

A Proof of the "Magicness" of the Siam Construction of a Magic Square

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Cover Page Footnote

I would like to thank Dr. Holder for all of her help with the mathematics and grammar in this paper. I would also like to thank the Rose-Hulman Mathematics Department for the funding to present this work at the University of Dayton's Undergraduate Mathematics Day.

Abstract. A magic square is an $n \times n$ array filled with n^2 distinct positive integers $1, 2, \dots, n^2$ such that the sum of the n integers in each row, column, and each of the main diagonals are the same. A Latin square is an $n \times n$ array consisting of n distinct symbols such that each symbol appears exactly once in each row and column of the square. Many articles dealing with the construction of magic squares introduce the Siam method as a “simple” construction for magic squares. Rarely, however, does the article actually prove that the construction yields a magic square. In this paper, we describe how to decompose a magic square constructed by the Siam method into two orthogonal Latin squares, which in turn, leads us to a proof that the Siam construction produces a magic square.

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1 Background

Magic squares of order n are $n \times n$ arrays filled with distinct positive integers $1, 2, \dots, n^2$ arranged in such a way that the sum of the integers in each row, column, and main diagonals are all the same. The sum of the entries in each row, column, and main diagonals is $\frac{n(n^2 + 1)}{2}$, which is called the *magic sum*.

Magic squares have a long history with the earliest known magic square being the Lo-Shu tortoise from ancient China. Over the years, many algorithms have been made to create magic squares. Usually these algorithms only work for certain order squares. For instance, the Siam method (which is discussed in the next section) only works for odd orders.

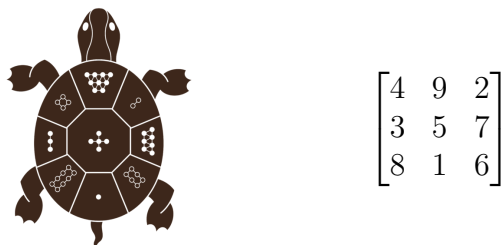


Figure 1: Lo-Shu Tortoise, the earliest known magic square. The number of dots on the tortoise correspond to the magic square on the right.

Latin squares of order n are $n \times n$ arrays with n distinct elements where each row and column have exactly one of each element. Two Latin squares are considered *orthogonal mates* if when superimposed, each of the possible n^2 ordered pairs occurs exactly once. That is, if L_1 and L_2 are two Latin squares of order n , then the $n \times n$ array (L_1, L_2) contains n^2 distinct entries given by

$$(L_1, L_2)_{i,j} = (L_1(i, j), L_2(i, j)). \quad (1)$$

This array is known as a Graeco-Latin square.

Example 1.1. Consider the two Latin squares L_1 and L_2 given below.

$$L_1 = \begin{bmatrix} 3 & 4 & 0 & 1 & 2 \\ 4 & 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 & 0 \\ 2 & 3 & 4 & 0 & 1 \end{bmatrix} \quad L_2 = \begin{bmatrix} 1 & 3 & 0 & 2 & 4 \\ 2 & 4 & 1 & 3 & 0 \\ 3 & 0 & 2 & 4 & 1 \\ 4 & 1 & 3 & 0 & 2 \\ 0 & 2 & 4 & 1 & 3 \end{bmatrix}$$

If we superimpose L_1 and L_2 to obtain

$$(L_1, L_2) = \begin{bmatrix} (3, 1) & (4, 3) & (0, 0) & (1, 2) & (2, 4) \\ (4, 2) & (0, 4) & (1, 1) & (2, 3) & (3, 0) \\ (0, 3) & (1, 0) & (2, 2) & (3, 4) & (4, 1) \\ (1, 4) & (2, 1) & (3, 3) & (4, 0) & (0, 2) \\ (2, 0) & (3, 2) & (4, 4) & (0, 1) & (1, 3) \end{bmatrix}$$

then we see that each ordered pair (a, b) with $0 \leq a, b \leq 4$ appears exactly once. For instance, observe that $(L_1, L_2)_{1,1} = (L_1(1, 1), L_2(1, 1)) = (3, 1)$ which appears exactly once in the Graeco-Latin square.

A *transversal* is a set of n cells on a Latin square of order n where each cell contains a different element and no two share a row or column. *Disjoint transversals* are transversals that do not overlap. A set of n disjoint transversals on a Latin square of order n can be relabeled to be a new Latin square. We use this fact in Section 3 to show that a magic square can be decomposed into two Latin squares. Finally, for any array, the diagonal containing the top left entry and the bottom right entry is referred to as the *main front diagonal*. Similarly, the diagonal containing the top right entry and the bottom left entry is referred to as the *main back diagonal*.

2 The Siam Method

The *Siam method* (also known as the Siamese method or the De la Loubère method) is an algorithm used to create magic squares of odd order. To create a magic square using this method, one places elements on an array starting with 1 and increasing by one until n^2 is reached. The first element is placed in the middle of the top row. Then elements are placed diagonally up and to the right (with the array looping around itself) until an already occupied location is encountered. When that happens, place the element one below where the last element was placed, and continue as before. The element you place will be the starting element of the next loop which we will refer to as a back diagonal.

Example 2.1. We show how to construct a magic square of order 5 using the Siam method by referring to the square in Figure 2. First the 1 is placed. Then, we follow the arrow up and to the right, looping down to the bottom right to place the 2. Following the arrow again, it loops to the middle left, which is where the 3 goes. Since, if we move diagonally again, we encounter an occupied cell containing a 1, we must instead place the 4 directly below the 3. The process is then repeated until the array is filled. Once the array is filled, the Siam method is finished and the result should be a magic square.

Now, each element can be uniquely expressed as $m_{i,j} = nb_{i,j} + u_{i,j}$ where $b_{i,j} = \lfloor \frac{m_{i,j}}{n} \rfloor$ and $u_{i,j} = m_{i,j} \bmod n$. We then form the arrays L_b and L_u by assigning $L_b(i, j) = b_{i,j}$ and $L_u(i, j) = u_{i,j}$. We can then show that both of these arrays are Latin squares. Given that S was constructed by increasing along the back diagonals as we move along the back diagonal, $u_{i,j}$ will increase, that is $u_{i,j} + 1 \equiv u_{i-1,j+1} \pmod{n}$. Since each back diagonal has exactly n elements, elements will not be repeated in the same back diagonal. The starting element of each back diagonal starts one column to the left and two rows down of the back diagonal before it. Since the Siam method only works for odd orders, the start of each back diagonal will be in a different row (and each will start in a different column as there are n back diagonals). Thus, the starting element of each back diagonal will be in a different row and column from the starting element of another back diagonal. This means that each element of the any back diagonal will be in a different row and column from that element in another back diagonal. Therefore, L_u will be a Latin square. Since each back diagonal has n different elements and do not overlap they will form a set of n disjoint transversals on L_u . If we reassign these transversals to their corresponding back diagonals (starting with 0) we obtain L_b (as the elements of each back diagonal are the elements of the last back diagonal plus n). Thus, L_b will also be a Latin square. Since M has elements $0, \dots, n^2 - 1$, every $(b_{i,j}, u_{i,j})$ pair is different. Thus the Latin squares L_b and L_u are orthogonal mates.

Example 3.2. When $n = 5$, we have that the Siam method produced the square

$$S = \begin{bmatrix} 17 & 24 & 1 & 8 & 15 \\ 23 & 5 & 7 & 14 & 16 \\ 4 & 6 & 13 & 20 & 22 \\ 10 & 12 & 19 & 21 & 3 \\ 11 & 18 & 25 & 2 & 9 \end{bmatrix}$$

which in turn yields the squares

$$S = \begin{bmatrix} 16 + 1 & 23 + 1 & 0 + 1 & 7 + 1 & 14 + 1 \\ 22 + 1 & 4 + 1 & 6 + 1 & 13 + 1 & 15 + 1 \\ 3 + 1 & 5 + 1 & 12 + 1 & 19 + 1 & 21 + 1 \\ 9 + 1 & 11 + 1 & 18 + 1 & 20 + 1 & 2 + 1 \\ 10 + 1 & 17 + 1 & 24 + 1 & 1 + 1 & 8 + 1 \end{bmatrix}$$

$$S = \begin{bmatrix} 16 & 23 & 0 & 7 & 14 \\ 22 & 4 & 6 & 13 & 15 \\ 3 & 5 & 12 & 19 & 21 \\ 9 & 11 & 18 & 20 & 2 \\ 10 & 17 & 24 & 1 & 8 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$S = 5 \begin{bmatrix} 3 & 4 & 0 & 1 & 2 \\ 4 & 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 & 0 \\ 2 & 3 & 4 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 3 & 0 & 2 & 4 \\ 2 & 4 & 1 & 3 & 0 \\ 3 & 0 & 2 & 4 & 1 \\ 4 & 1 & 3 & 0 & 2 \\ 0 & 2 & 4 & 1 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$S = 5L_b + L_u + J_5$$

4 Proof that the Siam Method Creates a Magic Square

In order to prove that a square constructed by the Siam Method is a magic square, we must show that each row, column, and main diagonals sum to the magic sum.

Label the rows and columns of all arrays of order n from 0 to $n - 1$, from top to bottom and left to right, like the example below.

$$\begin{bmatrix} (0,0) & (0,1) & \dots & (0,n-1) \\ (1,0) & (0,1) & \dots & (1,n-1) \\ \vdots & \vdots & \ddots & \vdots \\ (n-1,0) & (n-1,1) & \dots & (n-1,n-1) \end{bmatrix}$$

For S , let each back diagonal followed by the Siam method be given a number b , $b = 0, \dots, n - 1$, in the order that they were created. Let each element in each of those back diagonals be given a number u , $u = 0, \dots, n - 1$ in the order that they were created. Each new back diagonal starts 2 below and 1 to the left of the last one, each new entry of the back diagonal is up 1 and to the right 1. Since the first entry is at $(0, \frac{n-1}{2})$, we can define the position of any $m_{i,j} \in S$ as

$$(i, j) = (2, -1)b + (-1, 1)u + \left(0, \frac{n-1}{2}\right). \quad (2)$$

Lemma 1. *The sum of the entries of the main front diagonal of S is the magic sum $\frac{n(n^2 + 1)}{2}$.*

Proof. The square L_b is equivalent to a matrix created from the b values while L_u is equivalent to a matrix created from the u values. Thus, if we set $(i, j) = (k, k)$ we discover the nature of the main front diagonal of both our Latin squares.

$$\begin{aligned} k &= 2b - u, \\ k &= -b + u + \frac{n-1}{2}. \end{aligned} \quad (3)$$

When this system of equations is solved for b and u then we get (in mod n)

$$\begin{aligned} b &= 2k + \frac{1-n}{2}, \\ u &= 3k - n + 1. \end{aligned} \tag{4}$$

This gives us the values of L_b and L_u on their main front diagonals. Thus we can sum up the entries of the main front diagonal of $S = nL_b + L_u + J_n$ as follows by ranging k from 0 to $n-1$. Given $b = 2k + \frac{1-n}{2}$, $u = 3k - n + 1$, and $nL_b + L_u + J_n = S$ the sum of the main front diagonal will be

$$\begin{aligned} \sum_{k=0}^{n-1} S_{k,k} &= n \sum_{k=0}^{n-1} \left(2k + \frac{1-n}{2} \right) + \sum_{x=0}^{n-1} (3k - n + 1) + \sum_{k=0}^{n-1} 1 \\ &= \frac{n^2(n-1)}{2} + \frac{n(n-1)}{2} + n \\ &= \frac{n(n^2+1)}{2}. \end{aligned}$$

Thus, the main front diagonal of S sums up to the magic sum. □

Lemma 2. *The sum of the entries of the main back diagonal of S is the magic sum $\frac{n(n^2+1)}{2}$.*

Proof. To get the entries of the main back diagonal set $(i, j) = (k, n-1-k)$

$$\begin{aligned} k &= 2b - u, \\ n-1-k &= -b + u + \frac{n-1}{2}. \end{aligned} \tag{5}$$

When this system of equations is solved for b and u , we find that (in mod n)

$$\begin{aligned} b &= \frac{n-1}{2}, \\ u &= n-1-k. \end{aligned} \tag{6}$$

We can then similarly sum up the main back diagonal by ranging k from 0 to $n-1$. Given $b = \frac{n-1}{2}$, $u = n-1+k$, and $nL_b + L_u + J_n = S$ the sum of the main back diagonal will be

$$\begin{aligned} \sum_{k=0}^{n-1} S_{k,n-1-k} &= n \sum_{k=0}^{n-1} \frac{n-1}{2} + \sum_{k=0}^{n-1} (n-1-k) + \sum_{k=0}^{n-1} 1 \\ &= \frac{n^2(n-1)}{2} + \frac{n(n-1)}{2} + n \\ &= \frac{n(n^2+1)}{2}. \end{aligned}$$

Thus, the main back diagonal sums up to the magic sum. □

Finally, we need to show that the sum of the rows and columns of M are also equal to the magic sum.

Lemma 3. *The sum of the entries of the rows and columns of S is the magic sum $\frac{n(n^2 + 1)}{2}$.*

Proof. By the creation of the Latin squares, L_b and L_u , each row and column of the Latin squares contain the distinct entries $0, \dots, n - 1$ exactly once. By our construction, $S = nL_b + L_u + J_n$. Thus, the sum of each row and column is

$$\begin{aligned} n \sum_{k=0}^{n-1} k + \sum_{k=0}^{n-1} k + \sum_{k=0}^{n-1} 1 &= \frac{n^2(n-1)}{2} + \frac{n(n-1)}{2} + n \\ &= \frac{n(n^2 + 1)}{2}. \end{aligned}$$

Thus all the rows and columns sum up to the magic sum. □

Theorem 1. *The Siam method produces a magic square.*

Proof. The previous three lemmas prove that each row, column, and main diagonal sum to the magic sum of $\frac{n^2(n^2+1)}{2}$. Therefore, the magic square we constructed using the Siam method is in fact a magic square. □

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