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A STUDY OF THE SHORTEST PERIMETER POLYHEDRON

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Abstract. Melzak's Conjecture seeks the polyhedron with minimal perimeter for a given volume. In studying this problem, the first approach may be to compare different polyhedra with a unit volume. However, the wide array of volume formulas for polyhedra make a unit volume computationally cumbersome. Instead, a more efficient approach is to consider the ratio of edge length to volume: $\frac{P}{\sqrt[3]{V}}$. In this manner, one may assign the edge length or volume to conveniently fit the situation. This paper summarizes previous work on the problem and presents some experimental observations from recent research. Much work is presented on ideas centered around prisms and their properties, and ideas for future consideration are explained, offering contributions to the ongoing study of Melzak's Conjecture.

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1 Introduction

An intriguing property of circles is that the first derivative of their area yields circumference. Through clever use of this property, it is possible to attain similar results with polygons. The computations and techniques rely on undergraduate level calculus, fundamental geometry, and limited vector notation. This paper applies this concept in studying a geometric optimization problem known as Melzak's Problem. The problem seeks a polyhedron which minimizes the ratio

$$R = \frac{P}{\sqrt[3]{V}}$$

where P is the perimeter and V is the volume. In application, a package shaped as the shortest polyhedron would use minimal tape around the edges per package shipped. Thus, this problem is interesting both in theory and practice.

Melzak's problem is presented in the work of Dorff and Hall [2] following an extensive study of some relationships found among properties of geometric figures. Results from Dorff and Hall have been beneficial in examining this problem, which finds its origins in the work of Z.A. Melzak in the mid-1900s on polyhedra and pseudopolyhedra. Melzak proposed that the right triangular prism may be the minimizing solution to his problem. Computation confirms that this prism has a lower ratio, R, than the five Platonic Solids, however, the conjecture remains unproven.

In the next section, an analysis of regular polygons with an inscribed circle will provide a basis in $\mathbb{R}^2$ for the work in the rest of the paper. Section 3 develops these relationships in $\mathbb{R}^3$, with a focus on right-regular prisms. Section 4 builds on the idea from $\mathbb{R}^2$ of an inscribed circle, and applies it to $\mathbb{R}^3$ using an inscribed spheroid as a second method of proving that the right triangular prism has the minimum ratio R of all regular prisms. Finally, in an attempt to look to other types of polyhedra, a new approach in $\mathbb{R}^2$ is discussed in Section 5 in which changing the number of sides and the measures of the angles attempts to maximize the area while keeping a constant perimeter. Section 6 concludes with suggestions for further research.

2 Analysis of Polygons

The derivative of the area of a circle generates the circumference of that circle. Applying this property to regular polygons and parameterizing the polygon creates this property in all regular polygons. To exemplify this concept, consider a square with side lengths s and an inscribed circle of radius r, shown in Figure 1.

Differentiating the area of the square with respect to the side length, s, yields $\frac{dA}{ds} = 2s$, which does not correctly describe the perimeter of the square. However, by parameterizing the square with r, we find that for an inscribed circle, $r = \frac{1}{2}s$. Then, $A = (2r)^2 = 4r^2$, and differentiating with respect to the radius yields $\frac{dA}{dr} = 8r = 4s = P$. These observations are generalized in Theorem 1.

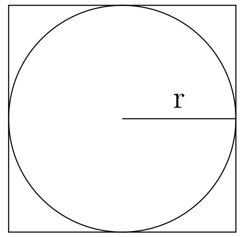

Figure 1: A square with inscribed circle of radius r.

Theorem 1. *For all regular polygons with area A, perimeter P, and an inscribed circle of radius r, the following property exists:*

$$\frac{dA}{dr} = P.$$

Proof. Let n be the number of sides. Drawing radii and connecting the vertices to the circle's center creates $2n$ congruent right triangles. In each triangle, note that a radius is the shorter leg and let ℓ be the second leg. The segments from the centroid to each vertex form the hypotenuses. Let θ be the central angle. Figure 2 demonstrates these values in an equilateral triangle.

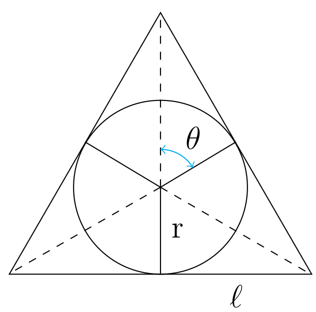

Figure 2: A triangle with inscribed circle of radius r.

Basic trigonometry yields $\theta = \frac{\pi}{n}$ and $\ell = r \cdot \tan\left(\frac{\pi}{n}\right)$. The total area is expressed by multiplying the area of one of the right triangles by $2n$, the number of right triangles:

$$A = \frac{1}{2}\ell r \cdot 2n = r^2 n \tan\left(\frac{\pi}{n}\right).$$

Differentiating yields

$$\begin{aligned} \frac{dA}{dr} &= 2rn \tan\left(\frac{\pi}{n}\right) \\ &= 2nr \tan\left(\frac{\pi}{n}\right) = P. \end{aligned}$$

□

These properties discovered in $\mathbb{R}^2$ are analogous to properties in $\mathbb{R}^3$ and provide insight to solving Melzak's Problem.

3 Analysis of Prisms

Transitioning into the discussion of polyhedra, it is noted that there are infinitely many types of polyhedra and their properties to consider. For the purpose of this article, only prisms will be discussed. This is done with the intention of being able to find the polyhedron which would prove Melzak's Conjecture. This article systematically discovers the minimizer of all prisms.

Even after restricting polyhedra to only prisms, there still exist many types of prisms to be observed. However, according to Berger, the minimizer may be more easily discovered because right-regular prisms are most efficient at minimizing total edge length among all prisms [1]. Therefore, considering only right-regular prisms in the succeeding calculations will yield this minimizer.

Until this point, we have related a figure's area, a quality in $\mathbb{R}^2$, to its perimeter in $\mathbb{R}^1$. The most natural three-dimensional analog is relating volume in $\mathbb{R}^3$ to surface area in $\mathbb{R}^2$. However, this study focuses on minimizing perimeter for a given volume, as Melzak's Conjecture centers around minimizing total edge length.

3.1 Establishing the Optimal Configuration

To aid in discovering which right-regular prism is the minimizer of total edge length for all prisms, the relationship among all of the individual edge lengths of the minimizer should be determined. Consider minimizing the ratio $R = \frac{P}{\sqrt[3]{V}}$ using standard calculus techniques where P is the total edge length of the prism. In doing so, consider volume, V, to be the area (as previously expressed) of the base face, with side length x, extended into three-space by an amount, y. Thus,

$$V = n \tan\left(\frac{\pi}{n}\right) r^2 y.$$

Then, it is possible to minimize R as follows:

$$R = \frac{P}{\sqrt[3]{V}} = (2nx + ny)\left(n \tan\left(\frac{\pi}{n}\right) r^2 y\right)^{-\frac{1}{3}}$$

$$\frac{dR}{dy} = (2nx + ny)\left(\frac{-1}{3}\left(n \tan\left(\frac{\pi}{n}\right) r^2 y\right)^{-\frac{4}{3}}\left(n \tan\left(\frac{\pi}{n}\right) r^2\right)\right) + n\left(n \tan\left(\frac{\pi}{n}\right) r^2 y\right)^{-\frac{1}{3}} = 0$$

$$\left(n \tan\left(\frac{\pi}{n}\right) r^2 y\right)^{-\frac{1}{3}}\left((2nx + ny)\left(-\frac{1}{3y}\right) + n\right) = 0$$

$$\frac{-2nx}{3y} - \frac{1}{3}n + n = 0$$

$$\frac{2}{3}n = \frac{2}{3}\frac{nx}{y}$$
$$y = x$$

From this process, it is observed that R is minimized when x and y are equal. In other words, a right-regular prism which has all equal edge lengths is the most efficient configuration for all right-regular prisms with respect to total edge length.

3.2 Proving Melzak's Conjecture for Prisms

Since it has been shown that a right-regular prism of equal edge lengths minimizes total edge length, consider all right-regular prisms of equal edge length as to find the minimizer of this set. First, an equation for total edge length solely in terms of n, the number of sides on the base face, is established. Using this equation, it can be determined which prism is the most optimal prism with respect to total edge length, given unit volume.

In order to eliminate all variables except for n, turn first to an equation for volume. Because the configuration in which all edge lengths are equal minimizes total edge length, x, the length of a side of the base face, and y, the length of a side of a lateral face, are equal. Consequently, one variable is eliminated, and it is then possible to determine a value for r, the apothem of the base face, in terms of n. This process is shown below:

$$V = 1 = A \cdot y$$
$$1 = \left[n \tan\left(\frac{\pi}{n}\right) r^2\right] \cdot \left[2r \tan\left(\frac{\pi}{n}\right)\right]$$
$$\Rightarrow r = \sqrt[3]{\frac{\cot^2(\frac{\pi}{n})}{2n}}$$

Return now to an equation for total edge length of a right-regular prism. Using this new information, an equation for total edge length in one variable is derived:

$$P = 2nx + ny$$
$$P = 2\left[2nr \tan\left(\frac{\pi}{n}\right)\right] + n\left[2r \tan\left(\frac{\pi}{n}\right)\right]$$
$$P = 6n\sqrt[3]{\frac{\cot^2\left(\frac{\pi}{n}\right)}{2n}} \tan\left(\frac{\pi}{n}\right)$$

Upon graphical analysis of this function (as seen in Figure 3) and considering only the positive integer values for $n \geq 3$, the function is determined to be increasing. Therefore, the lower bound located at $n = 3$ is the minimum of this sequence of total edge lengths.

From this, it may be concluded that a right-regular prism whose base face has three sides (a triangular prism) and whose edge lengths are all equal, is the total edge length minimizer

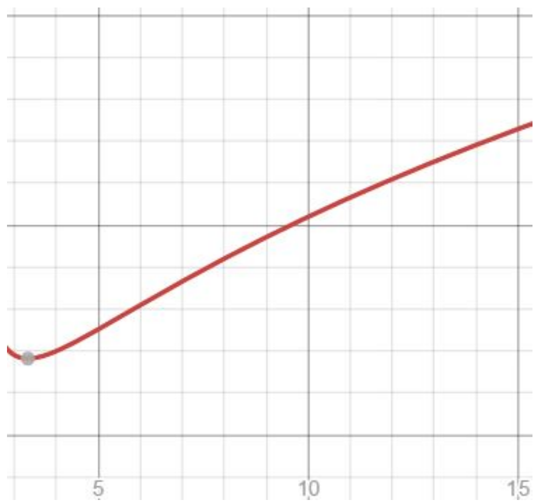

Figure 3: Graph of Total Edge Length with Respect to n.

for all prisms. While this does not solve Melzak's Conjecture for all polyhedra, this does provide insight as to why the triangular prism is speculated to be the overall minimizer and which properties that a minimizer must manifest.

4 Applications of Spheroids

Another approach to examining polyhedra with regards to Melzak's Conjecture involves the geometric figure known as the spheroid. Although the initial concept this approach developed from, in which a circle was inscribed in a polygon, worked for objects in $\mathbb{R}^2$, it can be manipulated to be used in $\mathbb{R}^3$. While it has been determined that the optimal prism has equal edge lengths, this does not mean that all radii of the spheroid inscribed within the prism will be equal.

Drawing from the initial concept, it would appear that minimizing the area of the circle inscribed polygon also minimizes the ratio of area to perimeter. By inscribing spheroids into prisms and other polyhedra, it should be possible to minimize the ratio of total edge length to volume utilizing a similar technique.

The volume of a spheroid, such as the one seen in Figure 4, is

$$V = \frac{4}{3}\pi a^2 b.$$

Consider the spheroid inscribed in a prism, where the bases of the prism are parallel to the circle with radius a. Then the volume for the spheroid can be written in the same terms as that of the prism itself:

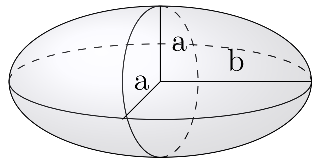

Figure 4: A spheroid with radii of a and b.

$$V = \frac{4}{3}\pi r^2 \left(\frac{1}{2}\right) y$$

$$V = \frac{4}{3}\pi r^2 \left(r \tan\left(\frac{\pi}{n}\right)\right)$$

$$V = \frac{4}{3}\pi \left(\sqrt[3]{\frac{\cot^2\left(\frac{\pi}{n}\right)}{2n}}\right)^2 \left(\sqrt[3]{\frac{\cot^2\left(\frac{\pi}{n}\right)}{2n}} \tan\left(\frac{\pi}{n}\right)\right)$$

$$V = \frac{4}{3}\pi \left(\frac{\cot^2\left(\frac{\pi}{n}\right)}{2n} \tan\left(\frac{\pi}{n}\right)\right)$$

$$V = \frac{2\pi}{3n} \left(\cot\left(\frac{\pi}{n}\right)\right)$$

Note that n cannot be less than 3, as there are no polyhedrons which can be constructed from polygons with less than 3 sides. Indeed, the minimum of this function is achieved when $n = 3$, as $\cot\left(\frac{\pi}{n}\right)$ increases at a faster rate than $\frac{2\pi}{3n}$ decreases, causing the volume of the spheroid to increase as the number of edges of the base increases. This approach supports the results generated from the analysis of prisms, and may provide an avenue to explore other types of polyhedra.

5 Moving Forward from Prisms

With extensive knowledge of prisms, an appropriate next step is to compare prisms with other polyhedra. One such comparison begins with an $\mathbb{R}^2$ allegory, comparing perimeter with area instead of volume.

Consider a square with rigid edges and flexible hinges on two opposite sides. These two hinges may fold outward to alter the shape of the square, and consequently, its area.

With this operation, the figure becomes two trapezoids of equal area. This operation preserves the square's perimeter while altering area. Let θ be the angle formed by the hinges and the transversal between them and let Δw be the horizontal displacement of each hinge. Finally, let h be the height of each trapezoid. Using the right triangles formed by the transversal, $h = \Delta w \tan\theta$, and $\Delta w = \frac{\cos\theta}{2}$. The area of this figure is given by

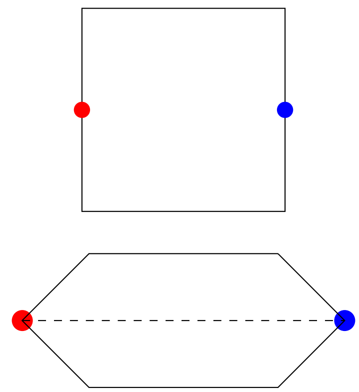

Figure 5: A compressed square.

$$\begin{aligned}
A &= 2 \cdot \frac{1 + (1 + 2\Delta w)}{2} h \\
&= (2 + 2\Delta w)\, h \\
&= (2 + 2\Delta w)\, \Delta w \tan\theta \\
&= (2 + \cos\theta)\, \frac{\cos\theta}{2} \tan\theta \\
&= \frac{1}{2}\,(2 + \cos\theta) \sin\theta \\
&= \sin\theta + \frac{\cos\theta \sin\theta}{2}.
\end{aligned}$$

Differentiating,

$$\begin{aligned}
\frac{dA}{d\theta} &= \cos\theta + \frac{1}{2}\left(\cos^2\theta - \sin^2\theta\right) \\
&= \cos\theta + \frac{1}{2}\left(\cos^2\theta - 1 + \cos^2\theta\right) \\
&= \cos^2\theta + \cos\theta - \frac{1}{2}.
\end{aligned}$$

By restricting results to convex polygons and applying the quadratic formula, this equation's real zero is at

$$\cos\theta = \frac{-1 + \sqrt{3}}{2}$$

and $\theta \approx 68°$.

This result indicates that the square's area increases as it compresses toward $\theta = 68°$. In $\mathbb{R}^3$, however, compressing a prism by any amount creates new edges around the middle to connect the hinges. Thus, in general cases, a proper right prism is a better minimizer than a compressed one.

6 Areas for Further Research

Other operations on right prisms may help eliminate classes of polyhedra from consideration in Melzak's Problem. One possibility is to rotate a base such that the connecting edges are not parallel. Another option is to merge polyhedra together and compare the merged volume/perimeter ratio with its two constituents. When two right square-base pyramids merge, their combined ratio is improved, but when the equilateral triangular prism merges with a square base pyramid, their combined ratio is higher than both individuals' parts.

Finally, a computational approach to Melzak's Problem may aid in understanding it. An ideal program would compare $R = \frac{P}{\sqrt[3]{V}}$ for different polyhedra and organize the data in a meaningful way. Analyzing the data, researchers may find trends that eliminate classes of polyhedra or lead to new theorems.

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[2] Dorff, M. & Hall, L.(2003).Solids in $\mathbb{R}^n$ Whose Area is the Derivative of the Volume. *The College Mathematics Journal, 34*(5), 1-16.