Rose-Hulman Undergraduate Mathematics Journal

Volume 21
Issue 1

Article 2

Iterated line graphs on bi-regular graphs and trees

Brenden Balch
University of Central Oklahoma, balch@math.colostate.edu

Cover Page Footnote
This project was funded by RCSA and STLR at the University of Central Oklahoma.

Rose-Hulman Undergraduate Mathematics Journal
VOLUME 21, ISSUE 1, 2020

Iterated Line Graphs on Bi-regular Graphs and Trees

By *Brenden Balch*

Abstract. In 1965, van Rooij and Wilf considered sequences of line graphs, in which they grouped sequences of line graphs into four categories. We'll add to their research by presenting results on sequences of line graphs for star graphs and bi-regular graphs. We will then investigate slight variations of star graphs.

1 Introduction

In 1965, van Rooij and Wilf considered sequences of iterated line graphs. Theorem 2 of [1] categorizes these sequences into four distinct cases. We state the theorem here for convenience.

Theorem 1. The sequence of graphs $G, L(G), L^2(G), \ldots$, where G is connected,

1. has ultimately steadily increasing numbers of vertices
2. is of the form $G, G, G, G, \ldots$
3. is of the form $G, L(G), L(G), L(G), \ldots$
4. is of the form $G_1, G_2, \ldots G_n, \varnothing, \varnothing, \varnothing, \ldots$

depending on whether

1. G has at least one $x \in V(G)$ with $deg\ x \geq 3$ and is not S_4,
2. G is a cycle,
3. G is S_4, or
4. G has $deg\ x \leq 2$ and some $deg\ x < 2$.

Mathematics Subject Classification. 05C30
Keywords. line graph, bi-regular graph, trees

The goal of this paper is to investigate part (a) of the theorem in more detail. In particular, we will find the order of the k^{th} line graph of stars with an edge subdivision and describe the structure of iterated line graphs up to third iteration.

We will assume that all graphs are connected, simple, and finite. The **line graph** of a graph G, denoted L(G), is the graph with V(L(G)) = E(G), and the adjacencies of vertices in G represents the adjacencies of the edges in L(G). By an **iterated line graph**, we mean $L^{k+1}(G) = L(L^k(G))$ for $k \geq 1$. Note that an edge, $e = uv$ in G, is adjacent to $(\deg u - 1) + (\deg v - 1)$ other edges in G. We use this fact so frequently that we state it as a lemma and introduce notation to avoid any ambiguities.

Lemma 1.1. *Suppose that* $e = uv$ *is an edge in* G. *Then* $\deg_{L(G)} e = \deg_G u + \deg_G v - 2$.

Furthermore, a vertex in G of degree d contributes $\binom{d}{2}$ (with the convention that $\binom{d}{2} = 0$ for $d = 0, 1$) edges in L(G). Then the number of edges in L(G) is given by Krausz's Lemma [2]:

Lemma 1.2 (Krausz's Lemma)**.** *If the degrees of the vertices of* G *are* $d_1, d_2, \ldots d_n$*, then the number of edges in* L(G) *is* $\sum_{j=1}^{n} \binom{d_j}{2}$.

Using his observation about the edges in a line graph, Krausz was able to characterize line graphs in the following theorem.

Theorem 2 (Krausz's Characterization of Line Graphs)**.** A graph L is the line graph of a connected graph if and only if the edges of L can be partitioned into complete subgraphs with every vertex in at most two of the subgraphs.

In the next section, we aim to produce a generalization of Krausz's Lemma in the setting where the line graph L is regular.

2 Stars, Regular Graphs, and Bi-regular Graphs

Let S_n denote the star graph on n vertices. Since every pair of edges is adjacent, we immediately have that $L(S_n) = K_{n-1}$, which is regular. Motivated by this, let R be a regular graph of order n and of degree d. Then L(R) is a regular graph of degree $d + d - 2 = 2(d-2) + 2$ and order of $\frac{nd}{2} = \frac{n(d-2)}{2} + n = n(2^{-1}(d-2) + 1)$. Induction on k, where k is the k^{th} line graph of R, produces the following:

Proposition 1. Let R be a regular graph with n vertices, each with degree d. Then $L^k(R)$

1. is $2^k d - \sum_{j=1}^{k} 2^j = 2^k(d-2) + 2$ regular.

2. has $n \prod_{j=0}^{k-1} (2^{j-1}(d-2) + 1)$ vertices.

More specifically, we get:

Corollary 1. The k^{th} iterated line graph of K_n

1. is $2^k(n-3)+2$ regular.

2. has $n\prod_{j=0}^{k-1}(2^{j-1}(n-3)+1)$ vertices.

Utilizing the fact that Proposition 1 and $L(S_n) = K_{n-1}$, we can immediately see that $L^k(S_n)$ is $2^{k-1}(n-4)+2$ regular and has $(n-1)\prod_{j=0}^{k-2}(2^{j-1}(n-4)+1)$ vertices. In section 3, we will extend the results of star graphs by considering different variations. However, in order to do this, we will first need to consider bi-regular graphs.

2.1 Bi-regular Graphs

A **bi-regular graph** is a bipartite graph partitioned into two sets, red and blue, such that all red vertices have the same degree and all blue vertices have the same degree. Now, let M be the set of red vertices and N be the set of blue vertices. We will denote these bi-regular graphs by $B_{m,n}(d_m, d_n)$, where m is the order of M, n is the order of N, and d_m and d_n are the degrees of the red and blue vertices, respectively. To avoid ambiguity, we will write $B_{m,n}(d_m, d_n)$ with $m \le n$ (note that if $d_m = d_n$, then the graph is regular). Counting the edges of a bi-regular graph is simple but extremely useful when considering variations of star graphs. Since every edge in B is adjacent to a vertex in M and a vertex in N, it follows that $|E(B)| = \sum deg_M u = (deg_M v)|M|$. Similarly, $|E(B)| = \sum deg_N v = (deg_N v)|N|$. We use this result often in the rest of the paper, so that we state it as a proposition:

Proposition 2. Let B be a nontrivial bi-regular graph with partitions M and N. Suppose $u \in M$ and $v \in N$. Then

$$(deg_M u)|M| = (deg_N v)|N| = |E(B)|.$$

We will now consider the line graph of an arbitrary bi-regular graph.

Proposition 3. Let $B = B_{m,n}(d_m, d_n)$ be any bi-regular graph. Then L(B) is $d_m + d_n - 2$ regular and has mK_{d_m} and nK_{d_n} as subgraphs.

Proof. Let $uv \in E(B)$ and suppose $deg\ u = d_m$. As B is bi-regular, we must have $deg\ v = d_n$. Then $deg\ uv = d_m + d_n - 2$ for every vertex in L(B). Now, the induced subgraph containing u, along with the d_m vertices that are adjacent to u, is S_{d_m+1}. Hence, $L(S_{d_m+1}) = K_{d_m}$. Since there are m vertices of degree d_m it follows that L(B) has mK_{d_m} as subgraphs. A similar argument shows that L(B) contains nK_{d_n} as subgraphs. □

Since the line graph of a bi-regular graph is regular, Proposition 1 applies. Ergo, we arrive at the following corollary.

Corollary 2.

1. $L^k(B_{m,n}(d_m, d_n))$ is $2^{k-1}(d_m + d_n - 4) + 2$ regular.

2. $|L^k(B_{m,n}(d_m, d_n))| = \frac{md_m + nd_n}{2} \prod_{j=0}^{k-2} (2^{j-1}(d_m + d_n - 4) + 1)$.

Proof. This fact follows from Propositions 1 and 3. □

We conclude this section with the following theorem:

Theorem 3. A connected graph G has a regular line graph if and only if G is regular or bi-regular.

Proof. The forward direction is Propositions 1 and 3. For the other direction, assume G is neither regular nor bi-regular. Then G has a vertex u adjacent to two other vertices, say v and w, such that $\deg v \neq \deg w$. It immediately follows from Lemma 1.1 that $\deg_{L(G)} uv \neq \deg_{L(G)} uw$ – meaning L(G) is not regular. □

We now use these basic results for bi-regular graphs to embark on our study of subdivisions of stars.

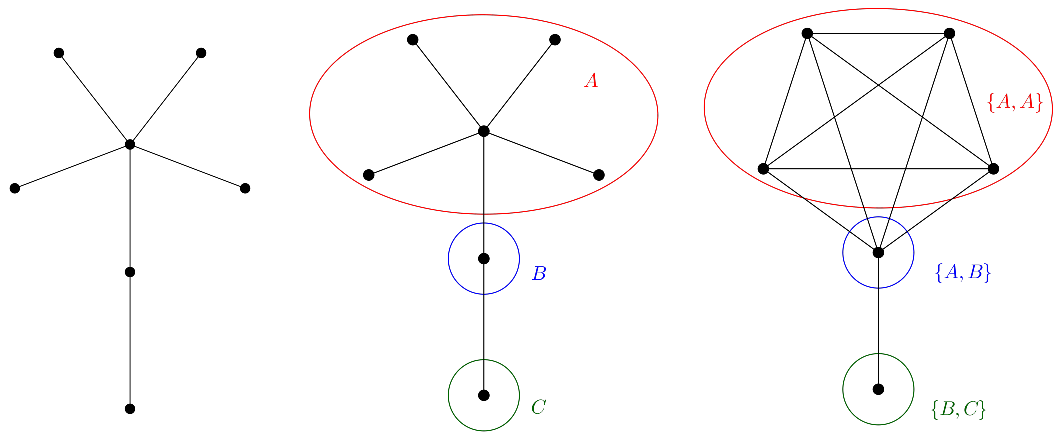
3 Subdivisions of Stars

For a graph G with $u, v \in V(G)$, if u and v are adjacent, we shall write $u \sim v$. Similarly, if $e_1, e_2 \in E(G)$ are adjacent to a common vertex, we will write $e_1 \sim e_2$. The notation will be clear with context.

Utilizing bi-regular graphs, we will now consider pendant subdivisions. Let $H = (W, F)$ be a graph. Suppose that $uw \in F$ and $v \notin W$. Let $G = (V, E)$, where $V = W \cup \{v\}$ and $E = (F - uw) \cup \{uv, vw\}$ is the set obtained from F by replacing uw with two new edges, uv and vw. Then G is a graph obtained from H by **subdividing** uw. A **subdivision** of a graph G is any graph that can be obtained from G by subdividing edges.

We will now consider graphs of the form $G_n = S_n + v$, where $S_n + v$ denotes a subdivision of S_n (Figure 1a). Observe that If the graph G has a pendant vertex and we attach a new vertex $v \notin V(G)$ with an edge, it is clear that this is isomorphic to a subdivision on the pendant vertex. Now, let $A = \{x \in S_n | x \neq w\}$, $B = \{v\}$, and $C = \{w\}$ as illustrated in Figure 1b. We will be looking at the intersection sets of $\{A, B, C\}$ as we take line graphs.

Let P_1 and P_2 be distinct intersection sets of G_n. Then one of two things can happen; either the intersection sets are disjoint, or they intersect. In the former case, if $x \in P_1$ then $x \nsim y$ for all $y \in P_2$. In the latter case, there exists vertices $x \in P_1$ and $y \in P_2$ such that $x \sim y$. We can use these intersection sets to form intersection graphs.



(a) G_6 (b) Partitions of G_6 (c) Partitions of $L(G_6)$

Figure 1

Proposition 4. The vertices of G_n partitioned with intersection sets A, B, and C, as described above, will give intersection graphs whose vertices are regular graphs and edges are bi-regular graphs.

This result follows from the construction of G_n and will be the primary tool for our investigation.

3.1 The First Iterated Line Graph of G_n

The intersection sets of $L(G_n)$ are $\{\{A, A\}, \{A, B\}, \{B, C\}\}$. It should be noted that the sets $\{A, B\}$ and $\{B, C\}$ of $L(G_n)$ are singleton sets due to the subdivision. We also do not have sets like $\{B, B\}$ or $\{C, C\}$ as B and C are singletons with no loops. Moreover, the set $\{A, C\}$ is always empty since the vertices of A and C are never adjacent (Figure 1b). As we continue, we will have additional nonempty, two-element subsets of the k^{th} iterated power set of $\{A, B, C\}$.

Keeping the partitions of the vertices of G_n in mind, we can look at at the intersection sets of the line graph of G_n. This will allow us to describe $L(G_n)$ in terms of its induced subgraphs and bi-regular graphs. First, we note that the set $\{B, C\}$ of $L(G_n)$ contains a single vertex of degree 1, and the set $\{A, B\}$ contains a single vertex of degree $n-1$.

Proposition 5. The structure of $L(G_n)$ is as follows:

1. The set $\{A, A\}$ of $L(G_n)$ contains K_{n-2} as the induced subgraph.

2. The bi-regular graph between $\{A, A\}$ and $\{A, B\}$ is $B_{1,n-2}(n-2, 1)$.

3. The bi-regular graph formed between the sets $\{A, B\}$ and $\{B, C\}$ consists of a single edge.

Proof. 1. The induced subgraph A of G_n is S_{n-1}. Now L(A) corresponds to {A, A}, and because $L(S_{n-1}) = K_{n-2}$, K_{n-2} is the induced subgraph of {A, A}.

2. The set {A, A} has $n-2$ vertices, and the set {A, B} has a single vertex. Now the vertex of {A, B} is adjacent to $n-1$ vertices, $n-2$ of which are from {A, B}. By Proposition 2, we find that the bi-regular graph between {A, A} and {A, B} is $B_{1,n-2}(n-2, 1)$.

3. This follows directly from the construction of G_n.

□

Note that Proposition 5 immediately yields

Corollary 3. $L(G_n)$ has n vertices.

Now that we have completely described the line graph of G_n, we can investigate further by considering $L^2(G_n)$.

3.2 Second Iterated Line Graph of G_n

In a similar manner as the last line graph, {{A, B}, {B, C}} will be a singleton and {{A, B}, {A, B}}, {{B, C}, {B, C}}, and {{A, A}, {A, C}} will all be empty. Let $\hat{A}$ = {{A, A}, {A, A}}, $\hat{B}$ = {{A, A}, {A, B}} and $\hat{C}$ = {{A, B}, {B, C}}.

Proposition 6. The structure of $L^2(G_n)$ is as follows:

1. The subset $\hat{A}$ of $L^2(G_n)$ contains $L(K_{n-2})$ as the induced subgraph.

2. The set $\hat{B}$ of $L^2(G_n)$ contains K_{n-2} as the induced subgraph.

3. The set $\hat{C}$ is a singleton of degree $n-2$.

4. The bi-regular graph formed between $\hat{A}$ and $\hat{B}$ is $B_{n-2, \frac{(n-2)(n-3)}{2}}(n-3, 2)$.

5. The sets $\hat{B}$ and $\hat{C}$ form $B_{1,n-2}(n-2, 1)$.

Proof. 1. Since {A, A} contains K_{n-2} as the induced subgraph and $\hat{A}$ corresponds to the line graph of K_{n-2}, the statement holds.

2. Note that because the bi-regular graph between {A, A} and {A, B} is $B_{1,n-2}(n-2, 1)$, $L(S_{n-1}) = K_{n-2}$ implies $L(B_{1,n-2}(n-2, 1)) = K_{n-2}$.

3. As noted earlier, {A, B} and {B, C} form a single edge, so its line graph is a single vertex. Let $x \in$ {A, B} and $y \in$ {B, C}. We have that $deg_{L(G_n)} x = n-1$ and $deg_{L(G)} y = 1$. Thus, if $e_{xy} = x, y \in V(L^2(G_n))$, $deg_{L^2(G_n)} e_{xy} = n-2$.

4. Let Γ denote the bi-regular graph between $\hat{A}$ and $\hat{B}$. It is clear that the number of vertices in $\hat{A}$ and $\hat{B}$ is $\frac{1}{2}(n-2)(n-3)$ and $n-2$, respectively. By Corollary 1, $L(K_{n-2})$ is $2n-4$ regular. Let $x \in \hat{A}$ and $y \in \hat{B}$. Then it can easily be shown that $deg_{L(G_n)}x = 2n-2$. Hence, $deg_\Gamma x = 2$. Proposition 2 can then be used to find that $deg_{\hat{B}} y = n-3$. Therefore, $\Gamma = B_{n-2, \frac{(n-2)(n-3)}{2}}(n-3, 2)$.

5. This follows directly.

□

To find the number of vertices in this line graph, we just have to add the vertices in each partition.In doing so, we get

Corollary 4. $L^2(G_n)$ has $\frac{1}{2}n^2 - \frac{3}{2}n + 2$ vertices.

3.3 Third Iterated Line Graph of G_n

Next, we consider the third line of G_n. It is at this point things get a little more complicated.

Proposition 7. The structure of $L^3(G_n)$ is as follows:

1. The set $\{\hat{A}, \hat{A}\}$ contains $L^2(K_{n-2})$ as the induced subgraph.

2. The set $\{\hat{A}, \hat{B}\}$ contains

$$L\left(B_{n-2, \frac{(n-2)(n-3)}{2}}(n-3, 2)\right)$$

as the induced subgraph.

3. The set $\{\hat{B}, \hat{B}\}$ contains $L(K_{n-2})$ as the induced subgraph.

4. The set $\{\hat{B}, \hat{C}\}$ contains K_{n-2} as the induced subgraph.

5. The bi-regular graph between the sets $\{\hat{A}, \hat{A}\}$ and $\{\hat{A}, \hat{B}\}$ is

$$B_{(n-2)(n-3), \frac{(n-2)^2(n-3)}{2}}(2n-4, 4)$$

6. The bi-regular graph that is formed between the sets $\{\hat{B}, \hat{B}\}$ and $\{\hat{B}, \hat{C}\}$ is

$$B_{n-2, \frac{(n-2)(n-3)}{2}}\left(\frac{(n-2)(n-3)}{2}, n-2\right).$$

7. The bi-regular graph formed between $\{\hat{A}, \hat{B}\}$ and $\{\hat{B}, \hat{C}\}$ is

$$B_{n-2,(n-2)(n-3)}(1, n-3).$$

8. The bi-regular graph that is formed between $\{\hat{A},\hat{B}\}$ and $\{\hat{B},\hat{B}\}$ is

$$B_{(n-2)(n-3),\frac{(n-2)(n-3)}{2}}(n-3,2(n-3)).$$

Proof. 1. The set $\hat{A}$ is $L(K_{n-2})$, so $\{\hat{A},\hat{A}\}$ corresponds to $L^2(K_{n-2})$.

2. The bi-regular graph between the set $\hat{A}$ and the set $\hat{B}$ is $B_{n-2,\frac{(n-2)(n-3)}{2},}(n-3,2)$. Therefore the set $\{\hat{A},\hat{B}\}$ contains $L\left(B_{n-2,\frac{(n-2)(n-3)}{2}}(n-3,2)\right)$ as the induced subgraph.

3. The set $\hat{B}$ has K_{n-2} as the induced subgraph so that $\{\hat{B},\hat{B}\}$ corresponds to $L(K_{n-2})$.

4. The bi-regular graph formed between sets $\hat{B}$ and $\hat{C}$ is $B_{n-2,1}(1,n-2)$. Observe that this is really just S_{n-1}, so the set $\{\hat{B},\hat{C}\}$ corresponds to K_{n-2}.

5. Let Γ denote the bi-regular graph between $\{\hat{A},\hat{A}\}$ and $\{\hat{A},\hat{B}\}$. By Proposition 1, the set $\{\hat{A},\hat{A}\}$ has $\frac{1}{2}(n-2)^2(n-3)$ vertices. By Proposition 2, the number of edges in $B_{n-2,\frac{(n-2)(n-3)}{2}}(n-3,2)$ is $(n-2)(n-3)$, which is the number of vertices in $L\left(B_{n-2,\frac{(n-2)(n-3)}{2}}(n-3,2)\right)$ and hence, the number of vertices in $\{\hat{A},\hat{B}\}$. A quick calculation shows that if $x\in\{\hat{A},\hat{B}\}$, then $deg_{\{\hat{A},\hat{A}\}}x = 4n-14$. Further, Corollary 1 tells us $deg_{\{\hat{A},\hat{A}\}}x = 2^2(n-3)+2 = 4n-10$. This implies that $deg_\Gamma x = 4$. In light of Proposition 2, we find that $d_n = 2n-4$. Ergo, $\Gamma = B_{(n-2)(n-3),\frac{(n-2)^2(n-3)}{2}}(2n-4,4)$.

6. Let Γ denote the bi-regular graph between $\{\hat{B},\hat{B}\}$ and $\{\hat{B},\hat{C}\}$. First, note that the set $\{\hat{B},\hat{B}\}$ has $\frac{1}{2}(n-2)(n-3)$ vertices and the set $\{\hat{B},\hat{C}\}$ has $n-2$ vertices. Let $x\in\hat{B}$ and $y\in\hat{C}$ such that $e=\{x,y\}$ is an edge. Then e is an edge of the bi-regular graph formed between $\hat{B}$ and $\hat{C}$. The reader will notice that e is adjacent to $\frac{1}{2}(n-2)(n-3)$ edges in the induced subgraph of $\hat{B}$, which is K_{n-2}. The edge e also corresponds to a vertex in $\{\hat{B},\hat{C}\}$ of Γ. Thus the $deg_\Gamma e = \frac{1}{2}(n-2)(n-3)$. By Proposition 2, d_n of Γ is $n-2$. Therefore, $\Gamma = B_{n-2,\frac{(n-2)(n-3)}{2}}\left(\frac{(n-2)(n-3)}{2},n-2\right)$.

7. Let Γ denote the bi-regular graph between $\{\hat{A},\hat{B}\}$ and $\{\hat{B},\hat{C}\}$. There are $(n-2)(n-3)$ vertices in $\{\hat{A},\hat{B}\}$ and $n-2$ vertices in $\{\hat{B},\hat{C}\}$. Let $x\in\hat{B}$ and $y\in\hat{C}$ such that $e=\{x,y\}$ is an edge. Note that e is an edge in the bi-regular graph formed between $\hat{B}$ and $\hat{C}$ and is adjacent to $n-3$ edges in the induced subgraph of $\hat{B}$, which is K_{n-2}. Since e corresponds to a vertex in Γ, with $deg_\Gamma e = n-3$. By Proposition 2, $deg_\Gamma y = 1$. Therefore, $\Gamma = B_{n-2,(n-2)(n-3)}(1,n-3)$.

8. Let Γ denote the bi-regular graph between $\{\hat{A},\hat{B}\}$ and $\{\hat{B},\hat{B}\}$. There are $(n-2)(n-3)$ vertices in $\{\hat{A},\hat{B}\}$ and $\frac{1}{2}(n-2)(n-3)$ vertices in $\{\hat{B},\hat{B}\}$. Let $x\in\hat{A}$ and $y\in\hat{B}$ such that $e=\{x,y\}$ is an edge of the bi-regular graph between $\hat{A}$ and $\hat{B}$. Now, e is adjacent to

$n-3$ edges in the induced subgraph of $\hat{\mathrm{B}}$. Hence, $deg_\Gamma e = n-3$. By Proposition 2, d_n of Γ is $2(n-3)$. Therefore, $\Gamma = \mathrm{B}_{(n-2)(n-3),\frac{(n-2)(n-3)}{2}}(n-3, 2(n-3))$.

□

Utilizing Corollary 1 shows that the set $\{\hat{\mathrm{A}}, \hat{\mathrm{A}}\}$ contains $\frac{1}{2}(n-2)(n-3)(n-4)$ vertices. By Corollary 2, we immediately see that $|\{\hat{\mathrm{A}}, \hat{\mathrm{B}}\}| = (n-2)(n-3)$ Observe that the set $\hat{\mathrm{B}}$ contains $\frac{1}{2}(n-2)(n-3)$ vertices, and that clearly $\{\hat{\mathrm{B}}, \hat{\mathrm{C}}\}$ has $n-2$ vertices. We are now in a position to calculate the order of $\mathrm{L}^3(\mathrm{G}_n)$.

Corollary 5. $\mathrm{L}^3(\mathrm{G}_n)$ has $\frac{1}{2}n^3 - 3n^2 + \frac{13}{2}n - 5$ vertices.

Computing the fourth line graph of G_n is possible; however, it would be extremely messy and time consuming with the techniques that have been utilized in this paper. Consequently, we are currently unable to find a nice way of writing k^{th} line graph of G_n. Nonetheless, by doing the first three iterations we have a way of understanding their structure through partitions.

References

[1] Rooij, A.C.M. van and Wilf, H.S. The interchange graph of a finite graph. *Acta Mathematica Academiae Scientiarum Hungaricae*, **16**(3–4):263–269, 1965.

[2] Krausz, József. Démonstration nouvelle d'une théoreme de whitney sur les réseaux. *Mat. Fiz. Lapok*, 50(1):75–85, 1943.

Brenden Balch
University of Central Oklahoma
balch@math.colostate.edu