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Abstract. Constructing a regular quadrilateral (square) and circle of equal area was proved impossible in Euclidean geometry in 1882. Hyperbolic geometry, however, allows this construction. In this article, we complete the story, providing and proving a construction for squaring the circle in elliptic geometry. We also find the same additional requirements as the hyperbolic case: only certain angle sizes work for the squares and only certain radius sizes work for the circles; and the square and circle constructions do not rely on each other.

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1 Introduction

In the *Rose-Hulman Undergraduate Math Journal*, 15 **1** 2014, Noah Davis demonstrated the construction of a hyperbolic circle and hyperbolic square in the Poincaré disk [1]. This type of construction, usually called, "squaring the circle," was proved impossible in Euclidean geometry in 1882. In this paper, we consider the problem in elliptic geometry and show that, as in hyperbolic, squaring the circle is possible. In Euclidean geometry exactly one line exists parallel to a given line through a point not on that given line, while in hyperbolic geometry many lines through a given point not on a given line exist which are parallel to the given line. In elliptic geometry, however, there are no lines parallel to a given line that pass through a point not on the line. Elliptic points exist in and on a disk of Euclidean points whose boundary is the Euclidean circle with center O, radius 1. So that two points determine one elliptic line, we have to stipulate that points which lie on the ends of a diameter are treated as the same point. These pairs of points are called antipodal points. Elliptic lines come in two forms, as shown in Figure 1. The first is simply a diameter of circle O. The second is an arc of a circle inside or on circle O which passes through antipodal points. When exactly one line exists parallel to a given line through a point not on that given line, we are in Euclidean geometry. In hyperbolic geometry, many lines through a given point not on a given line exist which are parallel to the given line. The third version states that through a point not on a given line, there are no lines parallel to the given line. This geometry is sometimes called elliptic geometry and we can square the circle here as well as hyperbolic. In this article, we present one such construction.

In Figure 1, we have elliptic line $\overleftrightarrow{OA}$, which is a Euclidean diameter, and elliptic line $\overleftrightarrow{AD}$. The point C is the Euclidean center of the Euclidean circle containing elliptic line $\overleftrightarrow{AD}$.

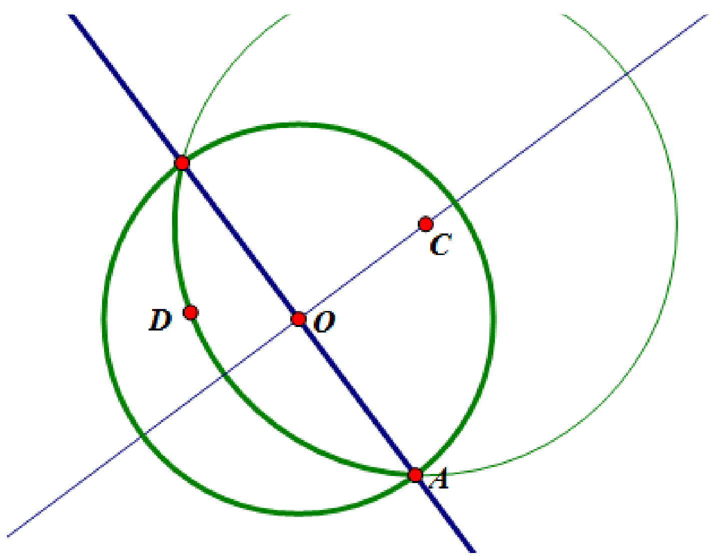

Figure 1. Elliptic lines $\overleftrightarrow{OA}$ and $\overleftrightarrow{AD}$.

Our regular quadrilateral, or square, will have sides on elliptic lines like $\overleftrightarrow{AD}$. We measure elliptic angles using tangents and that is how we will control the size of our vertex angles. To summarize, we will use the usual compass and straightedge to build Euclidean objects which have the elliptic interpretation of being a regular quadrilateral and circle with the same elliptic area.

Elliptic circles are just Euclidean circles. Unless the Euclidean center (the center we use for drawing the circle) is O, the Euclidean center is not the elliptic center. We will build our circles with center O to avoid this complication. One complication we cannot avoid is the preparation. We will see that squaring the circle has a strange property in both non-Euclidean geometries: we cannot build the circle from the square nor the square from the circle. This was a known property of hyperbolic geometry, as explained in both Davis [1] and Jagy [2]. The reason it is strange is that compass and straightedge constructions usually build on the given. For instance, given a Euclidean triangle, the perpendicular bisectors of two sides intersect at the circumcenter. Placing the point of the compass at the circumcenter and the pencil at any vertex, we construct the circumcircle. But when squaring the circle, even if the elliptic square is given, we must perform calculations and build the circle of equal area with no help from the square! And, depending the size of the vertex angles of the square, the circle might not even be constructible. Luckily, elliptic squares choose their vertex angle sizes between $\frac{\pi}{2}$ and π, with countably infinite many sizes available for construction. There's one more complication: an elliptic segment has both Euclidean size and elliptic size, and these only match for the degenerate segment of zero length.

We perform the calculations necessary for our construction at the start of the next section, in which we construct an elliptic square with known elliptic area. In the third section, we construct the elliptic circle with the same elliptic area and thus square the circle. We close with the implications of the surprising constraint that the square and the circle must be constructed without reference to each other.

2 Constructing the square

Before drawing anything, we need to perform the necessary calculations. The elliptic area formula for an elliptic square with vertex angle A is area $= 4A - 2\pi$. The area formula for the circle with elliptic radius r is $2\pi(1 - \cos r)$. An elliptic square and circle are shown in Figure 2.

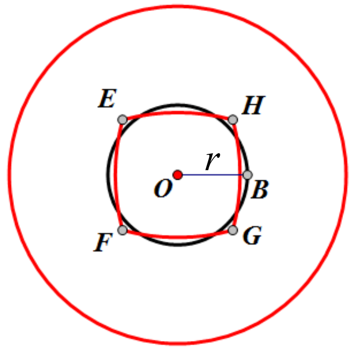

Figure 2. Elliptic square and circle.

Now, let's find an A and an r which will give us success. Since $A > \pi/2$, let's try $A = 2\pi/3$. Our two formulas imply that the area of the square will be $2\pi/3$ (just a coincidence) and we will need $\cos r = 2/3$. The procedures which follow have no particular mathematical merit except for the fact that they perform as required. The earnest do-it-yourselfer could think up new ways to make these sizes.

We are going to use a clever trick: we will build the square before we draw the boundary circle for elliptic space! This gives us the freedom to think of a way to get four angles of $2\pi/3$ each and we'll adjust the size of the boundary so that the Euclidean circles containing the sides pass through antipodal points. We will prove later that we should begin with an angle $A/8$, which is $\pi/12$. So as shown in Figure 3, we place a 30 degree angle, the undrawn $\angle HCO$ and bisect it, constructing $\angle OEC$. We mark $\overline{OD} \cong \overline{OC}$ so we can make the perpendicular bisector of the segment $\overline{CD}$, which intersects $\overrightarrow{CE}$ at E. The point E will be one of our four vertices of the elliptic square.

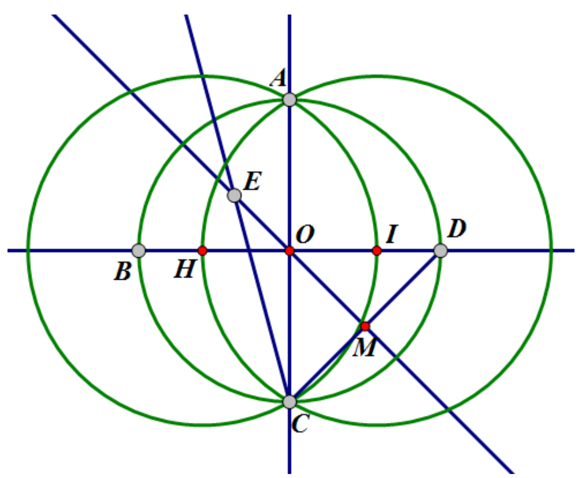

Figure 3. Elliptic square begun.

Reflect point E into the other three quadrants and these are the vertices of the desired elliptic square. Using the points A, B, C, D as centers of circles through F, G, H, E, respectively, we construct elliptic lines containing the sides of the elliptic square $EFGH$, as shown in Figure 4A.

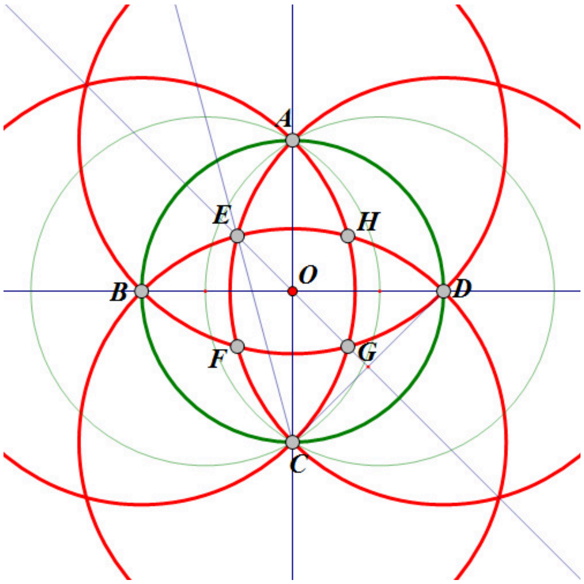

Figure 4A. Elliptic square complete.

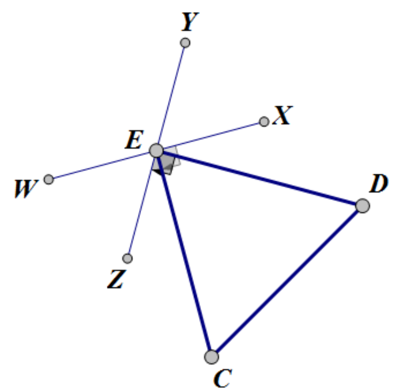

Figure 4B. Vertex angle E measurement.

We pause our construction to prove that each vertex angle of $EFGH$ has measure $2\pi/3$; but we will use $A/8 = \angle ECO$ in order to prove that our construction works in general. The isosceles triangle CDE has base angles $\frac{\pi}{4} + \frac{A}{8}$. This gives us $\angle CED = \frac{\pi}{2} - 2\frac{A}{8} = \frac{\pi}{2} - \frac{A}{4}$. By construction, in Figure 4B, we see $\angle ZEC = \angle XEC = \frac{\pi}{2} - \left(\frac{\pi}{2} - \frac{A}{4}\right) = \frac{A}{4}$. Since we measure angles by tangents, we measure the angle of the elliptic square at vertex E as $\frac{A}{4} + \frac{\pi}{2} - \frac{A}{4} + \frac{A}{4} = \frac{\pi}{2} + \frac{A}{4}$. For $A = \frac{2\pi}{3}$, $\angle E = \frac{\pi}{2} + \frac{1}{4}\left(\frac{2\pi}{3}\right) = \frac{2\pi}{3}$. This is the desired size in general because the elliptic square constructed in this way will have elliptic area $4\left(\frac{\pi}{2} + \frac{A}{4}\right) - 2\pi = A$, our desired elliptic area.

3 Constructing the circle

Having constructed our elliptic square, we now have to construct an elliptic circle with the same elliptic area. In the remaining figures, we have left the elliptic square visible, even though it does not help us in our construction of the circle. In order to have a circle centered at O with elliptic area $2\pi/3 = A$, we need an elliptic radius r such that $2\pi/3 =$

$A = 2\pi(1 - \cos r)$, so $\cos r = 2/3$. Figure 5A shows the key for achieving this radius: the elliptic triangle AOR which has right angles at O and R. The elliptic trigonometric formula $\cos A = \cos AOR \cos ARO + \sin AOR \sin ARO \cos OR$ simplifies to $\cos A = \cos OR$. So if we want $\cos r = \cos OR = 2/3$, we construct the Euclidean $\angle OAD$ such that $\cos OAD = 2/3$. By coincidence this time, the point D turns out to be on the boundary circle; in general, the point D lies on the horizontal line wherever we need it to be. The $\angle BAD$ has to be $\pi/2$ so that we may measure elliptic $\angle OAR$ using Euclidean $\overleftrightarrow{AD}$ as the tangent. Remember, we treat the elliptic disk as a Euclidean unit disk, so $OA = 1$, Euclidean measure. We also know $\angle OAD = \angle ABO$. Since we want $\cos ABO = 2/3$, we need $\overline{BO}$ to have Euclidean length $2/\sqrt{5}$. For other attempts at squaring the circle, this line of reasoning will work, provided the the desired radius is constructible.

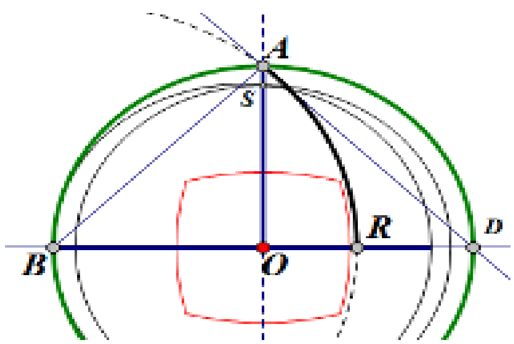

Figure 5A. Construction for radius.

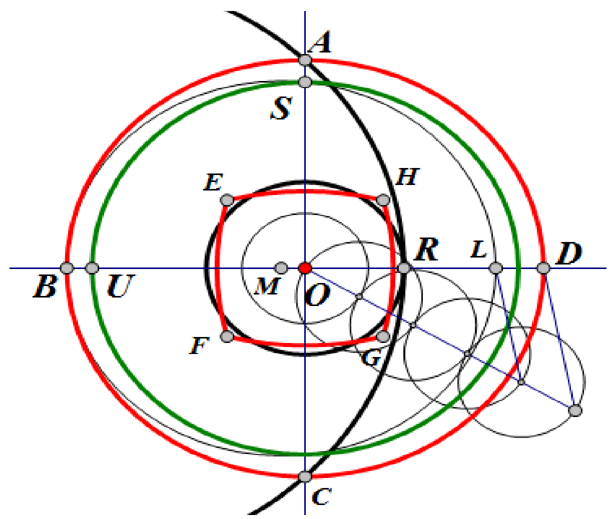

Figure 5B. The radius $\overline{OR}$.

In Figure 5B, the construction of the Euclidean length $2/\sqrt{5}$ begins with constructing an auxiliary ray $\overrightarrow{ON}$, cut into five equal pieces. Similar triangles gives the Euclidean length $OL = 4/5$. Since $OB = 1$, $OS = \sqrt{4/5} = 2/\sqrt{5}$. The circle with center O and radius $\overline{OS}$ intersects the horizontal axis at U. The arc with center U through A intersects the horizontal axis at point R so that $\overline{OR}$ has the required size to satisfy $\cos OAD = 2/3$. Figure 6 shows the finished product with construction marks for radius $\overline{OR}$ and we see an elliptic square and circle with the same elliptic area. We note that the circle cuts the corners of the square in the way we would expect.

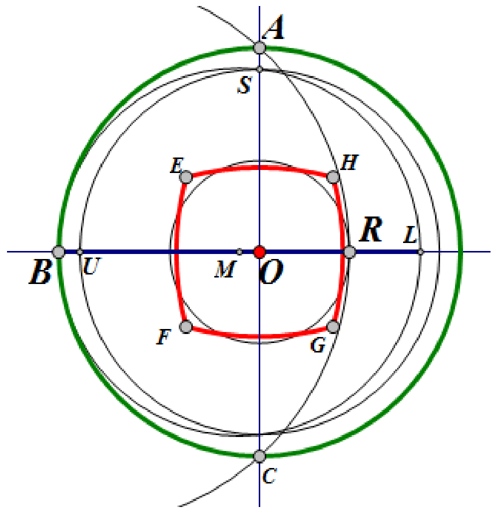

Figure 6. Elliptic square and circle.

Squaring the circle in hyperbolic shares some strategy with the work here: control of vertex angle size and use of Euclidean trigonometry to handle radius size. McDaniel [3] has the hyperbolic and elliptic squaring the circle presented together. Now that we have our constructions proved and an elliptic example done, we consider the general case.

4 The constraints

Squaring the circle excited mathematicians for millennia and even as we finish the story, we get a spicy twist at the end. Both Davis [1] and Jagy [2] explain some complexities of squaring the circle in hyperbolic geometry which we will prove happen when we square the circle in elliptic geometry. The first difficulty informed our constructions: the circle and the square have to be constructed without reference to each other. The second surprise is, not every constructible radius corresponds to a constructible vertex angle, and vice-versa. We will prove both these claims at the same time as follows.

Suppose we could construct the square from the circle. The constructible Euclidean radius $x = 19/\sqrt{427}$ corresponds to $\cos B = 19/28$, in the notation of our construction. Solving for the vertex angle, we find $\frac{9\pi}{14} = \frac{\pi}{2} + \frac{\pi}{7}$. The angle $\pi/7$ is not constructible. (Its construction would lead to the construction of a regular heptagon.) The unusual size $19/\sqrt{427}$ was calculated ahead of time so $9\pi/14$ would happen and it serves as a warning: not just any constructible length x will do!

Now, suppose we could construct the circle from the square. Let the vertex $\angle A = \pi/2 + \arctan 3$. The area of the square would be $4 \arctan 3$. Now we consider the circle of equal area as if we have constructed it: $4 \arctan 3 = 2\pi(1 - \cos r)$. Then $\cos r = 1 - \frac{2 \arctan 3}{\pi}$. From Theorem 1 in an article by Margolius[4], $\frac{\arctan 3}{\pi}$ is transcendental. Yet, from our construction proved above, we could construct $\cos r$ from constructible $\overline{OR}$. The constructible lengths, however, are algebraic, not transcendental. Therefore, we cannot construct the circle from the square.

The calculation of $\cos r$ denies us the use of constructible vertex angles which we could get from Pythagorean triples, and all the other arctangents of rational numbers except for $\frac{\pi}{2} + \arctan 1$. That is, we could construct the elliptic squares with such angles, but we are guaranteed to fail in constructing the elliptic circle of equal area. Similarly, any non-constructible Euclidean angle denies us the use of what could be (depending on the choice of angle - there's lots of ways to be non-constructible) a constructible radius. In summary, for those seeking vertex angles which will work, the angles $\frac{\pi}{2} + \theta$ where θ is a constructible modification (like angle bisection) of $\pi/4$ or $\pi/6$ cooperate, as we can see from our formulas in Section 3: square roots of integers are constructible. Vertex angles closer to $\pi/2$ have a little bit more workspace between the circle and square, but the figures themselves will be small: start with a big piece of paper so the Euclidean unit circle can be spacious.

Geometers maintain a list of geometrical characteristics which separate the non-Euclidean geometries from Euclidean geometries. For example, in non-Euclidean geometry, the sum of the angles of a triangle is not π, two triangles with three pairs of congruent angles are congruent, and not all triangles have circumcircles. To this list we add: we can square the circle.

References

[1] N. Davis, *Rose-Hulman Undergraduate Math Journal*, 15 **1** 2014.

[2] W. C. Jagy, Squaring circles in the hyperbolic plane, *The Mathematical Intelligencer,* **17**:31 - 36 (1995).

[3] M. McDaniel, *Geometry by Construction,* Universal Publishers, 2015.

[4] B. H. Margolius, Plouffe's constant is transcendental, www.lacim.uqam.ca/˜plouffe/articles/plouffe.pdf.