together with all the possible moves from one arrangement to another. These graphs are not only fascinating in their own right, but they shed considerable light on the nature of the puzzle itself. We will illustrate these graphs for different versions of the puzzle, as well as describe some important properties, such as planarity, of Hanoi graphs. Finally, we will also discuss random walks on Hanoi graphs.

1 The Tower of Hanoi

The Tower of Hanoi is a famous puzzle originally introduced by a "Professor Claus" in 1883. It was revealed in the following year that "Claus" was a reordering of the letters in number theorist Edouard Lucas' surname.

The Tower of Hanoi supposedly arose from the legend of the Tower of Brahma. This legend tells of three diamond needles and a tower of 64 golden disks, all on one needle, which a group of Brahmin monks must transfer to another needle one disk at a time. The disks are all of different diameters, and no disk may ever be placed atop a smaller one. Once the monks complete their task, the world will end. Thankfully, even moving one disk per second, and using the optimal strategy, it would well take over 40 times the currently accepted age of the universe to complete the task.

The setup of the traditional Tower of Hanoi puzzle consists of 3 wooden poles and n disks with differing diameters (Figure 1). These n disks are initially stacked on a single pole in order of (decreasing) diameter with the largest disk on the bottom. The goal of the puzzle is to move this tower of n disks to another pole one disk at a time while never placing a smaller disk on top of a larger disk. These two rules are identical to the rules the monks were to follow.

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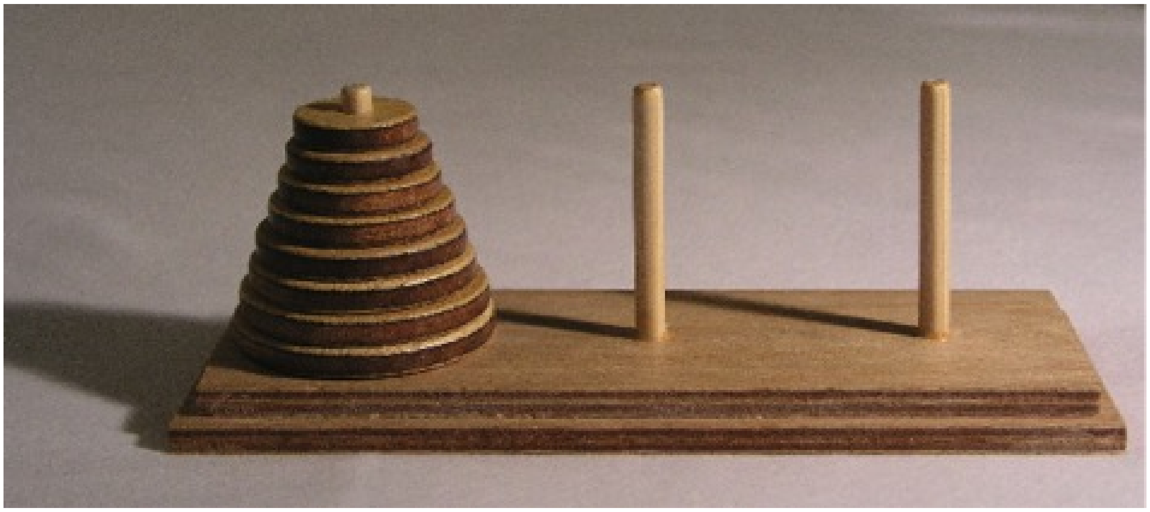


Figure 1: The Tower of Hanoi with three poles labelled 0, 1, and 2 corresponding to the leftmost, center, and rightmost pole, respectively.

In the puzzle's most basic form, the (minimum) number of moves needed to solve the puzzle for n disks can be found through a recurrence relation. The recurrence relation is as follows:

$$h_n = 2h_{n-1} + 1, \quad h_1 = 1. \tag{1}$$

Further, we can use induction to prove the closed form expression

$$h_n = 2^n - 1 \quad \text{for all integers } n = 1, 2, \ldots.$$

These are standard exercises in most discrete math textbooks [4]. However, this is not the end, but just the beginning of the exploration of the Tower of Hanoi. Our goal is to extend what we know about Hanoi graphs corrsponding to the traditional form of three poles to variations of the Tower of Hanoi. In particular, we will explore Hanoi graphs and random walks of the Tower of Hanoi with a fourth pole, then add an additional adjacency requirement to our variation, which restricts the legal moves of the puzzle.

2 Hanoi Graphs

2.1 Hanoi Graphs on Three Poles

A very fruitful tool in the study of the Tower of Hanoi is the *Hanoi graph*, which represents all of the possible states of the puzzle as well as the moves that can be made from one state to another. We will denote the Hanoi graph by H_n^k, where n is the number of disks and k is the number of poles in the puzzle. We begin by examining the traditional H_n^3 Hanoi graph on 3 poles. This graph's vertices display all legal configurations of the

n disks, totaling 3^n configurations, and its edges display all legal moves. Here "legal" means, as before, that one can only move one disk at a time and that no disk may be placed atop a smaller one. Following the notation of [7] and [6], we will label the poles 0, 1, and 2, corresponding to the leftmost, center, and rightmost poles, respectively. The disk with the largest diameter is labeled a_{n-1}, the remaining disks are labeled similarly, and the smallest would then be labeled a_0. This allows each vertex of the Hanoi graph to be labeled with an n-bit ternary string $a_{n-1}\dots a_1 a_0$, where $a_i \in \{0,1,2\}$ and $a_i = j$ if disk i lies on pole j (Figure 2).

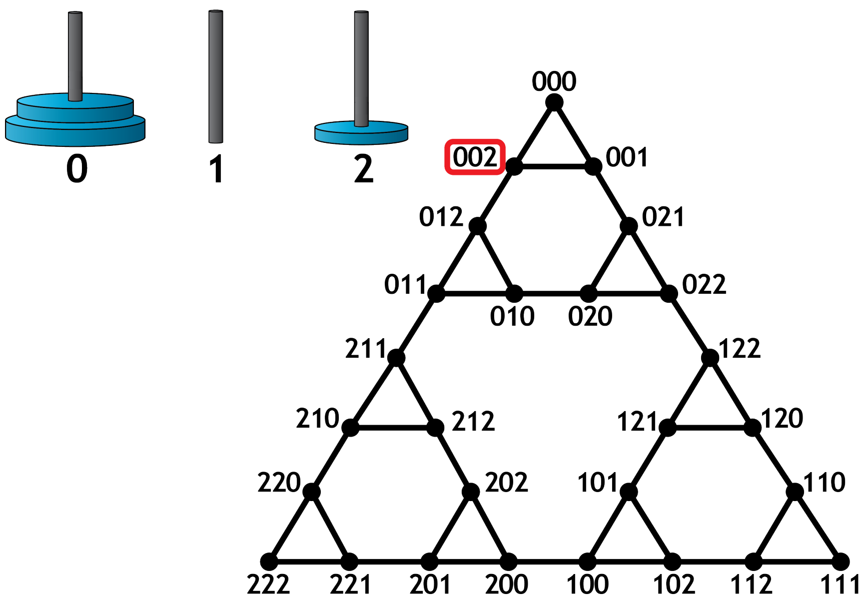


Figure 2: The Hanoi graph H_3^3. The highlighted vertex corresponds to the Tower of Hanoi puzzle in the state shown. The 3-bit ternary string 002 indicates that the largest disk a_2 and medium-sized disk a_1 lie on pole 0 and the smallest disk a_0 lies on pole 2.

Each resulting n-bit ternary string corresponds to a legal configuration. Two vertices are adjacent if and only if there is one legal move that takes one of the corresponding states of the puzzle to the other. Note that since legal moves are reversible, Hanoi graphs are *undirected*. Also, since it is possible to travel from any vertex to any other vertex along adjacent edges, Hanoi graphs are *connected*, as well. Solving the Tower of Hanoi puzzle in the fewest moves possible, then, corresponds to finding a shortest path between two vertices in the Hanoi graph. Figure 2 shows the shortest path in H_3^3 joining 000 and 222; this corresponds to all disks being initially stacked on the leftmost pole and then transferred to the rightmost pole using the smallest number of moves possible. The length of this path, namely seven edges, is consistent with our earlier result that the optimal number of moves is $h_3 = 2^3 - 1 = 7$.

The *substructure* $[j]$ is the set of vertices in H_n^k corresponding to legal configurations where the largest disk is on pole j. Note that H_3^3 is made up of the 3 substructures [0], [1], and [2], corresponding to the largest disk being fixed on each of the three poles. The positions of the remaining two disks correspond to the legal configurations of the Hanoi graph with 3 poles and 2 disks, H_2^3. These substructures are connected by *bridges*. An edge is a *bridge* if its endpoints lie in different substructures. By a combinatorial argument [9], the total number of bridges in H_n^k is given by

$$\binom{k}{2}(k-2)^{n-1}. \tag{2}$$

As a consequence, regardless of n, when there are $k = 3$ poles, the number of bridges is 3. Thus we can view H_n^3 recursively: H_n^3 consists of 3 copies of H_{n-1}^3 connected as substructures by three bridges. In this way, the Hanoi graphs H_n^3 display *self-similarity*, about which we say more below. A similar result holds for Hanoi graphs with $k > 3$ poles, but the number of bridges then depends on both the number k of poles and the number n of disks in the puzzle, and thus the the situation becomes more complex.

2.2 Hanoi Graphs on Four Poles

A variation on the Tower of Hanoi is the addition of a fourth pole. Figures 3 and 4 illustrate H_n^4 with $n = 2$ and $n = 3$, respectively. Note that H_2^4 is comprised of four copies of the H_1^4 graph connected by bridges; by formula (2), the number of bridges is given by

$$\binom{4}{2}(4-2)^{2-1} = 6 \cdot 2 = 12.$$

Similarly, H_3^4 consists of four copies of H_2^4; again, formula (2) gives the number of bridges:

$$\binom{4}{2}(4-2)^{3-1} = 6 \cdot 2^2 = 24.$$

This illustrates that the H_n^4 graphs display the same remarkable self-similarity seen in the H_n^3 graphs.

We have shown in H_n^3 there is a unique shortest path joining any two given perfect states, thus corresponding to the optimal solution of the Tower of Hanoi puzzle on three poles. It is noteworthy, however, that the H_n^4 graphs do not in general have unique shortest paths. For example, the paths

$$00 \to 01 \to 31 \to 33 \quad \text{and} \quad 00 \to 02 \to 32 \to 33$$

are both shortest paths from the perfect state 00 to the perfect state 33 (Figure 3).

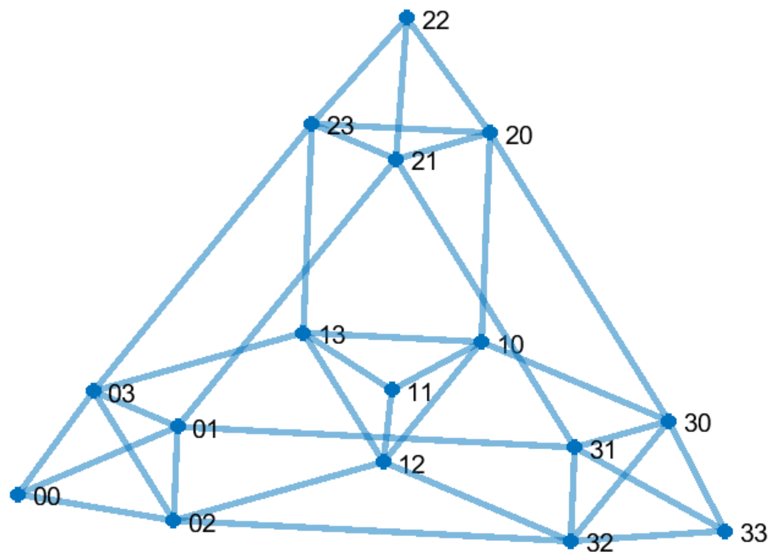


Figure 3: H_2^4 (drawn in MATLAB).

Just as H_n^3 has a total of 3^n vertices, so H_n^4 has 4^n. Indeed, in general, H_n^k has k^n vertices: we see this by examining the configurations of the corresponding Tower of Hanoi puzzle. If we take the disks from largest to smallest, there are k options, that is to say k poles, where we can place each disk. Because there are n disks, there are a total of k^n vertices.

A vertex represents a *perfect state* if for some j, all $a_i = j$. That is, all digits in the n-bit ternary string of the vertex are equal, and in the corresponding Tower of Hanoi puzzle, all disks are on the j^{th} pole. We note that all vertices in H_n^4 have degree at least 3, because the smallest disk can always be moved to each of the three other poles. Moreover, perfect states in H_n^4 have exactly this minimum degree of 3, because these moves are the only legal moves that can made. More generally, in the Hanoi graph H_n^k there are k perfect states each with degree $k-1$, regardless of n.

We can also give an upper bound for the degree of vertices in H_n^4. Taking $n=3$ for example, we see that the maximum degree for vertices in H_3^4 is 6, and this is achieved when each disk rests on a separate pole, leaving one empty pole. The smallest disk can move to each of the three poles on which it is not currently resting. The medium-sized disk can move either to the empty pole or atop the largest disk. Finally, the largest disk can move only to the empty pole. Thus we can make $3+2+1=6$ legal moves. We can apply this argument regardless of n, in fact. Again consider the smallest top three disks. The other disks would be trapped underneath these smallest three or be too large to legally move to another pole. So by the same logic as above, there are six possible moves.

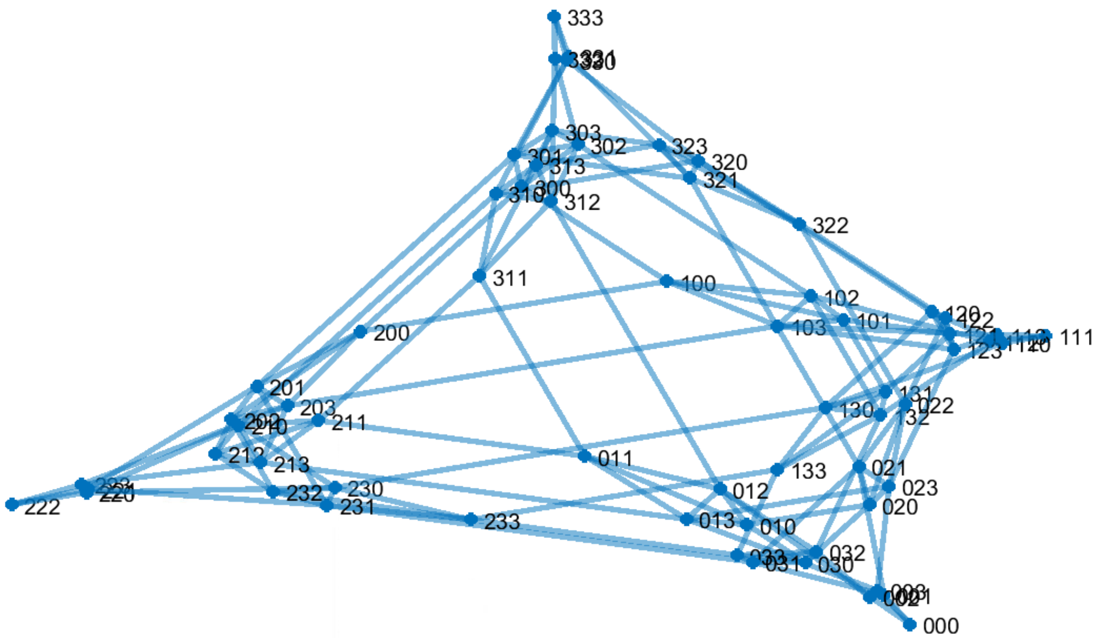


Figure 4: H_3^4 (drawn in MATLAB).

Finally, we note that the *only* possible degrees of vertices in H_n^4 are 3, 6, and 5. To see that 5 is the only possibility if the degree of the vertex is not 3 or 6, we divide the legal Tower of Hanoi configurations into four mutually exclusive cases. Either there are three, two, one, or zero empty poles. If there are three empty poles, then we have a perfect state, which as we noted has degree 3. If there are two empty poles, then each of the top disks on the other two poles can be moved to either of the empty poles, for a total of four legal moves. In addition, the smaller of the two top disks can be placed atop the larger of the disks, adding a fifth possible legal move. Thus, in this case, the degree of the corresponding vertex is five. If there is one empty pole, then the disks are spread out among three poles with three top disks that can be moved. By the same argument as above, these three top disks can each be moved in a total of six different ways. Finally, if there are zero empty poles, then once again n must be greater than 3 and there are four top disks. The largest of these has no legal moves, whereas the other three top disks create the same six moves as the case of one empty pole. Thus, the only possible degrees are 3, 5, and 6.

Now that we have established the number of vertices and the possible degrees for each vertex, we can calculate the number of edges in the graph. When $n = 1$, there are 4 vertices and each vertex is a perfect state with degree 3. Summing the degrees of each vertex counts every edge twice (this is sometimes known as the 'handshake theorem').

Thus, H^4_1 has $\frac{1}{2}\cdot(4\cdot 3) = 6$ edges. A similar calculation applies when $n = 2$: Figure 3 shows that 4 of the 16 vertices of H^4_2 are perfect states with degree 3 and the remaining 12 vertices have degree 5. Therefore, H^4_3 has $\frac{1}{2}\cdot[(4\cdot 3)+(12\cdot 5)] = 36$ edges. Finally, we see that H^4_3, shown in Figure 4, again has 4 perfect states of degree 3. To count the number of vertices that have degree 5 or 6, we first count 36 ternary strings that have exactly two numbers that are the same, e.g. 001, corresponding to two disks on the same pole and one on a different pole; as we noted earlier, such a vertex has degree 5. In the same way, we count 24 ternary strings that have three different digits, e.g. 012, which corresponds to three disks on separate poles and one empty pole; this vertex would have degree 6. All told, H^4_3 has $\frac{1}{2}\cdot[(4\cdot 3)+(36\cdot 5)+(24\cdot 6)] = 168$ edges.

We can also count the *faces* of planar graphs. Intuitively, the faces of a planar graph are the polygons bounded by the edges of the graph. To this we add, however, the *infinite face*, consisting of the exterior of the graph. (If we compactify the plane by stereographically projecting it onto a sphere, then the infinite face of the graph is simply the face containing the north pole.) With this understanding, the Euler-Descartes formula $V - E + F = 2$ allows us to calculate F given V and E.

All these results are summarized in Table 1; because H^4_3 is nonplanar, we do not use the Euler-Descartes formula to find the number of its faces.

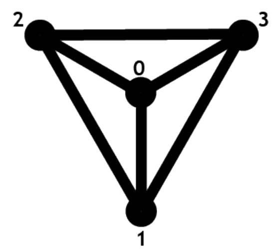
Table 1: V, E, and F for H^4_n for $n = 1, 2, 3$.

n	Vertices	Edges	Faces	Planar
1	4	6	4	Yes
2	16	36	22	Yes
3	64	168		No

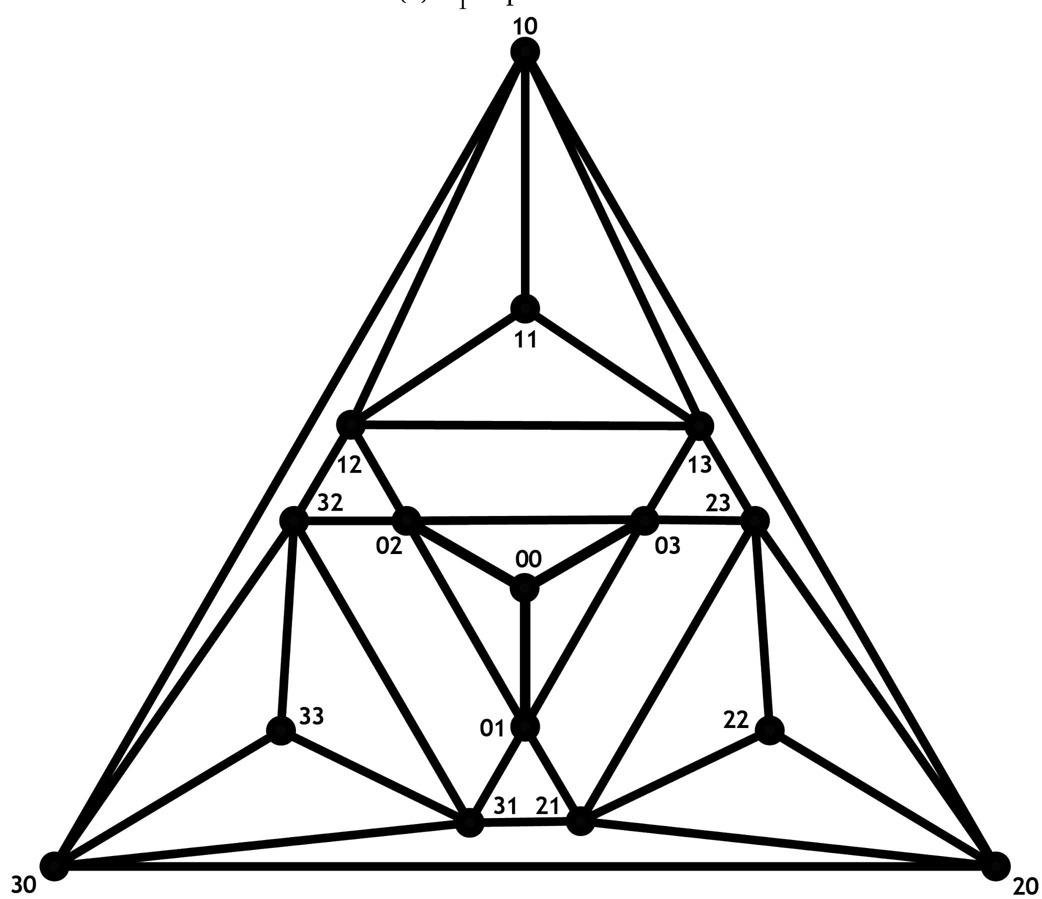
3 Planarity

The planarity of Hanoi graphs is a much-studied question. Of course, one way to establish that a graph is planar is to construct a planar embedding of the graph. Indeed, by this method, we can see that H^4_1 and H^4_2 are planar; Figures 5a and 5b illustrate planar embeddings of these graphs. Note also that H^4_1 is isomorphic to the planar complete graph K_4.

Demonstrating nonplanarity, however, can be more difficult. The Hanoi graph H^4_3 is notable for being nonplanar [5] while satisfying many of the standard necessary conditions for planarity. For example, one such criterion [8] says that a planar graph with V vertices and E edges with $V \geq 3$ satisfies $E \leq 3V - 6$. As Table 2 shows, each of the graphs H^4_1, H^4_2, and H^4_3 satisfies this condition, and yet H^4_1 and H^4_2 are planar whereas H^4_3 is not.



(a) H^4_1 in planar form



(b) H^4_2 in planar form

Figure 5: The planar Hanoi graphs H^4_1 and H^4_2

Table 2: Necessary condition for planarity $E \leq 3V - 6$.

n	$E \leq 3V - 6$	Satisfies Condition	Planar
1	$6 \leq 6 = 3(4) - 6$	Yes	Yes
2	$36 \leq 42 = 3(16) - 6$	Yes	Yes
3	$168 \leq 186 = 3(64) - 6$	Yes	No

Another condition states that if a graph G is a connected planar simple graph, then G contains at least one vertex of degree 5 or less. We see that this holds in H^4_n not only for $n = 1, 2,$ and 3, but for all n, because the minimum degree is always $4 - 1 = 3 < 5$.

In fact, every planar graph contains a vertex of degree five or less [2]. We see that for H^7_n, the minimum degree is always $7 - 1 = 6 > 5$, a contradiction. So, adding poles so that $k \geq 6$ means that these Hanoi graphs are nonplanar.

Further, Kuratowski's theorem states that a graph is planar if and only if it does not contain a subdivision of the complete graphs K_5 or $K_{3,3}$. However, in reality, it is difficult to check every single subdivision, so we will not rely on this theorem. Indeed, it would be a linear time algorithm [8].

Another way to examine planarity is to ask how close a graph is to being planar. That is, what is the number of times any two edges in the graph G cross each other? This is the *crossing number* of a graph, denoted $cr(G)$. Note that if $cr(G) = 0$, then no edges cross and G is planar. It is known [8] that for any graph G with V vertices and E edges, $cr(G) \geq E - 3V + 6$. Table 3 shows the results of applying this estimate to Hanoi graphs. Unfortunately, this does not reveal much about the planarity of these graphs, since we find only that the number of crossed edges is greater than or equal to a negative number, which is trivial.

Table 3: Crossing number estimates for Hanoi graphs

n	$cr(G) \geq E - 3V + 6$
1	$cr(H^4_1) \geq -3 = 3 - 3(4) + 6$
2	$cr(H^4_2) \geq -24 = 18 - 3(16) + 6$
3	$cr(H^4_3) \geq -102 = 84 - 3(64) + 6$

The results of applying these formulas to these three H^4_n graphs are not sufficient proof of planarity or nonplanarity. We only see that these graphs *might* be planar. However, there are many graphs that satisfy these inequalities that are not planar. By embedding the graphs in the plane, we know that H_1^4 and H_2^4 are planar. Further, H_3^4 is nonplanar, so H_n^4 where is also nonplanar for $n > 3$ because it contains H_3^4 as a subgraph.

As a side note, we can now examine circuits of the Hanoi graph H_4^n. First, we note that none of the H_n^4 graphs has an *Euler circuit*. An Euler circuit is a sequence of adjacent vertices and edges that starts and ends with the same vertex, uses every vertex of the graph at least once, and uses every edge exactly once. This means that every vertex of the graph must have even degree. However, we know that all perfect states in H_n^4 graphs have odd degree, namely 3. In fact, for $n = 2$, every vertex is of (odd) degree either 3 or 5. Thus, none of these graphs has an Euler circuit. It is possible to find Hamiltonian paths for Hanoi graphs, in which every vertex is used exactly once. Moreover, we can find a circuit in H_n^4 such that every vertex (except for the first and last) is used exactly once. This is called a Hamiltonian circuit. Figure 6 shows an example of a Hamiltonian circuit on the planar graph H_2^4. (The Hanoi graph H_3^3 of Figure 2 also has a Hamiltonian circuit, as Figure 7 illustrates).

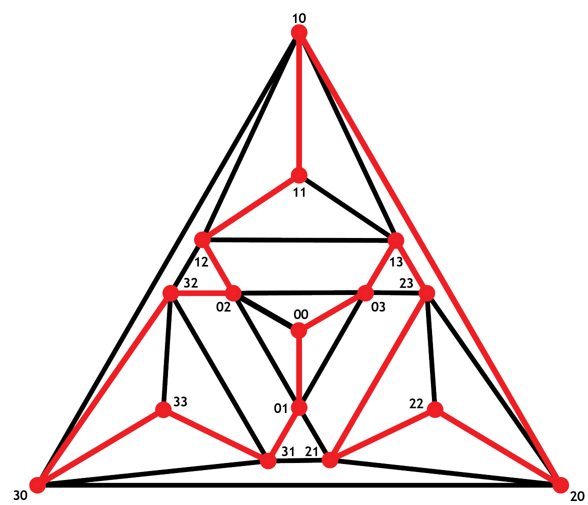


Figure 6: A Hamiltonian circuit for the planar graph H_2^4

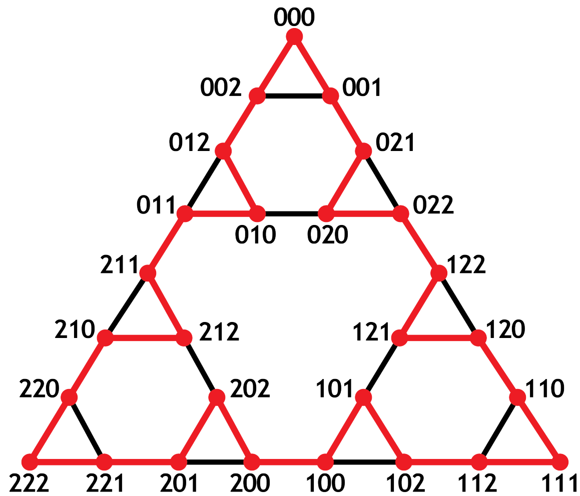


Figure 7: A Hamiltonian circuit for H_3^3

4 Pascal's Triangle Connection

It turns out that the graph H_n^3 is closely connected with Pascal's triangle. For example, the graph consisting of the odd entries in the first eight rows of Pascal's triangle is isomorphic to H_3^3 (Figure 8a and 8b). Indeed, the nature of this isomorphism can be used to address significant questions about the Tower of Hanoi [7]. Further, by taking additional rows of Pascal's triangle, we see that the Hanoi graphs H_n^3 are each isomorphic to subgraphs of Pascal's triangle. We can even see fractal structure in the graphs H_n^3 by allowing n to grow without bound; as n grows large, H_n^3 comes to resemble the famous Sierpinski triangle.

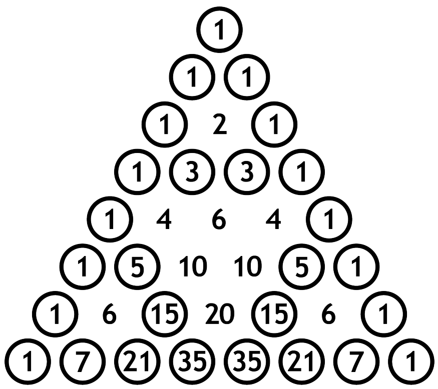
It is tempting to seek similar correspondences between other Hanoi graphs and various analogues of Pascal's triangle. For example, two structures related to Pascal's triangle are Pascal's hexagon (Figure 9) and Pascal's pyramid (Figure 10). We can add a degree of complexity to our Hanoi graph by increasing the number of poles from three to four. It is tempting to seek a correspondence between the Hanoi graph H_2^4, for example, and, say, Pascal's hexagon. Unfortunately, Pascal's hexagon, with the odd coefficients circled (Figure 9), has a few conspicuous features that prevent such a correspondence. If we work from the center of the graph outwards and let 00 be the center point, then the 1's in layer 0 of the three subgraphs can correspond to 01, 02, and 03. From there, each vertex can lead to two new vertices, just as Pascal's hexagon has two new 1's in layer 1. After layer 1, however, there is a problem in not generating enough 1's to correspond with the number of new vertices reached in the Hanoi graph. Further, and more importantly, the three subgraphs never connect up again, as is the case in the Hanoi graphs. So, there will be multiple odd numbers in Pascal's hexagon corresponding to the same vertex in the Hanoi graph, dashing our hopes of finding an isomorphism between the two structures.

Next we examine Pascal's pyramid, first alongside H_2^4 and then H_3^4. We compare these two layer-by-layer to more easily explore the connections of the 3D graph. Layers 0 and 1 line up in the same way we have seen in Pascal's triangle. Similarly, in layer 2, we circle the odd coefficients and let these correspond to the vertices 10, 20, and 30. Now, there is no single edge that connects the vertices in layer 1 to the vertices in layer 2. The six vertices that would be in this layer have a better fit in the next layer. In this layer 3, we also change the which numbers we circle from any odd number to any 3 present. This gives the six spots needed to correspond with the new vertices in the Hanoi graph. In the final layer, layer 4, we return to circling only the odd coefficients, and these three circled numbers correspond to the remaining perfect states in H_2^4. These results can be found in Figure 10. Note that every vertex in H_2^4 has a corresponding number in the first four layers of Pascal's pyramid (beginning at the 0^{th} layer). However, the edges that connect the vertices in the new order to correspond with Pascal's pyramid are not all straightforward as in the connection between H_n^3 and Pascal's pyramid, simply due to the vertices in layers 2 and 3 being switched.

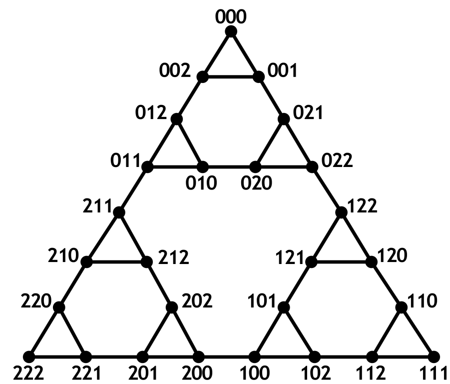
Pascal's pyramid, like Pascal's triangle, grows without bound. Again, it is tempting to think that we could extend the H_2^4 correspondence, even if it is not perfect, to include H_n^4. However, H_3^4 does not follow this correspondence. To see this, we note the number of new vertices in each layer of H_3^4 as compared to the odd numbers in Pascal's pyramid (Table 4). Note that in layers 0 and 1, the numbers match. However, in layer 2, the number of new vertices in H_3^4 is double that of the odd numbers in Pascal's Pyramid. Even worse, in layer 4, H_3^4 has 28 new vertices, while there are only 3 odd numbers in Pascal's Pyramid. So, there is no extension of the correspondence to include H_3^4.

Table 4: Comparing H_3^4 to Pascal's pyramid.

Layer	New Vertices in H_3^4	Odd numbers in Pascal's Pyramid
0	1	1
1	3	3
2	6	3
3	12	9
4	28	3
5	14	9



(a) Pascal's triangle with odd binomial coefficients encircled.



(b) The Hanoi graph H_3^3 with three disks on three poles.

Figure 8: The connection between the Pascal's triangle and Hanoi graphs on three poles.

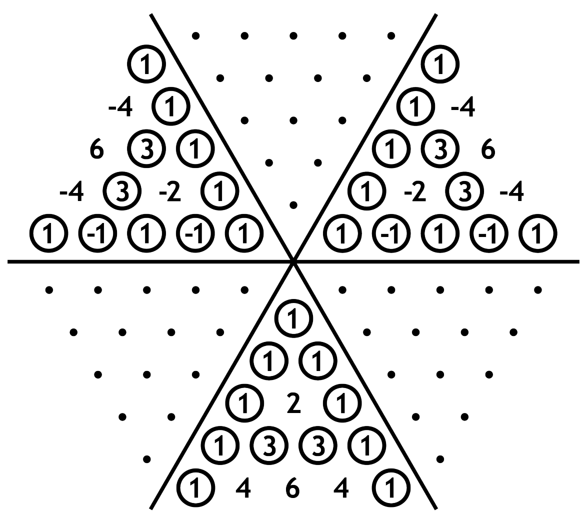


Figure 9: Pascal's Hexagon

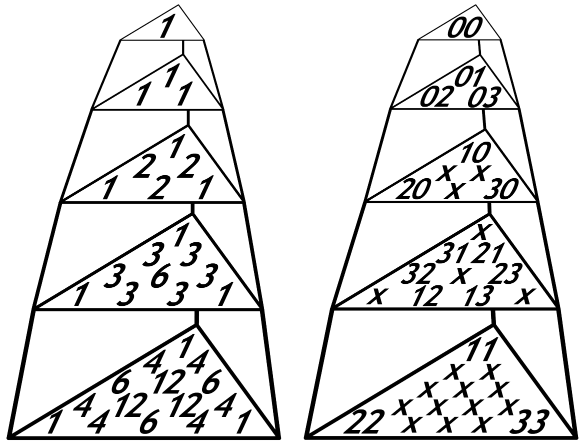


Figure 10: Pascal's Pyramid compared with H_2^4

5 Random Walks

We now explore solving the Tower of Hanoi on four poles with *random* moves; in other words, we will study *random walks* on H_n^4. We will assume that if a given vertex v is of degree n, then the probability of moving from v to any adjacent vertex is $\frac{1}{n}$. The only exception will be that in some cases we will treat the perfect states of the graph as *absorbing states*, which means that the probability of leaving them is 0.

There are many questions that can be addressed regarding random walks on Hanoi graphs, for example:

1. Given a starting vertex and a specified number of moves m, we can ask for the probability that we will be at any given vertex after the m^{th} move. Associated with this is the question of how quickly we approach the vertex.

2. If we designate the perfect states as the absorbing states, then starting from a nonperfect state, we can ask for the probability that we end in each of the perfect states.

As an example of the first question, we consider H_1^4, in which all states are perfect, and consider a random walk beginning at pole 0 and ending at pole 1. Since every vertex in H_1^4 has degree 3, the probability of moving to pole 1 from some other pole is $\frac{1}{3}$. Thus, the probability of arriving at pole 1 in 1 move is $\frac{1}{3}$. Then, the probability of arriving at pole 1 in 2 moves is the probability of *not* arriving in 1 move multiplied by the probability of then arriving at pole 1 in the next move, or $\frac{2}{3} \cdot \frac{1}{3} = \frac{2}{9}$. Similarly, the probability of arriving at pole 1 in 3 moves is $\frac{2}{3} \cdot \frac{2}{3} \cdot \frac{1}{3} = \frac{4}{27}$. Thus, the probability of arriving at pole 1 in m moves given is $\frac{2^{m-1}}{3^m}$. (See Figure 11 for an alternate illustration.)

To arrive at the expected number of moves, we first note that the geometric series

$$\sum_{m=1}^{\infty} \frac{2^{m-1}}{3^m} = \frac{1}{3} + \frac{2}{9} + \frac{4}{27} + \cdots = 1.$$

Now we calculate the expected number of moves:

$$\mathbb{E}(m) = \sum_{m=1}^{\infty} m \cdot \frac{2^{m-1}}{3^m} = 3.$$

Thus, it will take on average three moves on H_1^4 to arrive at the final state of pole 1, starting at pole 0. By symmetry, moreover, the beginning and final poles do not matter; this discussion shows that the expected number of moves to migrate from any given pole to any other given pole is 3, and that the probability of arriving at a given pole in m moves remains $\frac{2^{m-1}}{3^m}$.

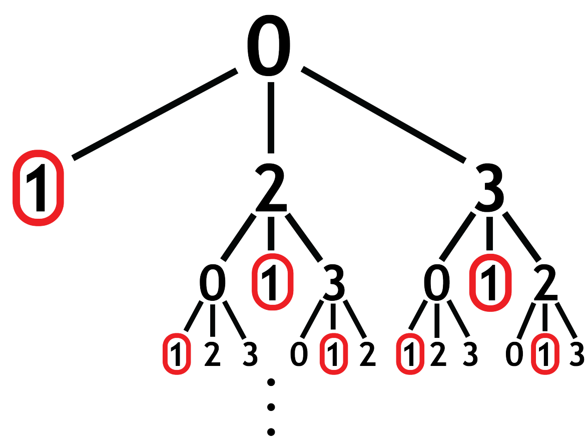


Figure 11: Illustration that the probability of beginning at the perfect state 0 in H_1^4 and ending at the perfect state 1 is $\frac{1}{3} + \frac{2}{9} + \frac{4}{27} + \cdots$.

Next we add a disk and consider random walks on H_2^4. Here, random walks are more complicated because clearly not every move leads to a perfect state. We recall that the vertices of H_2^4 have degree 3 (in the case of a perfect state) or 5 (otherwise); the corresponding probabilities of traversing an edge leading away from the vertex are thus $\frac{1}{3}$ and $\frac{1}{5}$.

We turn to the method of Markov Chains to address question 1 for H_2^4. Throughout the following, we will order the vertices in H_2^4 as

$$\{00, 01, 02, 03, 10, 11, 12, 13, 20, 21, 22, 23, 30, 31, 32, 33\}.$$

The entries of the 16×16 *transition matrix* $\mathbf{P}$, whose ij^{th} entry is the probability that we will move from the i^{th} vertex in the above list to the j^{th} vertex, appear in Figure 12.

$$\begin{bmatrix}
0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{5} & 0 & \frac{1}{5} & \frac{1}{5} & 0 & 0 & 0 & 0 & 0 & \frac{1}{5} & 0 & 0 & 0 & \frac{1}{5} & 0 & 0 \\
\frac{1}{5} & \frac{1}{5} & 0 & \frac{1}{5} & 0 & 0 & \frac{1}{5} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{5} & 0 \\
\frac{1}{5} & \frac{1}{5} & \frac{1}{5} & 0 & 0 & 0 & 0 & \frac{1}{5} & 0 & 0 & 0 & \frac{1}{5} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & 0 & 0 & 0 & \frac{1}{5} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{5} & 0 & \frac{1}{5} & \frac{1}{5} & 0 & \frac{1}{5} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{5} & 0 \\
0 & 0 & 0 & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & 0 & 0 & 0 & 0 & \frac{1}{5} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{5} & 0 & 0 & 0 & 0 & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & 0 & 0 & 0 \\
0 & \frac{1}{5} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{5} & 0 & \frac{1}{5} & \frac{1}{5} & 0 & \frac{1}{5} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{5} & 0 & 0 & 0 & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{5} & 0 & 0 & 0 & \frac{1}{5} & 0 & 0 & 0 & 0 & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\
0 & \frac{1}{5} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{5} & 0 & 0 & \frac{1}{5} & 0 & \frac{1}{5} & \frac{1}{5} \\
0 & 0 & \frac{1}{5} & 0 & 0 & 0 & \frac{1}{5} & 0 & 0 & 0 & 0 & 0 & \frac{1}{5} & \frac{1}{5} & 0 & \frac{1}{5} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0
\end{bmatrix}$$

Figure 12: Transition matrix $\mathbf{P}$ for the random walk H_2^4.

The method of Markov chains nicely answers our first question in terms of the powers $\mathbf{P}^m$ of the transition matrix $\mathbf{P}$. The entry $\mathbf{P}^m_{ij}$ is the probability that the random walk starting at the vertex i^{th} will be at vertex j^{th} after m moves. Calculation shows that taking $m = 25$ gives "steady state" values of the entries of $\mathbf{P}^m$ to at least four decimal places. Based on these values, the probability of ending in any perfect state is 0.0417 and the probability of ending in any nonperfect state is 0.0694 *regardless of the starting position.*

Similarly, when the transition matrix for H^4_1 is raised to powers $m \geq 9$, the probabilities also reach steady state up to at least four decimal places. Each probability is 0.25. Now, if we let the number of disks be 3, the transition matrix for H^4_3 raised to $m \geq 69$ also reaches steady state. All vertices with degree 3, which are the perfect states, have a probability of 0.0089. All vertices with degree 5 have probability 0.0149. Finally, all vertices with degree 6 have probability of 0.0179. Thus, as the degree of the ending vertex increases, so does the probability of arriving at that vertex, which agrees with intuition, since there are more paths to get to the vertex.

Interestingly, there is a pattern of the last vertices to reach steady state. In the transition matrix for H^4_2, there are 12 probabilities that are the last to change. They are the random walks from each perfect state to the vertex where the largest disk is on the same pole, but the smallest moves to a different pole:

$$00 \to 01, \quad 00 \to 02, \quad 00 \to 03$$

and similarly for the starting vertices of 11, 22, and 33.

When examining the transition matrix for H^4_3, however, there are 24 probabilities that are the last to change. These are all perfect states and each of those 24 vertices has degree 6. Further, the middle digit a_1 for these vertices in the ternary representation is the same as that for the starting vertex, but a_2 corresponding to the position of the largest disk, and a_0 corresponding to the position of the smallest disk, are different than a_1 and $a_2 \neq a_0$. One such example is $111 \to 012$.

Finally, we look to answer the second question. Alekseyev and Berger's paper [1] explored five variants of the Tower of Hanoi puzzle with n disks solved with random moves on 3 poles. One such variant is $r \to a$: the starting state is *random* (chosen uniformly from all 3^n states) and the end state is with all n disks on the same (*any*) pole. They were able to prove the following closed formula for this variation:

$$\mathbb{E}_{r\to a} = \frac{5^n - 2\cdot 3^n + 1}{4}.$$

Our goal is to extend their findings to The Tower of Hanoi on four poles. We will additionally stipulate that the random starting point be nonperfect and that the perfect states are absorbing states. The results are found in Table 5. We see that any starting vertex has the highest probability of arriving at the perfect state where the large disk is currently resting. Conversely, any starting vertex has the lowest probability of arriving at the perfect state where the small disk is currently resting. There is then equal probability of arriving at the perfect states where no disk is currently resting.

Table 5: Random walks on H_2^4 with absorbing states 00, 11, 22, and 33 details the probability of ending in each perfect state from all possible starting vertices.

2-bit string	$\mathbb{P}$(ending in 00)	$\mathbb{P}$(ending in 11)	$\mathbb{P}$(ending in 22)	$\mathbb{P}$(ending in 33)
00	1	0	0	0
01	0.4667	0.1333	0.2000	0.2000
02	0.4667	0.2000	0.1333	0.2000
03	0.4667	0.2000	0.2000	0.1333
10	0.1333	0.4667	0.2000	0.2000
11	0	1	0	0
12	0.2000	0.4667	0.1333	0.2000
13	0.2000	0.4667	0.2000	0.1333
20	0.1333	0.2000	0.4667	0.2000
21	0.2000	0.1333	0.4667	0.2000
22	0	0	1	0
23	0.2000	0.2000	0.4667	0.1333
30	0.1333	0.2000	0.2000	0.4667
31	0.2000	0.1333	0.2000	0.4667
32	0.2000	0.2000	0.1333	0.4667
33	0	0	0	1

6 Variation on the Tower of Hanoi: Adjacency Requirement and the Fourth Pole

An additional variation on the Tower of Hanoi with Four Poles is the *adjacency requirement*, which is sometimes referred to as *Straight-line Hanoi* or *Linear Hanoi*. The adjacency requirement is an additional rule requiring a disk to be placed only on an adjacent pole. Figures 13 and 14 show the Hanoi graphs for the adjacency requirement using $n = 2$ and $n = 3$ disks.

Note that in the case where $n = 1$, the Hanoi graph is a line and there is a unique shortest path joining any two vertices. Regardless of n, the first two moves (without retracing one's steps) are moving the smallest disk first to pole 1 and then to pole 2. Before this point, no other disk can be moved. Perhaps it is easiest to observe in Figure 15 that when $n > 1$, there is a unique shortest path in some cases and a non-unique paths in others. Further, length of the shortest path varies with the starting and ending states

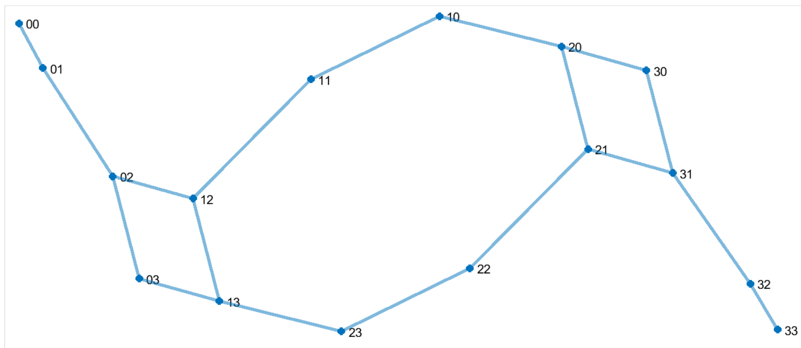


Figure 13: H_2^4 with adjacency requirement (drawn in MATLAB).

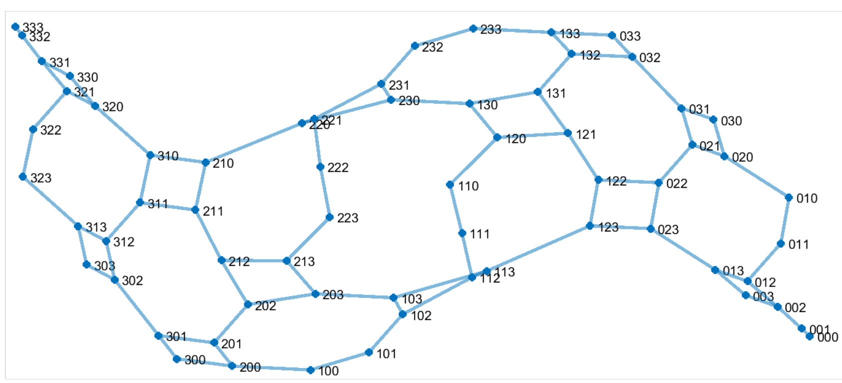


Figure 14: H_3^4 with adjacency requirement (drawn in MATLAB). Note the crossings near the vertices 112 and 113 in the lower center of the graph, reflecting the nonplanarity of this graph.

in the path. Whereas in Figure 3, which depicts H_2^4, the number of moves is the same regardless of the starting and ending perfect states and shortest paths are not unique. Note that the path $00 \to 11$ and $11 \to 00$ will have the same paths and therefore the same number of moves, but with the reverse order of vertices. The same is true for all the paths. These results are summarized in Table 6.

Table 6: Shortest Paths Between Perfect States When $n = 2$.

Path	With Adjacency Requirement		Without Adjacency Requirement	
	Number of Moves	Unique?	Number of Moves	Unique?
$00 \to 11$	4	Yes	3	No
$00 \to 22$	6	Yes	3	No
$00 \to 33$	10	No	3	No
$11 \to 22$	4	No	3	No
$11 \to 33$	6	No	3	No
$22 \to 33$	4	Yes	3	No

We can also examine the vertices, edges, and faces of these graphs. Note that the number of vertices will not change when adding the adjacency requirement, but the number of possible moves, or edges, will change. For $n = 1$, the Hanoi graph is a line, so we count the 3 edges that connect the 4 vertices. Then for $n = 2$ and $n = 3$, we sum the degrees of every vertex and divide by 2, as we have done before (Table 7). When comparing our new Table 7 that has the adjacency requirement to Table 1 for the H_n^4 graphs without the adjacency requirement, we see that the number of edges is exactly half.

Next we count the faces. For $n = 1$, whose corresponding graph is just a line, there is only one face, and that is the infinite face. We verify this by using the Euler-Descartes formula $V - E + F = 4 - 3 + 1 = 2$. Similarly, we can count 4 faces in Figure 13 of the graph H_2^4 and confirm this result with the formula $V - E + F = 16 - 18 + 4 = 2$.

Table 7: V, E, and F for Hanoi graphs with Four Poles and the Adjacency Requirement for $n = 1, 2, 3$.

n	Vertices	Edges	Faces	Planar
1	4	3	1	Yes
2	16	18	4	Yes
3	64	84		No

We return to the question of planarity with the adjacency requirement in force. We can easily embed both the $n = 1$ and $n = 2$ graphs in the plane (see Figure 15 for $n = 2$). However, as Figure 16 shows, $n = 3$ is once again an interesting case.

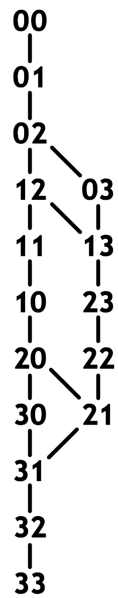


Figure 15: H_2^4 with adjacency requirement

We subject these graphs to two of our planarity tests, considering first the condition that assures planarity if $E \leq 3V - 6$. Referring to Table 7, we find that all three cases pass the test (Table 8). This makes sense because the right side of the inequality is the same as in H_n^4, while the left hand side of the inequality is halved and is supposed to be less than or equal to the right hand side. So the inequality still holds for all cases we tested.

Table 8: Necessary condition for planarity: $E \leq 3V - 6$ for graphs with the adjacency requirement.

n	$E \leq 3V - 6$	Satisfies Condition	Planar
1	$3 \leq 6 = 3(4) - 6$	Yes	Yes
2	$18 \leq 42 = 3(16) - 6$	Yes	Yes
3	$84 \leq 186 = 3(64) - 6$	Yes	No

Second, we examine the crossing numbers of the graphs by examining the graphs themselves. We see that our embedding of the graph with $n = 2$ (and $n = 1$) has a crossing number of 0 (Figure 15), which affirms our statement that these graphs are planar. Now, we look at Figure 16 and count the number of times two edges cross each other to see how close it is to being planar. Here, the crossing number is 1. With no better embeddings in the plane in sight, we can say that this graph is nonplanar.

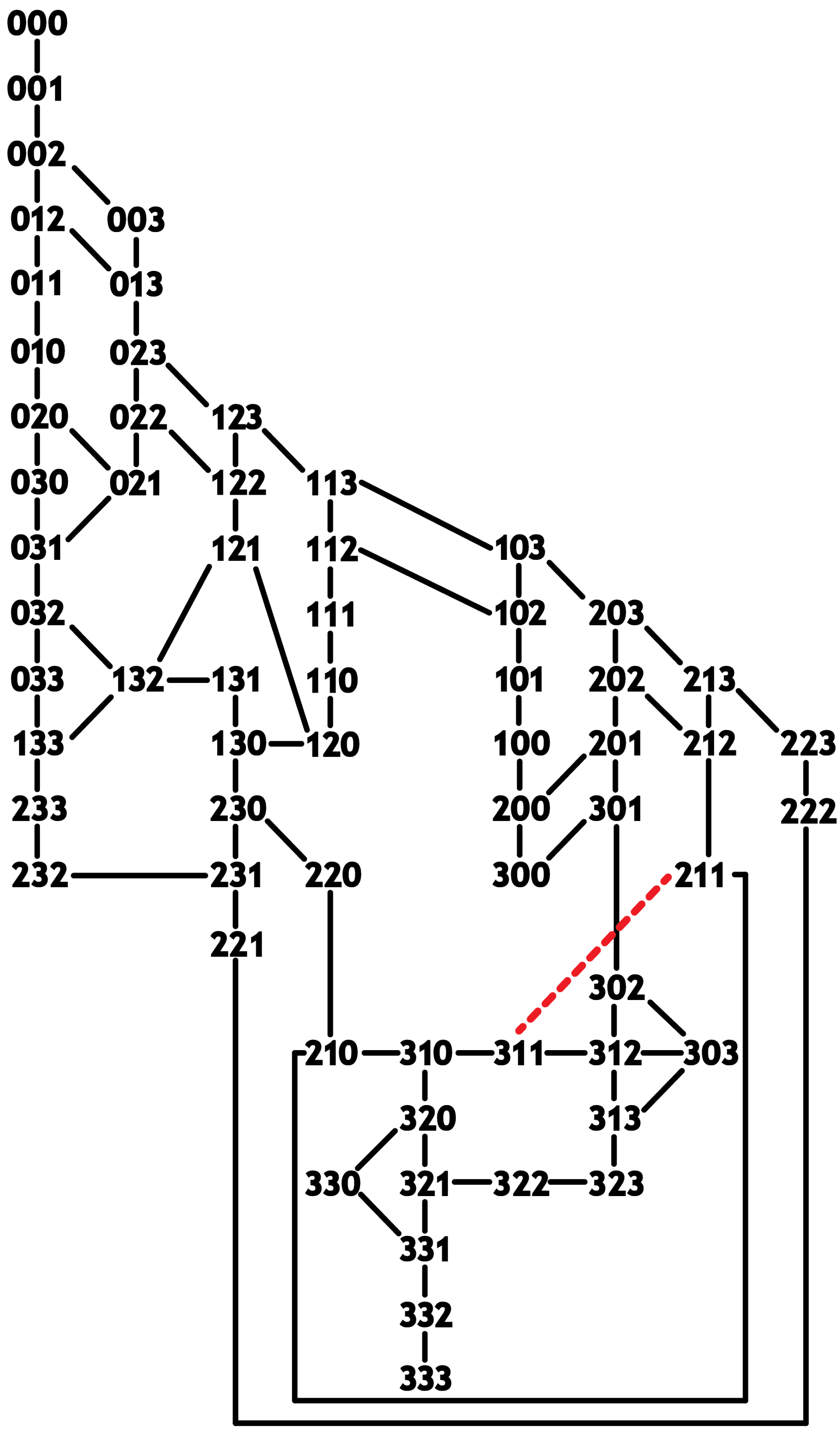


Figure 16: H_3^4 with adjacency requirement (drawn in MATLAB). Note dotted red line, reflecting the nonplanarity of this graph.

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