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Cover Page Footnote

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A Connection Between Quadratic Rational Maps and Linear Fractional Maps

By *Anna Marek*, *Laura Schlesinger*, *Ella White*, and *Danqi Yin*

Abstract. This research project is an investigation into quadratic rational maps, φ, of one complex variable that map the unit disk to itself. Previous research [7] shows that for each φ, a corresponding linear fractional map ζ can be found using the coefficients of φ, and this ζ can be used to characterize functions in the kernel of the adjoint of the composition operator with symbol φ, defined on a space of analytic functions. In this paper, we show sufficient conditions to ensure that certain cases of φ map the unit disk to itself and find all the forms of ζ. Finally, for a specific φ, we search for the kernel of its corresponding composition operator by calculating the matrices of C_φ and of the Toeplitz operator $T_{-z\zeta'(z)/\zeta(z)}$, defined by projecting $\frac{-z\zeta'(z)}{\zeta(z)} f(z)$ onto the space of analytic functions.

1 Introduction

A composition operator C_φ, for a fixed map φ referred to as the symbol, is a linear operator that is applied to a function f to produce the composition $f \circ \varphi$, i.e., $C_\varphi f = f \circ \varphi$. This type of operator can be studied using concepts from linear algebra and complex analysis, although we first introduce some ideas from operator theory for motivation.

Composition operators have been an object of formal study since the 1960s and have close ties to shift operators [3] and possibly to the famous open Invariant Subspace Problem [8] (see [9] for more details about the Invariant Subspace Problem). They also appear in physics as Koopman operators of dynamical systems [6]. In operator theory, it is natural to study an operator and its adjoint, which lead to inquiries about the spectrum (or the set of eigenvalues in linear algebra terms) and the kernel (or the nullspace) of the adjoint. In particular, understanding the adjoint of a composition operator helps uncover properties of the composition operator itself, including its norm [3].

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Ultimately, this project stems from research [7] aiming to classify functions (or vectors) in the kernel (or nullspace) of C_φ^*, the adjoint of C_φ, where $\varphi(z) = \dfrac{a_1z^2+b_1z+c_1}{a_2z^2+b_2z+c_2}$. Instead of the quadratic rational map φ, we use a functional equation involving the linear fractional map, $\zeta(z) = -\dfrac{(\overline{a_1}\,\overline{b_2}-\overline{b_1}\,\overline{a_2})+(\overline{a_1c_2}-\overline{c_1a_2})z}{(\overline{a_1c_2}-\overline{c_1a_2})+(\overline{b_1}\,\overline{c_2}-\overline{c_1}\,\overline{b_2})z}$, which depends on the coefficients of φ. However, the scope of this project, which took place over the course of a summer, was to find conditions on the coefficients of φ such that φ maps the unit disk to itself. As we will see in the definitions below, we require that φ map the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ to itself, that is, whenever $z \in \mathbb{D}$, then $\varphi(z) \in \mathbb{D}$. For example, we see from their images in Figures 1 and 2 that $\varphi_1(z) = \dfrac{z^2+1}{2}$ maps the unit disk to itself, but $\varphi_2(z) = \dfrac{4z}{z^2+3}$ does not. We wish to understand the relationship between the coefficients of each map and the image relative to the unit disk.

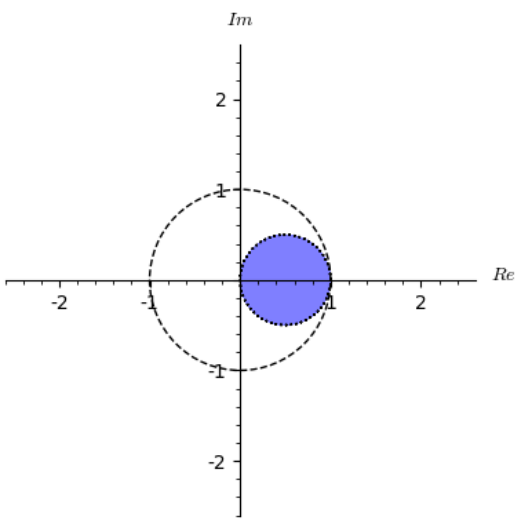

Figure 1: The shaded region is the image of the unit disk under the map $\varphi_1(z) = \dfrac{z^2+1}{2}$. The image lies entirely inside the unit disk, whose boundary is displayed for comparison.

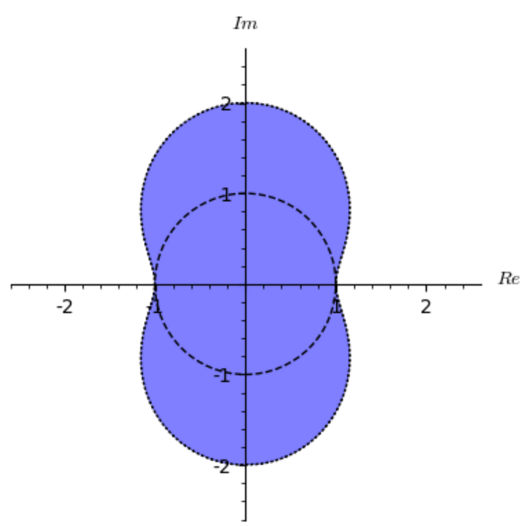

Figure 2: The shaded region is the image of the unit disk under the map $\varphi_2(z) = \dfrac{4z}{z^2+3}$. The image does not lie entirely inside the unit disk, whose boundary is displayed for comparison.

1.1 Background & Definitions

We begin by providing more details about the space of analytic functions that the composition operator C_φ acts on, formalizing definitions, and presenting a brief history of previous results in order to set the stage for new findings. The following definitions come from [3].

Definition 1. The Hardy space, H^2, is the Hilbert space of analytic functions for which $f(z)=\sum_{n=0}^{\infty} a_n z^n$ satisfies $\sum_{n=0}^{\infty} |a_n|^2<\infty$, where $z\in\mathbb{D}=\{z\in\mathbb{C}:|z|<1\}$ and $a_n\in\mathbb{C}$.

For example, the function $\varphi_1(z)=\frac{z^2+1}{2}$ is in H^2 because $a_0=\frac{1}{2}$, $a_1=0$, $a_2=\frac{1}{2}$, and $a_n=0$ for $n\ge 3$, and thus, $\sum_{n=0}^{\infty}|a_n|^2=\frac{1}{2}$ is finite. Also, the function $\varphi_2(z)=\frac{4z}{z^2+3}$ is in H^2. Notice that we can rewrite φ_2 as a power series:

$$\begin{aligned}\varphi_2(z)&=\frac{4z}{z^2+3}\\&=\frac{4z}{3}\cdot\frac{1}{1-\left(-\frac{z^2}{3}\right)}\\&=\frac{4z}{3}\cdot\sum_{n=0}^{\infty}\left(-\frac{z^2}{3}\right)^n\\&=\sum_{n=0}^{\infty}(-1)^n\frac{4z^{2n+1}}{3^{n+1}}\end{aligned}$$

where $\sum_{n=0}^{\infty}\left|(-1)^n\frac{4}{3^{n+1}}\right|^2\le 16\sum_{n=0}^{\infty}\frac{1}{9^{n+1}}$ which converges by the Geometric Series Test where the common ratio is $r=\frac{1}{9}<1$. However, the function $\varphi(z)=\frac{1}{1-z}$ is not in the H^2 because $\frac{1}{1-z}=\sum_{n=0}^{\infty}z^n$ where $a_n=1$ for all n and $\sum_{n=0}^{\infty}|a_n|^2$ is not finite.

Definition 2. The inner product for functions f,g in H^2, where $f(z)=\sum_{n=0}^{\infty}a_n z^n$ and $g(z)=\sum_{n=0}^{\infty}b_n z^n$, is defined as $\langle f,g\rangle=\sum_{n=0}^{\infty}a_n\overline{b_n}$.

For our purposes, we define analytic functions on $\mathbb{D}$ as functions that are differentiable on $\mathbb{D}$. Note that H^2 is spanned by the monomials, $\{1,z,z^2,\ldots\}$, which is in fact an orthonormal basis for H^2 and is referred to as the standard basis. It is clear that $\{1,z,z^2,\ldots\}$ spans H^2 since functions in H^2 can be written as power series. Just as in linear algebra, we can see this is an orthonormal basis for H^2 using the inner product. Let's consider the inner product $\langle z^j,z^k\rangle$ where $f(z)=z^j$ and $g(z)=z^k$. If $j=k$, then $a_n=1$ if $n=j$ and $a_n=0$ otherwise. Therefore, $\langle z^j,z^j\rangle=\sum_{n=0}^{\infty}a_n\overline{a_n}=1$, which shows that

$\{1, z, z^2, \ldots\}$ is normal. If $j \neq k$, then $a_n = 1, b_n = 0$ if $n = j$ and $a_n = 0, b_n = 1$ if $n = k$ and $a_n = 0, b_n = 0$ otherwise. Therefore, $\langle z^j, z^k \rangle = \sum_{n=0}^{\infty} a_n \overline{b_n} = 0$, which shows that $\{1, z, z^2, \ldots\}$ is orthogonal.

Definition 3. Given an analytic function $\varphi : \mathbb{D} \to \mathbb{D}$, the composition operator C_φ acts on H^2 and is defined by

$$C_\varphi f = f \circ \varphi$$

where φ is referred to as the symbol of the composition operator and $f \in H^2$.

Now, we define the adjoint of the operator C_φ in the usual linear algebra sense using the inner product.

Definition 4. Given an analytic function $\varphi : \mathbb{D} \to \mathbb{D}$, the adjoint, C_φ^*, of the composition operator acting on H^2 with symbol φ is defined by

$$\langle C_\varphi f, g \rangle = \langle f, C_\varphi^* g \rangle$$

where $f, g \in H^2$.

Since we are working over H^2, a vector space over the complex numbers, we notice that the definition of the adjoint operator is analogous to the conjugate transpose $\overline{A^T}$ of a matrix from linear algebra. See 2.3.2 for an example.

Here we note that we require φ to map $\mathbb{D}$ to $\mathbb{D}$ because the composition operator C_φ acts on the Hardy space where the functions $f \in H^2$ take inputs from the unit disk. Thus, the bulk of this research project is in identifying conditions to ensure φ maps the unit disk to itself. Throughout this paper, we only consider quadratic rational maps, that is, functions of the form $\varphi(z) = \dfrac{a_1 z^2 + b_1 z + c_1}{a_2 z^2 + b_2 z + c_2}$, where $a_1, b_1, c_1, a_2, b_2, c_2 \in \mathbb{C}$ with at least one nonzero quadratic term. We also require c_2 be nonzero, otherwise φ would have a pole (which is analogous to an infinite discontinuity) at $z = 0$ and thus would not map $\mathbb{D}$ to $\mathbb{D}$. For the purpose of convenience, we set $c_2 = 1$ so now $\varphi(z) = \dfrac{a_1 z^2 + b_1 z + c_1}{a_2 z^2 + b_2 z + 1}$.

We now turn our attention to different expressions for $C_\varphi^* f(z)$. The first expression for $C_\varphi^* f(z)$ comes from Cowen and Gallardo-Gutiérrez [2]. They show that deriving Formula 1 is immediate using properties of the Hardy space (that go beyond the scope of this paper), and they explain that it is "precise but difficult to use."

Theorem 1 (Cowen and Gallardo-Gutiérrez [2])**.** The integral form of the adjoint of a composition operator, C_φ^*, can be written:

$$C_\varphi^* f(z) = \int_{\partial \mathbb{D}} \frac{f(w)}{1 - \overline{\varphi(w)} z} \, dw \tag{1}$$

where $\partial \mathbb{D}$ is the boundary of $\mathbb{D}$, i.e., $\partial \mathbb{D} = \{z \in \mathbb{C} : |z| = 1\}$, the unit circle.

We provide Example 1 to help us understand how we might use Formula 1 to solve for the kernel of C_φ^* by searching for the functions f that make Formula 1 equal to 0.

Example 1. Consider $\varphi(z) = \dfrac{z^2+1}{2}$. Substituting φ into Formula 1 and simplifying gives us

$$\begin{aligned} C_\varphi^* f(z) &= \int_{\partial\mathbb{D}} \frac{f(w)}{1-\overline{\varphi(w)}z} dw \\ &= \int_{\partial\mathbb{D}} \frac{2f(w)}{2-\overline{w}^2 z - z} dw \end{aligned}$$

There is no straightforward method for finding solutions for f that make this integral equal to 0, and therefore, using Formula 1 to approach this problem is convoluted and not efficient.

Indeed, Formula 1 does not provide much insight to C_φ^* or its kernel due to the complicated nature of evaluating such an integral for a given $f \in H^2$, let alone an arbitrary $f \in H^2$. However, the work of Cowen and Gallardo-Gutiérrez [2] in 2006 followed by that of Hammond, Moorhouse, and Robbins [4] as well as that of Bourdon and Shapiro [1] in 2008 provides the explicit formula in Theorem 2 for C_φ^* on H^2 that is more user-friendly and motivates the remainder of the work in this paper.

Theorem 2 (Hammond, Moorhouse, and Robbins [4])**.** Let $\varphi : \mathbb{D} \to \mathbb{D}$ be a non-constant rational map, and let C_φ act on H^2. Set $\sigma(z) = \dfrac{1}{\overline{\varphi^{-1}(1/\overline{z})}}$, $\psi(z) = \dfrac{z\sigma'(z)}{\sigma(z)}$, and $\varphi(\infty) = \lim_{|z|\to\infty} \varphi(z)$. Then,

$$C_\varphi^* f(z) = \frac{f(0)}{1-\overline{\varphi(\infty)}z} + \sum \psi(z) f(\sigma(z)) \tag{2}$$

where the sum is taken over the branches of σ.

Our φ is a map of degree two, so its inverse φ^{-1} has two branches. Since σ depends on φ^{-1}, then σ also has two branches, say σ_1 and σ_2, which can be defined on the extended complex plane, i.e., the complex plane joined with the point at infinity, $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Therefore, for our quadratic rational map φ, Formula 2 gives

$$C_\varphi^* f(z) = \frac{f(0)}{1-\overline{\varphi(\infty)}z} + \sum_{j=1}^{2} \psi_j(z) f(\sigma_j(z)). \tag{3}$$

Recall $\varphi(z) = \dfrac{z^2+1}{2}$ from Example 1. We again try to solve for the kernel of C_φ^*, this time using Formula 3 and setting it equal to 0.

Example 2. Consider $\varphi(z) = \dfrac{z^2+1}{2}$. First, we compute $\sigma(z)$ by solving for $\varphi^{-1}(z)$, substituting in $\dfrac{1}{\overline{z}}$, taking the conjugate, and, finally, taking the reciprocal. Thus, the two branches of σ are $\sigma(z) = \pm\dfrac{1}{\sqrt{\frac{2}{z}-1}} = \pm\sqrt{\dfrac{z}{2-z}}$. We set $\sigma_1(z) = \sqrt{\dfrac{z}{2-z}}$ and $\sigma_2(z) = -\sqrt{\dfrac{z}{2-z}}$, and, therefore, $\psi_1(z) = \dfrac{z\sigma_1'(z)}{\sigma_1(z)} = \dfrac{1}{2-z}$ and $\psi_2(z) = \dfrac{z\sigma_2'(z)}{\sigma_2(z)} = \dfrac{1}{2-z}$. Next, we substitute these functions into Formula 3 to obtain

$$
\begin{aligned}
C_\varphi^* f(z) &= \frac{f(0)}{1-\overline{\varphi(\infty)}z} + \sum_{j=1}^{2} \psi_j(z) f(\sigma_j(z)) \\
&= \frac{f(0)}{1-\overline{\varphi(\infty)}z} + \psi_1(z) f(\sigma_1(z)) + \psi_2(z) f(\sigma_2(z)) \\
&= \frac{f(0)}{1-\overline{\varphi(\infty)}z} + \frac{1}{2-z}\cdot f\left(\sqrt{\frac{z}{2-z}}\right) + \frac{1}{2-z}\cdot f\left(-\sqrt{\frac{z}{2-z}}\right) \qquad (4)
\end{aligned}
$$

Note that $\varphi(\infty) = \lim\limits_{|z|\to\infty}\varphi(z) = \lim\limits_{|z|\to\infty}\dfrac{z^2+1}{2} = \infty$, and therefore, $\dfrac{f(0)}{1-\overline{\varphi(\infty)}z} = 0$.

Now, to find the kernel of C_φ^*, we set Equation 4 equal to 0 and solve for f. For convenience, we also make the substitution $w = \sqrt{\dfrac{z}{2-z}}$. We obtain the following equation:

$$
\frac{1}{2-z}\cdot f(w) + \frac{1}{2-z}\cdot f(-w) = 0
$$

which is equivalent to $f(w) = -f(-w)$ and shows that f is an odd function. Therefore, the kernel of C_φ^* is the set of odd functions.

Example 2 demonstrates that Formula 3 can be used to solve for the kernel of C_φ^* and is more direct than Formula 1. However, obtaining the functions σ and ψ may not necessarily be computationally efficient, and solving the resulting functional equation by setting Formula 3 equal to 0 is not always an obvious task. Instead of using Formula 3, we consider solving for the kernel of C_φ^*, $\ker(C_\varphi^*)$, using Miller's work [7] in which a different functional equation (derived from Equation 3) is introduced and only involves a linear fractional map that depends on the coefficients of φ.

Theorem 3 (Miller [7])**.** Let $\varphi(z) = \dfrac{a_1z^2+b_1z+c_1}{a_2z^2+b_2z+c_2}$ be a rational map of degree two mapping $\mathbb{D}$ into $\mathbb{D}$, and let C_φ act on H^2. Then, $f \in \ker(C_\varphi^*)$ if and only if

$$
\zeta(z)f(z) + z\zeta'(z)f(\zeta(z)) = 0. \qquad (5)
$$

where

$$\zeta(z) = -\frac{(\overline{a_1}\,\overline{b_2} - \overline{b_1}\,\overline{a_2}) + (\overline{a_1}\,\overline{c_2} - \overline{c_1}\,\overline{a_2})z}{(\overline{a_1}\,\overline{c_2} - \overline{c_1}\,\overline{a_2}) + (\overline{b_1}\,\overline{c_2} - \overline{c_1}\,\overline{b_2})z}. \tag{6}$$

Once more, we consider the function $\varphi(z) = \dfrac{z^2+1}{2}$ from Examples 1 and 2, and we solve for the kernel of C_φ^* using Equation 5.

Example 3. Consider $\varphi(z) = \dfrac{z^2+1}{2}$. Then, $\zeta(z) = -z$ using Formula 6, and thus $\zeta'(z) = -1$. Substituting ζ and ζ' into Equation 5, we have

$$-zf(z) + z(-1)f(-z) = 0$$

which can be simplified to be $f(z) = -f(-z)$, i.e., f is an odd function. Therefore, we again see that the kernel of C_φ^* is the set of odd functions.

Example 3 shows that Equation 5 provides a more efficient and concise approach to classifying the functions in the kernel of C_φ^* compared to Formulas 1 and 3. In fact, since Equation 5 only depends on ζ and not on φ, Example 3 solves for the kernel of C_φ^* for *any* function φ that has $\zeta(z) = -z$. Thus, we can conclude that the kernel of C_φ^* is the set of odd functions for any quadratic rational map φ that maps the unit disk to itself and has a corresponding $\zeta(z) = -z$.

To help further the research in finding the kernel of C_φ^* using Equation 5, we need to understand the relationship between the coefficients of φ in order for φ to map $\mathbb{D}$ to $\mathbb{D}$, which in turn, will determine ζ. We now turn our attention to our main objective of finding conditions on the coefficients of φ such that φ maps the unit disk to itself.

2 Findings

Recall that throughout this work, our interest is where quadratic rationals, which can be written in the form $\varphi(z) = \dfrac{a_1z^2+b_1z+c_1}{a_2z^2+b_2z+1}$ where $a_1, b_1, c_1, a_2, b_2 \in \mathbb{C}$ with at least one nonzero quadratic term, map the disk to itself.

2.1 Conditions on φ

The following propositions give sufficient conditions for various forms of φ to map the unit disk to itself.

Proposition 1. Let $\varphi(z) = a_1z^2 + b_1z + c_1$, where $a_1, b_1, c_1 \in \mathbb{C}$ and $a_1 \neq 0$. If the condition

$$|a_1| + |b_1| + |c_1| \leq 1$$

is satisfied, then $\left|\varphi(z)\right| < 1$ for all $z \in \mathbb{D}$, that is, φ maps $\mathbb{D}$ to itself.

Proof. By applying the triangle inequality and taking $|z| < 1$ for any $z \in \mathbb{D}$, we obtain the following:

$$\begin{aligned}\left|\varphi(z)\right| = \left|a_1 z^2 + b_1 z + c_1\right| &\leq \left|a_1 z^2\right| + |b_1 z| + |c_1| \\ &= |a_1||z|^2 + |b_1||z| + |c_1| \\ &< |a_1| + |b_1| + |c_1| \\ &\leq 1\end{aligned}$$

Therefore, $\left|\varphi(z)\right| < 1$ for $z \in \mathbb{D}$. □

Unfortunately, the converse of Proposition 1 is not true, as illustrated by $\varphi(z) = \frac{1}{2}z^2 + \frac{1}{2}z - \frac{1}{4}$ which maps the unit disk to itself as seen in Figure 2.1 but $\frac{5}{4} = |a_1| + |b_1| + |c_1| > 1$. While our investigation into necessary conditions for the coefficients of φ to map the unit disk to itself was not fruitful within the time constraints of this project, we do propose that such conditions be researched further.

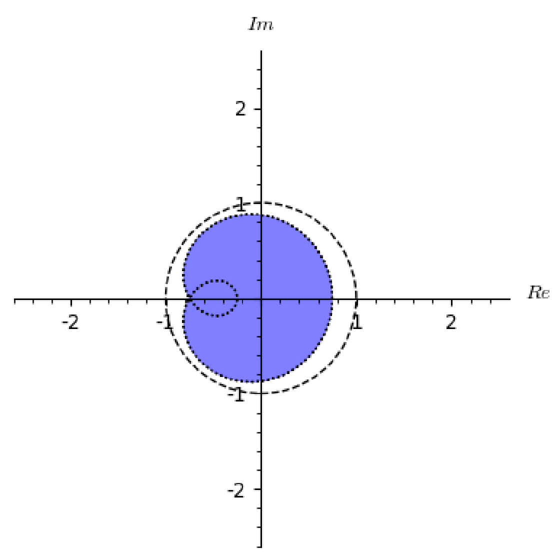

Figure 3: The shaded region is the image of the unit disk under the map $\varphi(z) = \frac{1}{2}z^2 + \frac{1}{2}z - \frac{1}{4}$. The image lies entirely inside the unit disk, whose boundary is displayed for comparison.

Proposition 2. Let $\varphi(z) = \dfrac{a_1 z^k}{a_2 z^2 + 1}$ where $a_1, a_2 \in \mathbb{C}$ are nonzero, $\dfrac{1}{|a_2|} > 1$, and $k = 0, 1, 2$.

If one of the conditions

$$\begin{cases} \dfrac{1}{\frac{1}{|a_2|}-1} < 1 \quad \text{and} \quad \left|\dfrac{a_1}{a_2}\right| \leq 1 \\[2ex] \dfrac{1}{\frac{1}{|a_2|}-1} < 1 \quad \text{and} \quad 1 < \left|\dfrac{a_1}{a_2}\right| \leq \dfrac{1}{|a_2|} - 1 \\[2ex] \dfrac{1}{\frac{1}{|a_2|}-1} \geq 1 \quad \text{and} \quad \left|\dfrac{a_1}{a_2}\right| < \dfrac{1}{|a_2|} - 1 \end{cases}$$

is satisfied, then $\left|\varphi(z)\right| < 1$ for all $z \in \mathbb{D}$, that is, φ maps $\mathbb{D}$ to itself.

Proof. First, in order for $\varphi(z) = \dfrac{a_1 z^k}{a_2 z^2 + 1}$ to be analytic on $\mathbb{D}$, its poles must lie outside of $\mathbb{D}$. The poles of φ, which are the roots of the denominator, are $\pm i\sqrt{\dfrac{1}{a_2}}$. If $\left|\pm i\sqrt{\dfrac{1}{a_2}}\right| = \left|\sqrt{\dfrac{1}{a_2}}\right| > 1$, or equivalently, $\dfrac{1}{|a_2|} > 1$, then the poles of φ lie outside of $\mathbb{D}$, and thus φ is analytic on $\mathbb{D}$ as desired. Now, we factor a_2 from the denominator to obtain $\varphi(z) = \dfrac{a_1}{a_2} \cdot \dfrac{z^k}{z^2 + \frac{1}{a_2}}$. Ignoring the factor of $\dfrac{a_1}{a_2}$ for now, we focus on the simplified function $\widetilde{\varphi}(z) = \dfrac{z^k}{z^2 + \frac{1}{a_2}}$. Again, since we require the poles of φ, and hence the poles of $\widetilde{\varphi}$, to lie outside of the unit disk, then $\dfrac{1}{|a_2|} > 1$.

Next, we search for all such functions $\widetilde{\varphi}$ that map the unit disk to itself, that is, all forms of the function $\widetilde{\varphi}$ such that $\left|\widetilde{\varphi}(z)\right| < 1$ for all $z \in \mathbb{D}$. By applying the reverse triangle inequality and taking $|z| < 1$ for any $z \in \mathbb{D}$, we obtain

$$\begin{aligned} \left|\widetilde{\varphi}(z)\right| &= \left|\frac{z^k}{z^2 + \frac{1}{a_2}}\right| \\ &= \frac{|z|^k}{\left|z^2 - \left(-\frac{1}{a_2}\right)\right|} \\ &\leq \frac{1}{\frac{1}{|a_2|} - |z|^2} \\ &< \frac{1}{\frac{1}{|a_2|} - 1} \end{aligned}$$

We consider the two cases when $\dfrac{1}{\frac{1}{|a_2|}-1} < 1$ and when $\dfrac{1}{\frac{1}{|a_2|}-1} \geq 1$.

Case 1. If $\dfrac{1}{\frac{1}{|a_2|}-1} < 1$, then $\left|\widetilde{\varphi}(z)\right|$ for all $z \in \mathbb{D}$ is guaranteed to be less than 1, and thus $\widetilde{\varphi}$ maps $\mathbb{D}$ to itself. We consider the two subcases when $\left|\dfrac{a_1}{a_2}\right| \leq 1$ and $\left|\dfrac{a_1}{a_2}\right| > 1$.

Subcase (i). If $\left|\dfrac{a_1}{a_2}\right| \leq 1$, then the modulus of $\varphi(z) = \dfrac{a_1}{a_2} \cdot \widetilde{\varphi}(z)$ is less than 1 for all $z \in \mathbb{D}$. Thus, φ maps $\mathbb{D}$ to itself.

Subcase (ii). If $1 < \left|\dfrac{a_1}{a_2}\right| \leq \dfrac{1}{|a_2|} - 1$, then

$$\begin{aligned}
\left|\varphi(z)\right| &= \left|\frac{a_1}{a_2}\right| \cdot \left|\widetilde{\varphi}(z)\right| \\
&< \left(\frac{1}{|a_2|} - 1\right) \cdot \frac{1}{\frac{1}{|a_2|}-1} \\
&= 1
\end{aligned}$$

Thus, φ maps $\mathbb{D}$ to itself.

Case 2. If $\dfrac{1}{\frac{1}{|a_2|}-1} \geq 1$, and additionally, if $\left|\dfrac{a_1}{a_2}\right| < \dfrac{1}{|a_2|} - 1$, then

$$\begin{aligned}
\left|\varphi(z)\right| &= \left|\frac{a_1}{a_2}\right| \cdot \left|\widetilde{\varphi}(z)\right| \\
&< \left(\frac{1}{|a_2|} - 1\right) \cdot \frac{1}{\frac{1}{|a_2|}-1} \\
&= 1
\end{aligned}$$

Thus, φ maps $\mathbb{D}$ to itself.

□

Proposition 3. Let $\varphi(z) = \dfrac{a_1 z^2}{b_2 z + 1}$ where $a_1, b_2 \in \mathbb{C}$ are nonzero and $\dfrac{1}{|b_2|} > 1$. If one of the conditions

$$\begin{cases} \dfrac{1}{\frac{1}{|b_2|}-1} < 1 \quad \text{and} \quad \left|\dfrac{a_1}{b_2}\right| \le 1 \\ \dfrac{1}{\frac{1}{|b_2|}-1} < 1 \quad \text{and} \quad 1 < \left|\dfrac{a_1}{b_2}\right| \le \dfrac{1}{|b_2|} - 1 \\ \dfrac{1}{\frac{1}{|b_2|}-1} \ge 1 \quad \text{and} \quad \left|\dfrac{a_1}{b_2}\right| < \dfrac{1}{|b_2|} - 1 \end{cases}$$

is satisfied, then $|\varphi(z)| < 1$ for all $z \in \mathbb{D}$, that is, φ maps $\mathbb{D}$ to itself.

Proof. This proof follows similarly to that of Proposition 2 since the pole $z = -\dfrac{1}{b_2}$ of φ lies outside of the unit disk and $\varphi(z) = \dfrac{a_1}{b_2} \cdot \dfrac{z^2}{z + \frac{1}{b_2}}$. □

Proposition 4. Let $\varphi(z) = \dfrac{a_1 z^2 + b_1 z^k}{a_2 z^l + 1}$ where $a_1, b_1, a_2 \in \mathbb{C}$ are nonzero, $\dfrac{1}{|a_2|} > 1$, $k = 0, 1$, and $l = 1, 2$. If one of the conditions

$$\begin{cases} \dfrac{1+\left|\frac{b_1}{a_1}\right|}{\frac{1}{|a_2|}-1} < 1 \quad \text{and} \quad \left|\dfrac{a_1}{a_2}\right| \le 1 \\ \dfrac{1+\left|\frac{b_1}{a_1}\right|}{\frac{1}{|a_2|}-1} < 1 \quad \text{and} \quad 1 < \left|\dfrac{a_1}{a_2}\right| \le \dfrac{\frac{1}{|a_2|}-1}{1+\left|\frac{b_1}{a_1}\right|} \\ \dfrac{1+\left|\frac{b_1}{a_1}\right|}{\frac{1}{|a_2|}-1} \ge 1 \quad \text{and} \quad \left|\dfrac{a_1}{a_2}\right| < \dfrac{\frac{1}{|a_2|}-1}{1+\left|\frac{b_1}{a_1}\right|} \end{cases}$$

is satisfied, then $|\varphi(z)| < 1$ for all $z \in \mathbb{D}$, that is, φ maps $\mathbb{D}$ to itself.

Proof. As show in the proofs of Propositions 2 and 3, φ is analytic on $\mathbb{D}$ when the poles of φ lie outside of the unit disk, that is, when $\dfrac{1}{|a_2|} > 1$. Following similar steps as in the rest of the proof of Proposition 2, we begin by factoring a_1 from the numerator and a_2 from the denominator to rewrite φ as $\varphi(z) = \dfrac{a_1}{a_2} \cdot \dfrac{z^2 + \frac{b_1}{a_1} z^k}{z^l + \frac{1}{a_2}}$. Setting $\widetilde{\varphi}(z) = \dfrac{z^2 + \frac{b_1}{a_1} z^k}{z^l + \frac{1}{a_2}}$, we apply the triangle inequality in the numerator and the reverse triangle inequality in the denominator, and take $|z|<1$ for any $z\in\mathbb{D}$ to obtain

$$\begin{aligned}
\left|\widetilde{\varphi}(z)\right| &= \left|\frac{z^2+\frac{b_1}{a_1}z^k}{z^l+\frac{1}{a_2}}\right| \\
&= \left|\frac{z^2+\frac{b_1}{a_1}z^k}{z^l-\left(-\frac{1}{a_2}\right)}\right| \\
&\le \frac{|z|^2+\left|\frac{b_1}{a_1}\right||z|^k}{\frac{1}{|a_2|}-|z|^2} \\
&< \frac{1+\left|\frac{b_1}{a_1}\right|}{\frac{1}{|a_2|}-1}
\end{aligned}$$

We consider the two cases when $\dfrac{1+\left|\frac{b_1}{a_1}\right|}{\frac{1}{|a_2|}-1}<1$ and when $\dfrac{1+\left|\frac{b_1}{a_1}\right|}{\frac{1}{|a_2|}-1}\ge 1$.

Case 1. If $\dfrac{1+\left|\frac{b_1}{a_1}\right|}{\frac{1}{|a_2|}-1}<1$, then $\left|\widetilde{\varphi}(z)\right|$ for all $z\in\mathbb{D}$ is guaranteed to be less than 1, and thus $\widetilde{\varphi}$ maps $\mathbb{D}$ to itself. We consider the two subcases when $\left|\dfrac{a_1}{a_2}\right|\le 1$ and $\left|\dfrac{a_1}{a_2}\right|>1$.

Subcase (i). If $\left|\dfrac{a_1}{a_2}\right|\le 1$, then the modulus of $\varphi(z)=\dfrac{a_1}{a_2}\cdot\widetilde{\varphi}(z)$ is less than 1 for all $z\in\mathbb{D}$. Thus, φ maps $\mathbb{D}$ to itself.

Subcase (ii). If $1<\left|\dfrac{a_1}{a_2}\right|\le\dfrac{\frac{1}{|a_2|}-1}{1+\left|\frac{b_1}{a_1}\right|}$, then

$$\begin{aligned}
\left|\varphi(z)\right| &= \left|\frac{a_1}{a_2}\right|\cdot\left|\widetilde{\varphi}(z)\right| \\
&< \frac{\frac{1}{|a_2|}-1}{1+\left|\frac{b_1}{a_1}\right|}\cdot\frac{1+\left|\frac{b_1}{a_1}\right|}{\frac{1}{|a_2|}-1} \\
&= 1
\end{aligned}$$

Thus, φ maps $\mathbb{D}$ to itself.

Case 2. If $\dfrac{1+\left|\frac{b_1}{a_1}\right|}{\frac{1}{|a_2|}-1} \geq 1$, and additionally, if $\left|\dfrac{a_1}{a_2}\right| < \dfrac{\frac{1}{|a_2|}-1}{1+\left|\frac{b_1}{a_1}\right|}$, then

$$\begin{aligned}
\left|\varphi(z)\right| &= \left|\frac{a_1}{a_2}\right| \cdot \left|\widetilde{\varphi}(z)\right| \\
&< \frac{\frac{1}{|a_2|}-1}{1+\left|\frac{b_1}{a_1}\right|} \cdot \frac{1+\left|\frac{b_1}{a_1}\right|}{\frac{1}{|a_2|}-1} \\
&= 1
\end{aligned}$$

Thus, φ maps $\mathbb{D}$ to itself.

□

Proposition 5. Let $\varphi(z) = \dfrac{b_1 z + c_1}{a_2 z^2 + 1}$ where $b_1, c_1, a_2 \in \mathbb{C}$ are nonzero and $\dfrac{1}{|a_2|} > 1$. If one of the conditions

$$\begin{cases}
\dfrac{1+\left|\frac{c_1}{b_1}\right|}{\frac{1}{|a_2|}-1} < 1 \quad \text{and} \quad \left|\dfrac{b_1}{a_2}\right| \leq 1 \\[2ex]
\dfrac{1+\left|\frac{c_1}{b_1}\right|}{\frac{1}{|a_2|}-1} < 1 \quad \text{and} \quad 1 < \left|\dfrac{b_1}{a_2}\right| \leq \dfrac{\frac{1}{|a_2|}-1}{1+\left|\frac{c_1}{b_1}\right|} \\[2ex]
\dfrac{1+\left|\frac{c_1}{b_1}\right|}{\frac{1}{|a_2|}-1} \geq 1 \quad \text{and} \quad \left|\dfrac{b_1}{a_2}\right| < \dfrac{\frac{1}{|a_2|}-1}{1+\left|\frac{c_1}{b_1}\right|}
\end{cases}$$

is satisfied, then $\left|\varphi(z)\right| < 1$ for all $z \in \mathbb{D}$, that is, φ maps $\mathbb{D}$ to itself.

Proof. This proof follows similarly to that of Proposition 4 since the poles $z = \pm i\sqrt{\dfrac{1}{a_2}}$ of φ lie outside of the unit disk and $\varphi(z) = \dfrac{b_1}{a_2} \cdot \dfrac{z + \frac{c_1}{b_1}}{z^2 + \frac{1}{a_2}}$.

□

Note, in this paper, we have not addressed the conditions for the coefficients of quadratic rational maps φ whose numerator or denominator is of the form $az^2 + bz + c$, where a, b, c are all nonzero, and for which φ maps $\mathbb{D}$ to itself.

Type	I	II	III	IV	V
ζ	$-z$	$-\dfrac{A}{z}$	$-(A+z)$	$-\dfrac{z}{1-Bz}$	$-\dfrac{A+z}{1+Bz}$

Table 1: ζ assumes one of the five types given here for any quadratic rational map φ. Note that $A, B \in \mathbb{C}$ are nonzero constants.

2.2 Possible Forms of ζ

From the coefficients of φ, we are able to identify all possible forms of ζ, from Formula 6. We categorize them into five types shown in Table 1.

Although we have not found conditions for the coefficients of every quadratic rational map φ that maps $\mathbb{D}$ to itself, we are still able to find their ζ types. Table 2 lists all the forms of the function φ with their corresponding ζ. Note that the coefficients for each φ are nonzero. This convention diverges from Proposition 1 where we did not require that all of the coefficients in $a_1 z^2 + b_1 z + c_1$ be nonzero, but notice in Table 2 that $a_1 z^2$ and $a_1 z^2 + b_1 z$ (with $b_1 \neq 0$) have different ζ types.

$\varphi(z)$	$\zeta(z)$	Type
$a_1 z^2$	$-z$	I
$a_1 z^2 + c_1$		
$\dfrac{a_1 z^2}{a_2 z^2 + 1}$		
$\dfrac{c_1}{a_2 z^2 + 1}$		
$\dfrac{a_1 z^2 + c_1}{a_2 z^2 + 1}$		
$\dfrac{b_1 z}{a_2 z^2 + 1}$	$\dfrac{\overline{a_2}}{z}$	II
$\dfrac{b_1 z}{a_2 z^2 + b_2 z + 1}$		
$\dfrac{a_1 z^2}{b_2 z + 1}$	$-\left(\overline{b_2} + z\right)$	III
$\dfrac{a_1 z^2}{a_2 z^2 + b_2 z + 1}$		
$a_1 z^2 + b_1 z$	$-\dfrac{\overline{a_1} z}{\overline{a_1} + \overline{b_1} z}$	IV
$a_1 z^2 + b_1 z + c_1$		
$\dfrac{c_1}{a_2 z^2 + b_2 z + 1}$	$-\dfrac{\overline{a_2} z}{\overline{a_2} + \overline{b_2} z}$	

$\varphi(z)$	$\zeta(z)$	Type
$\dfrac{a_1 z^2 + b_1 z}{a_2 z^2 + 1}$	$\dfrac{\overline{b_1 a_2} - \overline{a_1} z}{\overline{a_1} + \overline{b_1} z}$	V
$\dfrac{b_1 z + c_1}{a_2 z^2 + 1}$	$\dfrac{\overline{b_1 a_2} + \overline{c_1 a_2} z}{-\overline{c_1 a_2} + \overline{b_1} z}$	
$\dfrac{a_1 z^2 + b_1 z + c_1}{a_2 z^2 + 1}$	$\dfrac{\overline{b_1 a_2} - (\overline{a_1} - \overline{c_1 a_2}) z}{(\overline{a_1} - \overline{c_1 a_2}) + \overline{b_1} z}$	
$\dfrac{a_1 z^2 + c_1}{b_2 z + 1}$	$-\dfrac{\overline{a_1 b_2} + \overline{a_1} z}{\overline{a_1} - \overline{c_1 b_2} z}$	
$\dfrac{a_1 z^2 + b_1 z}{b_2 z + 1}$	$-\dfrac{\overline{a_1 b_2} + \overline{a_1} z}{\overline{a_1} + \overline{b_1} z}$	
$\dfrac{a_1 z^2 + b_1 z + c_1}{b_2 z + 1}$	$-\dfrac{\overline{a_1 b_2} + \overline{a_1} z}{\overline{a_1} + (\overline{b_1} - \overline{c_1 b_2}) z}$	
$\dfrac{a_1 z^2 + b_1}{a_2 z^2 + b_2 z + 1}$	$-\dfrac{(\overline{a_1 b_2} - \overline{b_1 a_2}) + \overline{a_1} z}{\overline{a_1} + \overline{b_1} z}$	
$\dfrac{a_1 z^2 + c_1}{a_2 z^2 + b_2 z + 1}$	$-\dfrac{\overline{a_1 b_2} + (\overline{a_1} - \overline{c_1 a_2}) z}{(\overline{a_1} - \overline{c_1 a_2}) - \overline{c_1 b_2} z}$	
$\dfrac{b_1 z + c_1}{a_2 z^2 + b_2 z + 1}$	$-\dfrac{\overline{b_1 a_2} + \overline{c_1 a_2} z}{\overline{c_1 a_2} - (\overline{b_1} - \overline{c_1 b_2}) z}$	
$\dfrac{a_1 z^2 + b_1 z + c_1}{a_2 z^2 + b_2 z + 1}$	$-\dfrac{(\overline{a_1 b_2} - \overline{b_1 a_2}) + (\overline{a_1} - \overline{c_1 a_2}) z}{(\overline{a_1} - \overline{c_1 a_2}) + (\overline{b_1} - \overline{c_1 b_2}) z}$	

Table 2: Each form of φ corresponds to a particular type of ζ found by using Formula 6.

We can now classify the kernel of C_φ^* for entire groups of functions φ, namely those functions φ that correspond to ζ of Type I. As we remarked after Example 3, it follows from solving Equation 5 that $\ker(C_\varphi^*) = \{f \in H^2 : f(z) = -f(-z)\}$ for each φ that has the corresponding $\zeta(z) = -z$.

2.3 An Interesting Example

While we have seen the ease of using Equation 5 to find the kernel of C_φ^* for some functions φ, we turn our attention to an example for which Equation 5 does *not* yield a straightforward classification of functions in $\ker(C_\varphi^*)$. We consider $\varphi(z) = \dfrac{z^2+z}{2}$ (seen in Figure 4) and attempt to investigate the nature of the kernel of C_φ^* using Equation 5.

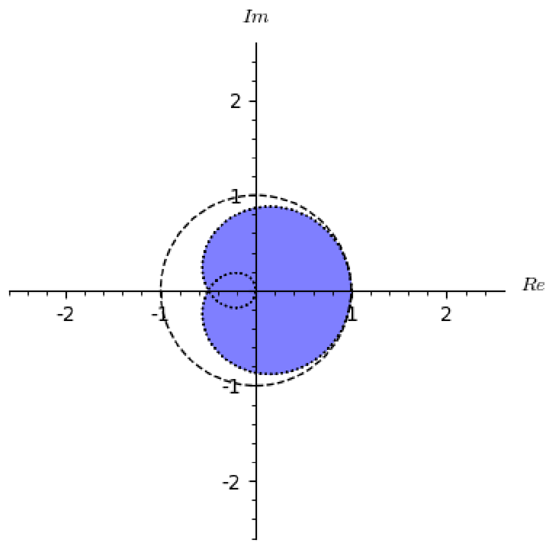

Figure 4: The shaded region is the image of the unit disk under the map $\varphi(z) = \dfrac{z^2+z}{2}$. The image lies entirely inside the unit disk, whose boundary is displayed for comparison.

For this φ, we find $\zeta(z) = -\dfrac{z}{1+z}$ and $\zeta'(z) = \dfrac{-1}{(1+z)^2}$. Notice that substituting ζ and ζ' into Equation 5 and solving for $f(z)$ gives

$$-\frac{z}{1+z} \cdot f(z) + z \cdot \frac{-1}{(1+z)^2} \cdot f\left(-\frac{z}{1+z}\right) = 0$$

$$\Rightarrow \qquad f(z) = -\frac{1}{1+z} \cdot f\left(-\frac{z}{1+z}\right) \tag{7}$$

Unfortunately, we are unable to readily identify which functions f satisfy Equation 7, and so we proceed in trying to find the kernel of C_φ^* in two alternative ways.

2.3.1 First Attempt. Our first attempt at finding the kernel of C_φ^* still begins by rearranging Equation 5 in the following way:

$$\begin{aligned} & \zeta(z)f(z) + z\zeta'(z)f(\zeta(z)) = 0 \\ \Leftrightarrow\quad & z\zeta'(z)f(\zeta(z)) = -\zeta(z)f(z) \\ \Rightarrow\quad & \frac{-z\zeta'(z)}{\zeta(z)}f(\zeta(z)) = f(z) \\ \Leftrightarrow\quad & \frac{-z\zeta'(z)}{\zeta(z)}C_\zeta f(z) = f(z) \end{aligned} \tag{8}$$

Before we proceed further, we introduce another type of operator [3] which offers a more convenient notation for Equation 8.

Definition 5. Let ρ be a bounded analytic function on $\mathbb{D}$. The Toeplitz operator T_ρ, with symbol ρ, acting on H^2 is defined by

$$T_\rho h(z) = (\rho h)(z) = \rho(z)h(z)$$

for $z \in \mathbb{D}$ and $h \in H^2$.

Using the Toeplitz operator T_ρ with symbol $\rho = \dfrac{-z\zeta'}{\zeta}$, we can rewrite Equation 8 as

$$T_{-z\zeta'/\zeta}C_\zeta f(z) = f(z). \tag{9}$$

Recall from Theorem 3, $f \in \ker(C_\varphi^*)$ if and only if f satisfies Equation 5, which is equivalent to Equation 9. Notice that in Equation 9, $f(z)$ is an eigenvector of the operator $T_{-z\zeta'/\zeta}C_\zeta$ and has an associated eigenvalue of 1. Therefore, $\ker(C_\varphi^*)$ is the eigenspace of the operator $T_{-z\zeta'/\zeta}C_\zeta$ associated with the eigenvalue of 1.

In order to find the eigenspace of the operator $T_{-z\zeta'/\zeta}C_\zeta$ associated with the eigenvalue 1, we proceed by finding the matrix representation, M, of the operator $T_{-z\zeta'/\zeta}C_\zeta$ with respect to the standard basis $\{1, z, z^2, \ldots\}$ in H^2. The vectors in this eigenspace represent the power series coefficients of functions in $\ker(C_\varphi^*)$, that is, if the vector $\langle a_n\rangle$ is in this eigenspace of M, then $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is in $\ker(C_\varphi^*)$.

For our symbol $\varphi = \dfrac{z^2+z}{2}$ with $\zeta(z) = -\dfrac{z}{1+z}$ and $\zeta'(z) = \dfrac{-1}{(1+z)^2}$, we begin constructing the matrix representation, M, of the operator $T_{-z\zeta'/\zeta}C_\zeta$ by simplifying the expression for the symbol of the Toeplitz operator:

$$\frac{-z\zeta'(z)}{\zeta(z)} = -z\cdot -\frac{1}{(1+z)^2}\cdot -\frac{1+z}{z} = -\frac{1}{1+z} \tag{10}$$

Now, we find the image of each standard basis vector of H^2 under $\mathrm{T}_{-z\zeta'/\zeta}\mathrm{C}_\zeta$ using

$$\mathrm{T}_{-z\zeta'/\zeta}\mathrm{C}_\zeta f(z) = \frac{-z\zeta'(z)}{\zeta(z)} f(\zeta(z)) = -\frac{1}{1+z} \cdot f\left(-\frac{z}{1+z}\right). \tag{11}$$

After substituting the first standard basis vector, $f(z) = 1$, into Equation 11 we obtain

$$\begin{aligned}
\mathrm{T}_{-z\zeta'/\zeta}\mathrm{C}_\zeta f(z) &= -\frac{1}{1+z} \cdot 1 \\
&= -\frac{1}{1+z} \\
&= \sum_{n=0}^{\infty} (-1)^{n+1} z^n = -1 + z - z^2 + - \cdots
\end{aligned}$$

In the last step, we rewrite $-\dfrac{1}{z+1}$ as its power series expansion centered about $z = 0$. We use the general form of the Geometric Power Series, $\dfrac{1}{1-u} = \sum_{n=0}^{\infty} u^n = 1 + u + u^2 + u^3 + \cdots$, and we rewrite $-\dfrac{1}{1+z}$ as $-\dfrac{1}{1-(-z)} = \sum_{n=0}^{\infty} (-1)^{n+1} z^n = -1 + z - z^2 + - \cdots$. The coefficients of the terms in the series make up the entries for the first column of the matrix of $\mathrm{T}_{-z\zeta'/\zeta}\mathrm{C}_\zeta$, which is

$$\begin{bmatrix} -1 \\ 1 \\ -1 \\ \vdots \end{bmatrix}.$$

Substituting the second basis vector, $f(z) = z$, into Equation 11 we obtain

$$\begin{aligned}
\mathrm{T}_{-z\zeta'/\zeta}\mathrm{C}_\zeta f(z) &= -\frac{1}{1+z} \cdot -\frac{z}{1+z} \\
&= \frac{z}{(1+z)^2} \\
&= \sum_{n=0}^{\infty} (-1)^{n+1} n z^n = 0 + z - 2z^2 + 3z^3 - + \cdots
\end{aligned}$$

Again, we rewrite $\dfrac{z}{(1+z)^2}$ as its power series representation centered about $z = 0$. We can find this expansion by considering

$$\begin{aligned}\frac{z}{(1+z)^2} &= z\cdot\frac{d}{dz}\left(-\frac{1}{1+z}\right)\\ &= z\cdot\frac{d}{dz}\left(\sum_{n=0}^{\infty}(-1)^{n+1}z^n\right)\\ &= z\cdot\sum_{n=0}^{\infty}(-1)^{n+1}nz^{n-1}\\ &= z\cdot\sum_{n=0}^{\infty}(-1)^{n+1}nz^{n}\end{aligned}$$

Therefore, $\frac{z}{(1+z)^2} = \sum_{n=0}^{\infty}(-1)^{n+1}nz^n$. The coefficients of the series give us the entries of the second column of M:

$$\begin{bmatrix} 0\\ 1\\ -2\\ 3\\ -4\\ \vdots \end{bmatrix}$$

Substituting the third basis vector, $f(z) = z^2$, in Equation 11 we obtain

$$\begin{aligned}\mathrm{T}_{-z\zeta'/\zeta}\mathrm{C}_{\zeta}f(z) &= -\frac{1}{1+z}\cdot f\left(-\frac{z}{1+z}\right)\\ &= -\frac{1}{1+z}\cdot\left(-\frac{z}{1+z}\right)^2\\ &= -\frac{z^2}{(1+z)^3}\\ &= \sum_{n=0}^{\infty}(-1)^{n+1}\frac{1}{2}n(n-1)z^n = 0+0z-z^2+3z^3-6z^4+-\cdots\end{aligned}$$

Again, using the derivative of the previous series, we can write the last line in the form of a series expansion, following the work below:

$$\begin{aligned}-\frac{z^2}{(1+z)^3} &= \frac{1}{2}z^2\cdot\frac{d^2}{dz^2}\left(-\frac{1}{1+z}\right)\\ &= \frac{1}{2}z^2\cdot\sum_{n=0}^{\infty}(-1)^{n+1}n(n-1)z^{n-2}\\ &= \sum_{n=0}^{\infty}(-1)^{n+1}\frac{1}{2}n(n-1)z^n\end{aligned}$$

The coefficients of the series give us the entries for the third column of M:

$$\begin{bmatrix} 0 \\ 0 \\ -1 \\ 3 \\ -6 \\ \vdots \end{bmatrix}$$

We continue in this way and find that the matrix follows the familiar pattern of Pascal's triangle:

$$M = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & \dots \\ -1 & -2 & -1 & 0 & 0 & \dots \\ 1 & 3 & 3 & 1 & 0 & \dots \\ -1 & -4 & -6 & -4 & -1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Since M is a lower-triangular matrix with 1 and -1 along the main diagonal, the eigenvalues of M are 1 and -1. However, as this is an infinite matrix, we are not able to determine the eigenspace associated to 1 given the tools for finite-dimensional matrices from a typical undergraduate linear algebra course. Instead, we investigate the behavior of the eigenvalue 1 and its associated eigenspace for different square blocks of the matrix M. We first consider the 5×5 block of M, M_5. After row reducing $M_5 - I_5$, we obtain

$$M_5 - I_5 = \begin{bmatrix} -2 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ -1 & -2 & -2 & 0 & 0 \\ 1 & 3 & 3 & 0 & 0 \\ -1 & -4 & -6 & -4 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -2 & -1 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

which has 2 free variables (as seen from the 2 columns that do not have leading ones), i.e., the eigenspace associated to 1 has dimension 2. If we repeat this process for the 7×7 block of M, M_7, then we obtain

$$M_7 - I_7 = \begin{bmatrix} -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -2 & -2 & 0 & 0 & 0 & 0 \\ 1 & 3 & 3 & 0 & 0 & 0 & 0 \\ -1 & -4 & -6 & -4 & -2 & 0 & 0 \\ 1 & 5 & 10 & 10 & 5 & 1 & 0 \\ -1 & -6 & -15 & -20 & -15 & -6 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 5 & 6 & 2 \\ 0 & 0 & 1 & 0 & -5 & -6 & -2 \\ 0 & 0 & 0 & 1 & 3 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

which has 3 free variables (as seen from the 3 columns that do not have leading ones), i.e., the eigenspace associated to 1 has dimension 3. Experimentally, the number of free variables increases when investigating larger blocks of M, corroborating with [7] that the eigenspace associated to 1, and hence $\ker(C_\varphi^*)$, is infinite-dimensional.

2.3.2 Second Attempt. Our second approach to this investigation is to find $\ker(C_\varphi^*)$ more directly by obtaining the matrix representation with respect to the standard basis of H^2 of C_φ^* and solving for its null space. We construct the matrix for C_φ^* by finding the matrix representation, N, with respect to the standard basis of H^2 for C_φ and taking its conjugate transpose, $\overline{N^\top}$, that is, we transpose the matrix N and then take the complex conjugate of each entry [5].

We begin by finding the images of the basis vectors of H^2 under the composition operator C_φ, and we recall one more time that $\varphi(z) = \dfrac{z^2+z}{2} = \dfrac{1}{2}z + \dfrac{1}{2}z^2$. For the first basis vector $f(z) = 1$, we obtain

$$C_\varphi f(z) = f(\varphi(z)) = 1.$$

Similar to our first approach, the columns of the matrix come from the coefficients of the linear combinations of the standard basis $\{1, z, z^2, \ldots\}$. We see the first column of N is

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{bmatrix}.$$

For the second basis vector, $f(z) = z$, we obtain

$$C_\varphi f(z) = f(\varphi(z)) = \varphi(z) = \frac{1}{2}z + \frac{1}{2}z^2$$

and so the second column of N is

$$\begin{bmatrix} 0 \\ 1/2 \\ 1/2 \\ 0 \\ \vdots \end{bmatrix}.$$

For the third basis vector, $f(z) = z^2$:

$$C_\varphi f(z) = f(\varphi(z)) = \varphi(z)^2 = \left(\frac{1}{2}z + \frac{1}{2}z^2\right)^2 = \frac{1}{4}z^2 + \frac{1}{2}z^3 + \frac{1}{4}z^4$$

Then the third column of N is

$$\begin{bmatrix} 0 \\ 0 \\ 1/4 \\ 1/2 \\ 1/4 \\ \vdots \end{bmatrix}.$$

For the fourth basis vector, $f(z) = z^3$, we will use the Binomial Theorem to expand $\varphi(z)^3$.

$$C_\varphi f(z) = f(\varphi(z)) = \varphi(z)^3 = \left(\frac{1}{2}z + \frac{1}{2}z^2\right)^3 = \frac{1}{8}z^3 + \frac{3}{8}z^4 + \frac{3}{8}z^5 + \frac{1}{8}z^6$$

Hence the fourth column of N is

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1/8 \\ 3/8 \\ 3/8 \\ 1/8 \\ \vdots \end{bmatrix}.$$

In fact, since the basis vectors of H^2 are the increasing powers of z, the images of the basis vectors under C_φ are the increasing powers of the binomial $\varphi(z) = \frac{1}{2}z + \frac{1}{2}z^2$, so we are able to find the rest of the columns of N via the Binomial Theorem to obtain

$$N = \begin{bmatrix} 1 & 0 & 0 & 0 & \dots \\ 0 & 1/2 & 0 & 0 & \dots \\ 0 & 1/2 & 1/4 & 0 & \dots \\ 0 & 0 & 1/2 & 1/8 & \dots \\ 0 & 0 & 1/4 & 3/8 & \dots \\ 0 & 0 & 0 & 3/8 & \dots \\ 0 & 0 & 0 & 1/8 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Now, the matrix for C_φ^* is found by taking the conjugate transpose of N:

$$\overline{N^\top} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1/2 & 1/2 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1/4 & 1/2 & 1/4 & 0 & 0 & \dots \\ 0 & 0 & 0 & 1/8 & 3/8 & 3/8 & 1/8 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Note that since all of the entries of N are real, all of the entries of $\overline{\mathrm{N}^\top}$ are also real. At this point, we seek the nullspace of $\overline{\mathrm{N}^\top}$, but are unable to determine it using the finite-dimensional tools for a typical undergraduate linear algebra course. Corroborating with [3], our experiment suggests that the eigenvalues are the non-negative integer powers of $\frac{1}{2}$.

While we have not yet had success in finding $\ker(C_\varphi^*)$ for $\varphi(z) = \dfrac{z^2+z}{2}$, we find it curious that Pascal's triangle and the binomial coefficients appear in the matrices obtained by these two approaches. We also wonder if there are other connections between these two approaches.

3 Conclusion and Further Research

Our research has led to the identification of conditions for different forms of a degree-two rational map φ, as well as the classification of five ζ types. The results provide sufficient conditions for certain φ to map the unit disk to itself and the classification of the functions in the kernel of C_φ^* in the Hardy space using Miller's approach [7]. Additionally, for an interesting case of φ where the approach is not straightforward, we attempted to find $\ker(C_\varphi^*)$ in two different ways, resulting in the discovery of some interesting patterns in the generated matrices.

However, our research brought us to the limits of our knowledge of infinite-dimensional matrices. Nonetheless, we would like to investigate the properties of the matrix for $T_{-z\zeta'/\zeta}C_\zeta$ that follows Pascal's triangle. In the future, we hope to apply our methods to investigate other cases of φ by finding sufficient conditions for all possible forms of degree-two rational maps from the unit disk to itself. It would be ideal to also provide necessary conditions for quadratic rational maps from the unit disk to itself. Ultimately, we seek to identify functions in $\ker(C_\varphi^*)$ for the different forms of φ and the corresponding five types of ζ to gain a better understanding of the properties of C_φ^*. For each type of ζ, we would consider general forms of the function f, such as polynomials or linear fractional transformations, and attempt to solve Equation 5. We hope this process would provide insight into solving for power series representations of functions f that solve Equation 5, i.e., functions f that are in $\ker(C_\varphi^*)$.

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