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Abstract. In number theory, difference tables can be constructed to develop and visualize patterns within a given sequence that can sometimes lead to constructing a closed form representation of the sequence. Analyzing patterns in difference tables and corresponding diagonals for various classes of functions, including exponential functions and combinations of these exponentials, reveals a significant amount of structure. This paper serves to explain some of the most notable patterns along diagonals of difference tables and difference tables formed from the diagonals of the previous difference tables of a given sequence of numbers. These diagonals, which can be manipulated to find closed forms for functions of the form $f(n) = n^k j^n$ where k and j are natural numbers by using a corresponding leading coefficient on a polynomial equivalent to the $(j-1)$th diagonal, can be found with a simple algorithm.

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1 Introduction

Difference tables are often used in elementary number theory to explore patterns in sequences of integers. Suppose we have a function, $f(n)$, where n is a nonnegative integer. A difference table includes values for n and $f(n)$ in its first two rows. The entry in the nth column of the third row is defined as $f(n)-f(n-1)$ for all $n \geq 1$ with the convention that the first entry of the third row is 0. Values in each subsequent row are formed by similarly taking differences of the values in the preceding row, adding leading 0s as placeholders. For example, the difference table for $f(n) = n^2 + 4$ is shown in Figure 1.

n	0	1	2	3	4	5	6	7
$f(n)$	4	5	8	13	20	29	40	53
Δ^1	0	1	3	5	7	9	11	13
Δ^2	0	0	2	2	2	2	2	2
Δ^3	0	0	0	0	0	0	0	0

Figure 1: Difference table for $f(n) = n^2 + 4$.

Notice that this difference table eventually culminates in a row with constant values. We will call this the first constant row (FCR). It turns out that this is true of all polynomial functions, and that the polynomial can be reconstructed from the difference table in the following way. The degree of the polynomial's first term is the number of differences it takes to create the FCR, and the leading coefficient is the FCR's value divided by the degree factorial [2]:

$$\begin{aligned}\text{Degree} &= \text{Position FCR}\\ \text{Leading Coefficient} &= \text{Value FCR / Degree!}\end{aligned}$$

Using these recursively, after determining the degree and leading coefficient from the original difference table, we can construct a new difference table by subtracting the value of the leading term from the original $f(n)$ sequence for each value of n. So in general, if $f(n) = a_n j^n + a_{n-1} j^{n-1} + \cdots + a_1 j + a_0$, we create the difference table for $f(n) - a_n j^n = a_{n-1} j^{n-1} + \cdots + a_1 j + a_0$, allowing us to find the next nonzero a_i. We repeat this process to find all coefficients of $f(n)$.

For example, applying this process to the difference table in Figure 1 shows that the first term in $f(n)$ is n^2, because the degree is the position of the FCR (two in this case), and the leading coefficient is the value of the FCR (two) divided by the degree factorial (again two), yielding 1. Since the first term is n^2, we subtract those values from $f(n)$, resulting in $n^2 + 4 - n^2 = 4$. So the next difference table will have fours in the first row. The next term we find is four, so we subtract four from what remains of $f(n)$: $4 - 4 = 0$ and the process is complete. For more information, see Cuoco's *Mathematical Connections* [2].

In this paper, we extend the study of difference tables from polynomials to exponential functions. Unlike difference tables for polynomials, exponential difference tables do not reduce to a FCR. For example, the start of the difference table for $f(n) = 4^n$ is shown in Figure 2.

n	0	1	2	3	4	5	6	7
$f(n)$	1	4	16	64	256	1024	4096	16384
Δ^1	0	3	12	48	192	768	3072	12288
Δ^2	0	0	9	36	144	576	2304	9216
Δ^3	0	0	0	27	108	432	1728	6912
Δ^4	0	0	0	0	81	324	1296	5184
Δ^5	0	0	0	0	0	243	972	3888
Δ^6	0	0	0	0	0	0	729	2916

Figure 2: Difference table for $f(n) = 4^n$.

We will show that Pascal's triangle appears in the difference table of any function, and this will lead us to a way to characterize the difference table of functions of the form $f(n) = a^n$, $f(n) = a^n + c2^n$, and $f(n) = n^k j^n$. Specifically, we'll start in Section 2 by showing the following theorem:

Theorem 1. *Let f be a function on the nonnegative integers with $f(i) = a_i$. Coefficients on a_i within the difference table for f correspond to Pascal's triangle along every diagonal of a difference table, up to a negative sign. More precisely, the coefficient on a_i in column n, row Δ^r is $(-1)^{(n-i)} \begin{pmatrix} r \\ n-i \end{pmatrix}$, where the row corresponding to Δ^0 is $f(n)$ itself.*

n	0	1	2	3	4 ...
$f(n)$	a_0	a_1	a_2	a_3	a_4
Δ^1		$a_1 - a_0$	$a_2 - a_1$	$a_3 - a_2$	$a_4 - a_3$
Δ^2			$a_2 - 2a_1 + a_0$	$a_3 - 2a_2 + a_1$	$a_4 - 2a_3 + a_2$
Δ^3				$a_3 - 3a_2 + 3a_1 - a_0$	$a_4 - 3a_3 + 3a_2 - a_1$
Δ^4					$a_4 - 4a_3 + 6a_2 - 4a_1 + a_0$

Figure 3: Difference table for an arbitrary function f with $f(i) = a_i$.

This theorem is illustrated by the difference table in Figure 3. Comparing with Pascal's triangle in Figure 4, we see the pattern of the coefficients of each entry in Figure 3 follows that of the rows of Pascal's triangle, with the appropriate negative signs included. As corollaries, we will obtain the following characterizations. Here $Diag(n)$ refers to the nth entry on the main non-zero diagonal through the difference table:

$$
\begin{array}{cccccccccccc}
 & & & & & \binom{0}{0} & & & & & \\
 & & & & \binom{1}{0} & & \binom{1}{1} & & & & \\
 & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & \\
 & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & \\
 & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} & \\
\binom{5}{0} & & \binom{5}{1} & & \binom{5}{2} & & \binom{5}{3} & & \binom{5}{4} & & \binom{5}{5}
\end{array}
$$

Figure 4: Pascal's Triangle.

Corollary 1. *For n a nonnegative integer, $f(n) = a^n$ if and only if $Diag(n) = (a-1)^n$.*

Corollary 2. *For n a nonnegative integer, $f(n) = a^n + 2^n c$ if and only if $Diag(n) = (a-1)^n + c$.*

We continue in Section 3 by considering a "diagonals of diagonals" approach to characterize difference tables of functions of the form $f(n) = n^k j^n$. "Diagonals of diagonals" is the process of taking a difference table and pulling the diagonal values from it to construct the first line to in a new difference table from which we pull the diagonals to again create another difference table, and so on. This leads to the following conjecture, that for any unknown function of the form $f(n) = n^k j^n$, there exists a unique polynomial which represents the values found along the $(j-1)$th diagonal of the unknown function. Furthermore, this unique polynomial's leading term will always be $j^k n^k$. We conclude in Section 4 with a few thoughts on further research.

2 Analysis

We begin this section with a proof of Theorem 1 followed by proofs of Corollaries 1 and 2. It may be helpful to refer to Figure 3 as an example and visual for parts of this proof.

Proof of Theorem 1. We proceed by induction:

Let n denote the column of the difference table and r the row in the difference table corresponding to Δ^r. We need to show that the coefficient on a_i is $(-1)^{(n-i)} \left(\begin{array}{c} r \\ n-i \end{array} \right)$.

Base Case: The formula $(-1)^{(n-i)}\begin{pmatrix} r \\ n-i \end{pmatrix}$ defines the coefficients on a_i for all values of i and n in row zero, since

$$(-1)^{(n-i)}\begin{pmatrix} r \\ n-i \end{pmatrix} = (-1)^{(n-i)}\begin{pmatrix} 0 \\ n-i \end{pmatrix} = \begin{cases} 1, \text{ where } n = i \\ 0, \text{ otherwise.} \end{cases}$$

Inductive Case: Assume the formula works for rows 1 through r. We want to show the a_i coefficient in column n, row $r+1$, is $(-1)^{(n-i)}\begin{pmatrix} r+1 \\ n-i \end{pmatrix}$. Using the method by which difference tables are constructed, we see that this means we want to show

$$(-1)^{(n-i)}\begin{pmatrix} r+1 \\ n-i \end{pmatrix} a_i = (-1)^{(n-i)}\begin{pmatrix} r \\ n-i \end{pmatrix} a_i - (-1)^{((n-1)-i)}\begin{pmatrix} r \\ (n-1)-i \end{pmatrix} a_i.$$

Substituting $\begin{pmatrix} k \\ j \end{pmatrix} = \frac{k!}{(k-j)!j!}$, and dividing by $a_i(-1)^{(n-i)}$, we have

$$\frac{(r+1)!}{(r+1-(n-i))!(n-i)!} = \frac{r!}{(r-(n-i))!(n-i)!} - (-1)\frac{r!}{(r-(n-1-i))!(n-1-i)!}.$$
$$\frac{(r+1)r!}{(r+1-(n-i))!(n-i)!} = \frac{r!}{(r-(n-i))!(n-i)!} + \frac{r!(n-i)}{(r+1-(n-i))!(n-i)!}.$$

Upon subtraction, what we need to show is

$$\frac{(r+1)r! - r!(n-i)}{(r+1-(n-i))!(n-i)!} = \frac{r!}{(r-(n-i))!(n-i)!}.$$

But this is in fact true because

$$\frac{r!(r+1-(n-i))}{(r+1-(n-i))!(n-i)!} = \frac{r!}{(r+1-(n-i)-1)!(n-i)!} = \frac{r!}{(r-(n-i))!(n-i)!}.$$

$\square$

We will now prove Corollaries 1 and 2, showing that Theorem 1 results in a new method for finding closed form solutions to exponential sequences.

Proof of Corollary 1. This follows from creating a difference table from the function $f(n) = a^n$ and factoring the resulting diagonal entries:

n	0	1	2	3	4
$f(n)$	1	a	a^2	a^3	a^4
Δ^1	0	$a-1$	a^2-a	a^3-a^2	a^4-a^3
Δ^2	0	0	a^2-2a+1	a^3-2a^2+a	$a^4-2a^3+a^2$
Δ^3	0	0	0	a^3-3a^2+3a-1	$a^4-3a^3+3a^2-a$
Δ^4	0	0	0	0	$a^4-4a^3+6a^2-4a+1$

$$\begin{aligned} a-1 &\Rightarrow (a-1)^1 \\ a^2-2a+1 &\Rightarrow (a-1)^2 \\ a^3-3a^2+3a-1 &\Rightarrow (a-1)^3 \\ a^4-4a^3+6a^2-4a+1 &\Rightarrow (a-1)^4 \end{aligned}$$

Clearly, the function $f(n) = (a-1)^n$ exhibits the same values showing up along the first diagonal of the difference table for $f(n) = a^n$. Theorem 1 and the Binomial Theorem combine to ensure that this pattern continues indefinitely.

□

The power function a^n is fairly easy to recognize without the use of difference tables, but Corollary 2 describes a case where a more complicated function can be identified using difference tables.

Proof of Corollary 2. Since difference tables are closed under addition and scalar multiplication, we can consider the above terms individually. (These functions work similarly to how we broke down the polynomials term by term using difference tables.) Applying Corollary 1 to a^n results in $(a-1)^n$ on the diagonal. Similarly, applying Corollary 1 to 2^n gives $(2-1)^n = (1)^n = 1$ on the diagonal. Multiplying by 2^n by c and combining the two terms gives the result.

□

3 Diagonals of diagonals

In this section we present a method for analyzing functions of the form $f(n) = n^k j^n$ where k and j are any combination of natural numbers. The idea for this function comes from Corollary 2: imagine 2^n in Corollary 2 being j^n and c in Corollary 2 being n^k.

We analyze $f(n) = n^k j^n$ using an approach we call "diagonals of diagonals." Diagonals of diagonals is the process of taking a difference table and pulling the diagonal values from it to construct another difference table from which we pull the diagonals to again create another difference table. Here is an example of "diagonals of diagonals." We start with the difference table for $f(n) = n^2 \cdot 3^n$:

n	0	1	2	3	4	5	6
$f(n)$	0	3	36	243	1296	6075	26244
Δ^1	0	3	33	207	1053	4779	20169
Δ^2	0	0	30	174	846	3726	15390
Δ^3	0	0	0	144	672	2880	11664
Δ^4	0	0	0	0	528	2208	8784
Δ^5	0	0	0	0	0	1680	6576
Δ^6	0	0	0	0	0	0	4896

Now we take the diagonal elements: 0, 3, 30, 144, ..., and then make a new difference table:

n	0	1	2	3	4	5	6
$f(n)$	0	3	30	144	528	1680	4896
Δ^1	0	3	27	114	384	1152	3216
Δ^2	0	0	24	87	270	768	2064
Δ^3	0	0	0	63	183	498	1296
Δ^4	0	0	0	0	120	315	798
Δ^5	0	0	0	0	0	195	483
Δ^6	0	0	0	0	0	0	288

Repeating the process again yields a polynomial difference table. This example indicates the polynomial $f(n) = 9n^2 - 6n$ corresponds to the original exponential $f(n) = n^2 \cdot 3^n$.

n	0	1	2	3	4	5	6
$f(n)$	0	3	24	63	120	195	288
Δ^1	0	3	21	39	57	75	93
Δ^2	0	0	18	18	18	18	18
Δ^3	0	0	0	0	0	0	0
Δ^4	0	0	0	0	0	0	0
Δ^5	0	0	0	0	0	0	0
Δ^6	0	0	0	0	0	0	0

The number of difference tables constructed in all is shown in the following table under the heading *iterations*:

$f(n)$	iterations	diagonal value
2^n	1	1
$n2^n$	1	$2n$
$n^2 2^n$	1	$4n^2 - 2n$
$n^3 2^n$	1	$8n^3 - 12n^2 + 6n$
$n^4 2^n$	1	$16n^4 - 48n^3 + 60n^2 - 26n$
$n^5 2^n$	1	$32n^5 - 160n^4 + 360n^3 - 380n^2 + 150n$
$n^6 2^n$	1	$64n^6 - 480n^5 + 1680n^4 - 3120n^3 + 2940n^2 - 1082n$
$\vdots$		
3^n	1	2^n
$n3^n$	2	$3n$
$n^2 3^n$	2	$9n^2 - 6n$
$n^3 3^n$	2	$27n^3 - 54n^2 + 30n$
$n^4 3^n$	2	$81n^4 - 324n^3 + 468n^2 - 222n$
$n^5 3^n$	2	$243n^5 - 1620n^4 + 4320n^3 - 5130n^2 + 2190n$
$n^6 3^n$	2	$729n^6 - 7290n^5 + 30780n^4 - 65610n^3 + 68400n^2 - 27006n$
$\vdots$		
4^n	1	3^n
$n4^n$	3	$4n$
$n^2 4^n$	3	$16n^2 - 12n$
$n^3 4^n$	3	$64n^3 - 144n^2 + 84n$
$n^4 4^n$	3	$256n^4 - 1152n^3 + 1776n^2 - 876n$
$n^5 4^n$	3	$1024n^5 - 7680n^4 + 22080n^3 - 27600n^2 + 12180n$
$n^6 4^n$	3	$4096n^6 - 46080n^5 + 211200n^4 - 478080n^3 + 520560n^2 - 211692n$
$\vdots$		
5^n	1	4^n
$n5^n$	4	$5n$
$n^2 5^n$	4	$25n^2 - 20n$
$n^3 5^n$	4	$125n^3 - 300n^2 + 180n$
$n^4 5^n$	4	$625n^4 - 3000n^3 + 4800n^2 - 2420n$
$n^5 5^n$	4	$3125n^5 - 25000n^4 + 75000n^3 - 96500n^2 + 43380n$
$n^6 5^n$	4	$15625n^6 - 187500n^5 + 900000n^4 - 2107500n^3 + 2351400n^2 - 972020n$
$\vdots$		
6^n	1	5^n
$n6^n$	5	$6n$
$n^2 6^n$	5	$36n^2 - 30n$
$n^3 6^n$	5	$216n^3 - 540n^2 + 330n$
$n^4 6^n$	5	$1296n^4 - 6480n^3 + 10620n^2 - 5430n$
$n^5 6^n$	5	$7776n^5 - 64800n^4 + 199800n^3 - 261900n^2 + 119130n$
$n^6 6^n$	5	$46656n^6 - 583200n^5 + 2883600n^4 - 6901200n^3 + 7821180n^2 - 3267030n$
$\vdots$		
$n^k j^n$	$(j-1)$for $k > 0$	polynomial (conjecture)

The above chart was created using *Mathematica* calculations and was checked by hand. This chart leads to the following conjecture.

Conjecture 1. *For any function of the form $f(n) = n^k j^n$, there exists a unique polynomial which represents the values found along the $(j-1)$th diagonal of the function's difference table constructed from the diagonals of diagonals method. Furthermore, this unique polynomial's leading term will always be $j^k n^k$. Similarly, the second term will always be $-\frac{1}{2}(j-1)j^{k-1}n^{k-1}(k^2-k)$.*

Whether the results of diagonals of diagonals continue to be polynomials or not, the following algorithm illustrates a simple idea that provides evidence of the uniqueness of diagonals of diagonals: Diagonals of diagonals will iterate difference tables for a function until there's a constant row. Then we may construct summation tables (the same amount of times as difference tables) until we arrive at exactly our original function. Thus, diagonals of diagonals is reversible.

In summary, when dealing with a function of the form $f(n) = n^k j^n$, to find the closed form the key is to find the leading term of the polynomial that indicates j and k and appears along the $j-1$th diagonal, which the preceding algorithm will find. Note, the number of difference tables it took to get to the FCR is j and the difference row on which it reduced to a constant is the value for k. Also, by subtracting values from the leading term in the original sequence and repeating, we find not only the leading term, but the entire closed form as well. This is a highly programmable and reversible algorithm (taking a difference table from diagonal values of a sum table reverses the process and so does taking a sum table with the diagonal values from a difference table).

4 Conclusion and further questions

From a general function's difference table we have found that Pascal's triangle manifests itself in the difference table down each diagonal. While observing and proving the relationships between Pascal's triangle and difference tables of exponentials, we found several of the relationships we use to find closed forms for polynomials and exponential functions from difference tables follows from a proof of Pascal's triangle's presence in difference tables.

We also see that a unique polynomial corresponds to these exponential functions in the "diagonals of diagonals" approach above. A more comprehensive study on the third and fourth terms of these polynomials may lend itself to a greater over-arching pattern among those sequences, which would potentially yield a closed form for the entire polynomial along the $(j-1)$th diagonal for any function of the form $f(n) = n^k j^n$.

A few questions remain. For example, functions such as $f(n) = n^k j^n + 1$ will not be as easily recognized with the diagonals of diagonals approach. So, can difference tables and subsequent difference tables constructed from the diagonals of difference tables of unknown sequences be used to construct closed form solutions to other, more complicated, general

classes of functions? What about functions of more than one variable, perhaps with a three dimensional version of a difference table?

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