

## The Isoperimetric Inequality: Proofs by Convex and Differential Geometry

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## The Isoperimetric Inequality: Proofs by Convex and Differential Geometry

### Cover Page Footnote

I wish to express my sincere thanks to Professor Dr. Carla Cederbaum for her continuing guidance and support. I am grateful to Dr. Armando J. Cabrera Pacheco for all his suggestions to improve my writing style. And I am also thankful to Michael and Anja for correcting my written English. In addition, I am also grateful to the referee for his positive and detailed report.

## The Isoperimetric Inequality: Proofs by Convex and Differential Geometry

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By *Penelope Gehring*

**Abstract.** The Isoperimetric Inequality has many different proofs using methods from diverse mathematical fields. In the paper, two methods to prove this inequality will be shown and compared. First the 2-dimensional case will be proven by tools of elementary differential geometry and Fourier analysis. Afterwards the theory of convex geometry will briefly be introduced and will be used to prove the Brunn–Minkowski-Inequality. Using this inequality, the Isoperimetric Inequality in  $n$  dimensions will be shown.

### 1 Introduction

The Isoperimetric Problem (*isos* is the ancient Greek word for equal and *perimetron* for perimeter) is the search for the biggest possible area of planar domains with fixed perimeter. The ancient Greeks wanted to measure the size of islands by calculating the time needed to circumnavigate it by ship. Already at that time, they understood that this method is insecure, because an island with bigger perimeter does not need to have bigger area (see [6]). Out of this historical context, the famous 2-dimensional Isoperimetric Inequality was developed. It states that  $4\pi A \leq U^2$ , where  $U$  is the perimeter and  $A$  is the area of a domain. Equality holds if and only if the area is a circle. While the ancient Greeks had previously understood that for fixed perimeter the circle has the biggest possible area, it was not until the 19th century that the inequality was proven. The first proofs used calculus of variations, but many different kinds of proofs and generalizations to  $n$  dimensions were developed.

In this paper, we will show two different approaches to prove the Isoperimetric Inequality. First, we will present a proof of the 2-dimensional case, which uses methods from differential geometry. Because we want to use tools from differential geometry, we only look at domains with boundaries that can be described by continuously differentiable curves. The perimeter of these domains can be identified with the length

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of the boundary curve. In addition, we can use the parametrization of the curve to write formulas for the enclosed area, so that we can convert the problem into a simpler analytic question. Secondly, the proof for  $n$  dimensions uses convex geometry and looks at the domain itself instead of the boundary. Here, we require the sets to be convex but this is not restrictive at all. In two dimensions, this can be seen by mirroring the dents, such that the perimeter stays the same but the area increases (see Figure 1). In  $n$  dimensions this argument is replaced by Steiner's Symmetrization [2].

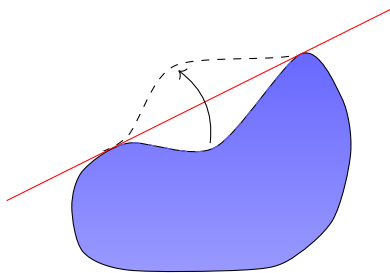


Figure 1: The dent

We will prove the  $n$ -dimensional Isoperimetric Inequality

$$S(K) \geq nV_n(B_n)^{1/n}V_n(K)^{1-1/n},$$

where  $S$  is the surface measure,  $V_n$  is the  $n$ -dimensional volume measure and  $K$  is a convex, compact set with non-empty interior  $\overset{\circ}{K}$ . In this case, the equality holds if and only if  $K$  is a ball.

## 2 Notation

In this section, we will briefly discuss the basic notation. More specific notation will be defined when it is needed. For  $z \in \mathbb{C}$ , we denote the imaginary part of  $z$  by  $\Im(z)$  and the real part by  $\Re(z)$ . Furthermore, we denote the closed  $n$ -dimensional unit ball by  $B^n$  and its topological boundary as  $S^{n-1}$ . Let  $A \subset \mathbb{R}^n$  be an arbitrary set, then the topological closer of  $A$  is denoted by  $\overline{A}$  and the topological interior by  $\overset{\circ}{A}$ . For two points  $x, y \in \mathbb{R}^n$ , we write  $[x, y] = \{z \in \mathbb{R}^n; \exists t \in [0, 1] \text{ with } z = (1-t)x + ty\}$ .

## 3 The Isoperimetric Inequality in differential geometry

In the proof of the Isoperimetric Inequality in two dimensions, we will look at domains with continuously differentiable boundary. We will identify the Euclidean plane with  $\mathbb{C}$  and we will consider the boundary as a periodic function from  $\mathbb{R}$  to  $\mathbb{C}$ , so that we are able to use Fourier analysis. In the next two subsections, we will give a brief introduction of Fourier analysis and differential geometry.

### 3.1 Fourier series

Fourier series expand periodic functions from  $\mathbb{R}$  to  $\mathbb{C}$ . We will use this to simplify the formulas of area by rewriting it using Fourier coefficients. Following [4], we will introduce the theory behind the Fourier series and answer the question of existence and convergence of Fourier expansions.

**Definition 3.1.** A function  $f : \mathbb{R} \rightarrow \mathbb{C}$  is called *periodic* with period  $L \in \mathbb{R}$  if it satisfies

$$f(x + L) = f(x)$$

for all  $x \in \mathbb{R}$ .

**Remark 3.2.** We will concentrate on functions with period  $2\pi$ . We can transform a periodic function  $f$  with period  $L$  to a function with period  $2\pi$  using  $F(x) := f(\frac{L}{2\pi}x)$ .

**Definition 3.3.** Let  $f : \mathbb{R} \rightarrow \mathbb{C}$  be a periodic function that is Riemann integrable on the interval  $[0, 2\pi]$ . Then the numbers

$$c_k := \frac{1}{2\pi} \int_0^{2\pi} f(x)e^{-ikx} dx, \quad k \in \mathbb{Z}$$

are called the *Fourier coefficients* of  $f$ . With the partial sums  $\mathcal{F}_n[f](x) = \sum_{k=-N}^N c_k e^{ikx}$ , we define the formal limit of the partial sums

$$\mathcal{F}[f](x) := \lim_{n \rightarrow \infty} \mathcal{F}_n[f](x) = \sum_{k=-\infty}^{\infty} c_k e^{ikx}$$

as the *Fourier series* of  $f$ .

**Theorem 3.4.** Let  $f : \mathbb{R} \rightarrow \mathbb{C}$  be a periodic function that is piecewise continuously differentiable. Then the Fourier series of  $f$  converges uniformly to  $f$ .

*Proof.* Can be found in [4] (Chapter 23, Theorem 3). □

With the theorem above, we can identify the boundary curve of the domain in the Isoperimetric Inequality with its Fourier series.

### 3.2 Plane curves

In this subsection, we consider curves in the Euclidean plane. Following [1] we will introduce briefly the language of the differential geometry of curves.

**Definition 3.5.** Let  $I \subseteq \mathbb{R}$  be an interval. A *plane parameterized curve* is a continuously differentiable map  $c : I \rightarrow \mathbb{R}^2$ . A plane parameterized curve is called *regular*, if for all  $t \in I$  the derivative of  $c$ , denoted by  $\dot{c} = (c'_1, c'_2)$ , satisfies  $\dot{c}(t) \neq 0$ .

**Note:** From now on, all our curves will be regularly parametrized curves.

**Definition 3.6.** Let  $c : I \rightarrow \mathbb{R}^2$  be a curve. A *transformation of parameter* of  $c$  is a diffeomorphism  $\varphi : J \rightarrow I$  with an interval  $J \subseteq \mathbb{R}$ . The curve  $\tilde{c} = c \circ \varphi : J \rightarrow \mathbb{R}^2$  is called *reparametrization*.

**Definition 3.7.** A curve  $c : \mathbb{R} \rightarrow \mathbb{R}^2$  is called

- (a) *periodic with period*  $L$ , if it holds that
  - (i)  $\exists L > 0 : c(t + L) = c(t) \forall t \in \mathbb{R}$ , and
  - (ii)  $\forall L' \text{ with } c(t + L') = c(t) \forall t \in \mathbb{R} : L \leq L'$ .
- (b) *closed* if it has a periodic regular parametrization.
- (c) *simply closed* if it has a periodic regular parametrization  $c$  with period  $L$  so that  $c|_{[0,L]}$  is injective.

**Remark 3.8.** The definition of a period above is a special case of Definition 3.1. Contrary to definition 3.1, we require that the period  $L$  is the smallest possible number with the needed properties. Later, this will be important because we want to know at which time the curve finishes one round around the domain. Furthermore, we need a boundary curve of a domain and, if we have a curve that is not simply closed, our enclosed set is not connected.

### 3.3 Length and area

In this subsection, we will show a relation between the parametrization of a curve and the enclosed area. First, we will introduce a few properties on the length of a curve that can be shown with simple computation; hence, we will skip the proofs. We will use the notation of [1].

**Definition 3.9.** A *unit speed curve* is a regular curve  $c : I \rightarrow \mathbb{R}^2$  satisfying  $\|\dot{c}(t)\| = 1 \forall t \in I$ , where  $\|\cdot\|$  is the Euclidean norm.

**Proposition 3.10.** *For all regular curves  $c$ , there exists a transformation of parameter  $\varphi$  so that the reparametrization  $c \circ \varphi$  is a unit speed curve.*

**Definition 3.11.** Let  $c : [a, b] \rightarrow \mathbb{R}^2$  be a curve. Then  $L[c] := \int_a^b \|\dot{c}(t)\| dt$  is called the *length* of  $c$ .

**Lemma 3.12.** *The length of a curve is invariant under reparametrization.*

**Remark 3.13.** Before we prove the Isoperimetric Inequality we will show a way to compute the area of a domain using the parametrization of the boundary curve.

**Lemma 3.14.** *Let  $G \subseteq \mathbb{R}^2$  be a bounded domain such that the boundary is a simply closed curve with parametrization  $c(t) = (x(t), y(t))^T$  and period  $L$ . Then*

$$A[G] = - \int_0^L \dot{x}(t)y(t) dt = \int_0^L x(t)\dot{y}(t) dt = \frac{1}{2} \int_0^L (x(t)\dot{y}(t) - \dot{x}(t)y(t)) dt.$$

*Proof.* Let  $c$  be the curve mentioned above. Without loss of generality, assume that  $c$  is a unit speed curve and is positively oriented. Let  $n(t)$  be the inner unit normal of  $c(t)$  and define  $id_{\mathbb{R}^2} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  with  $id_{\mathbb{R}^2}(x, y) = (x, y)$ . Using Gauss theorem and the fact that the curve is a unit speed curve, we get

$$\int_G \left( \frac{\partial(id_{\mathbb{R}^2})^1}{\partial x} + \frac{\partial(id_{\mathbb{R}^2})^2}{\partial y} \right) dx dy = \int_G \operatorname{div} id_{\mathbb{R}^2} dx dy = - \int_0^L \langle id_{\mathbb{R}^2}(c(t)), n(c(t)) \rangle dt.$$

First look at the left side of this equation

$$\int_G \left( \frac{\partial(id_{\mathbb{R}^2})^1}{\partial x} + \frac{\partial(id_{\mathbb{R}^2})^2}{\partial y} \right) dx dy = \int_G \left( \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} \right) dx dy = 2 \int_G dx dy = 2A[G].$$

For the right side, we compute

$$\begin{aligned} - \int_0^L \langle id_{\mathbb{R}^2}(c(t)), n(c(t)) \rangle dt &= - \int_0^L (-x(t)\dot{y}(t) + \dot{x}(t)y(t)) dt \\ &= \int_0^L (x(t)\dot{y}(t) - \dot{x}(t)y(t)) dt. \end{aligned}$$

Putting both sides together, we get

$$2A[G] = \int_0^L (x(t)\dot{y}(t) - \dot{x}(t)y(t)) dt \Leftrightarrow A[G] = \frac{1}{2} \int_0^L (x(t)\dot{y}(t) - \dot{x}(t)y(t)) dt.$$

Using partial integration, it follows

$$\int_0^L x(t)\dot{y}(t) dt = \underbrace{[x(t)y(t)]_0^L}_{=0} - \int_0^L \dot{x}(t)y(t) dt = - \int_0^L \dot{x}(t)y(t) dt.$$

□

### 3.4 The proof of the Isoperimetric Inequality I

After introducing some notations of the theory of Fourier analysis and differential geometry, we are now able to prove the Isoperimetric Inequality. This proof can also be found in [1].

**Theorem 3.15.** (*Isoperimetric Inequality*)

Let  $G \subseteq \mathbb{R}^2$  be a bounded domain such that the boundary is a simply closed curve with period  $L$ . Let  $A[G]$  be the area and  $U[G]$  be the perimeter of  $G$ . Then

$$4\pi \cdot A[G] \leq U[G]^2.$$

Equality holds if and only if  $G$  is a circle.

*Proof.* Without loss of generality, assume  $c$  is a unit speed curve. Consider the injective restriction  $c|_{[0,L]}$  and by abusing notation call it also  $c$ . Let  $c(t) = (x(t), y(t))^T$  and consider the complex valued function

$$z: \mathbb{R} \rightarrow \mathbb{C}, \quad z(t) = x\left(\frac{L}{2\pi}t\right) + iy\left(\frac{L}{2\pi}t\right).$$

The function  $z$  parametrizes the curve  $c$ . Now, we can look at the Fourier series of  $z$ , written (see Definition 3.3) as follows

$$\mathcal{F}_n[z](t) = \sum_{k=-\infty}^{\infty} c_k e^{ikt},$$

where  $c_k$  are the Fourier coefficients. Because  $z$  is a periodic, continuously differentiable function, with Theorem 3.4 we can conclude that

$$z(t) = \sum_{k=-\infty}^{\infty} c_k e^{ikt}.$$

*Step 1.* We want to express the length of  $c$  by the Fourier coefficients. For this, we first compute

$$\int_0^{2\pi} |\dot{z}(t)|^2 dt = \int_0^{2\pi} \left(\frac{L}{2\pi}\right)^2 \underbrace{\left\| \dot{c}\left(\frac{Lt}{2\pi}\right) \right\|^2}_{=1 \text{ unit speed}} dt = \frac{L^2}{2\pi}, \quad (1)$$

where the derivative of  $z$  is given by  $\dot{z}(t) = \sum_{k=-\infty}^{\infty} c_k i k e^{ikt}$ . Hence, we get

$$\begin{aligned} \int_0^{2\pi} |\dot{z}(t)|^2 dt &= \int_0^{2\pi} \dot{z}(t) \overline{\dot{z}(t)} dt = \int_0^{2\pi} \sum_{k,l=-\infty}^{\infty} c_k \bar{c}_l k l e^{i(k-l)t} dt \\ &= 2\pi \sum_{k=-\infty}^{\infty} |c_k|^2 k^2. \end{aligned}$$

Equation (1) implies that for the length of  $c$

$$L[c]^2 = (2\pi)^2 \sum_{k=-\infty}^{\infty} k^2 |c_k|^2.$$



*Step 2.* Next, we want to express the area using the Fourier coefficients. From Lemma 3.14 it follows that

$$\begin{aligned} 2A[G] &= \int_0^{2\pi} \Re(z(s) \cdot i \cdot \overline{\dot{z}(s)}) ds \\ &= \int_0^{2\pi} \Re\left(\sum_{k=-\infty}^{\infty} c_k e^{iks} \cdot i \cdot \sum_{l=-\infty}^{\infty} \overline{c_l} (-i) l e^{-ils}\right) ds \\ &= \Re\left(\int_0^{2\pi} \sum_{k,l=-\infty}^{\infty} l c_k \overline{c_l} e^{i(k-l)s} ds\right) = 2\pi \sum_{k=-\infty}^{\infty} k |c_k|^2 \end{aligned}$$

*Step 3.* Now, we put the first and the second step together. From

$$\frac{A[G]}{\pi} = \sum_{k=-\infty}^{\infty} k |c_k|^2 \leq \sum_{k=-\infty}^{\infty} k^2 |c_k|^2 = \frac{L[c]^2}{4\pi^2}$$

follows  $4\pi A[G] \leq L[c]^2$ . In two dimensions we have  $U[G] = L[c]$ . Thus, we showed the Isoperimetric Inequality. Equality holds if and only if

$$\sum_{k=-\infty}^{\infty} k |c_k|^2 = \sum_{k=-\infty}^{\infty} k^2 |c_k|^2.$$

This is equivalent to  $c_k = 0 \forall k \neq 0, 1$ . This is satisfied if and only if we have  $z(t) = c_0 + c_1 \cdot e^{it}$ , i.e., a circle.  $\square$

In the next section, we introduce convex geometry, which looks at the Isoperimetric Problem from a different point of view.

## 4 The Isoperimetric Inequality In convex geometry

In the convex geometry approach, we consider convex, compact sets in  $\mathbb{R}^n$  and their volumes. Our main goal is to prove the theorem of Brunn–Minkowski, which states that the  $n$ -th root of the volume functional  $V_n$  is concave for convex, compact sets. More precisely, for convex, compact sets  $K_1, K_2 \subseteq \mathbb{R}^n$  and  $\lambda \in [0, 1]$  it holds

$$V_n((1 - \lambda)K_0 + \lambda K_1)^{\frac{1}{n}} \geq (1 - \lambda)V_n(K_0)^{\frac{1}{n}} + \lambda V_n(K_1)^{\frac{1}{n}}.$$

The idea of the proof for this theorem is to show the inequality above using induction on  $n$ . Here, we will use the Integral Theorem of Fubini to establish induction step. In this chapter, some statements of the theory of convex, compact sets will be introduced as well as properties of the volume functional.

Later, we will show the Isoperimetric Inequality using the theorem of Brunn–Minkowski. We will follow [8].

#### 4.1 Basics in convex geometry

In this subsection, we will introduce the necessary language and notation, which will be used in the following sections.

**Definition 4.1.** A set  $A \subseteq \mathbb{R}^n$  is called *convex* if all  $x, y \in A$  satisfy

$$(1 - \lambda)x + \lambda y \in A \quad 0 \leq \lambda \leq 1.$$

**Remark 4.2.** Intersections of convex sets, and for affine maps the pre-image and the image of convex sets are also convex. Let  $A, B \subset \mathbb{R}^n$  be convex and  $\lambda \in \mathbb{R}$ . Then

$$\begin{aligned} \lambda B &= \{x \in \mathbb{R}^n; \exists b \in B \text{ with } x = \lambda \cdot b\}, \text{ and} \\ A + B &= \{x \in \mathbb{R}^n; \exists a \in A, b \in B \text{ with } x = a + b\} \end{aligned}$$

are also convex.

**Definition 4.3.** Let  $A \subseteq \mathbb{R}^n$  be non-empty, compact and convex. Then  $A$  is called a *convex body*. The set of all convex bodies of  $\mathbb{R}^n$  is denoted by  $\mathcal{K}^n$  and the set of all convex bodies in  $\mathbb{R}^n$  with non-empty interior is denoted by  $\mathcal{K}_0^n$ .

**Definition 4.4.**

1. Define  $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty, -\infty\}$ .
2. A function  $f : \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$  is called *proper* if it satisfies  $\{f = -\infty\} = \emptyset$  and  $\{f = \infty\} \neq \mathbb{R}^n$ .
3. A function  $f : \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$  is called *convex* if it is proper and it satisfies

$$f((1 - \lambda)x + \lambda y) \leq (1 - \lambda)f(x) + \lambda f(y)$$

for all  $x, y \in \mathbb{R}^n$  and  $0 \leq \lambda \leq 1$ .

4. A function  $f : D \rightarrow \overline{\mathbb{R}}$  with  $D \subseteq \mathbb{R}^n$  is called *convex* if its expansion on  $\mathbb{R}^n$

$$\tilde{f}(x) := \begin{cases} f(x) & \text{for } x \in D \\ \infty & \text{otherwise} \end{cases}$$

is convex.

5. A function  $f$  is *concave* if  $-f$  is convex.

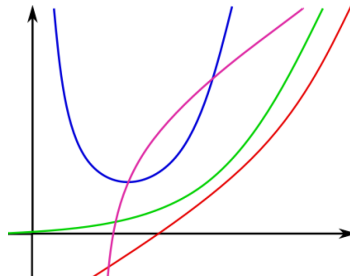


Figure 2: The blue, green and red graphs are associated to convex functions, but the pink graph is associated to a concave function.

### 4.2 The support function

The following section introduces the support function, which can be used to ascertain the "height" of a convex body.

**Definition 4.5.** Let  $K \subseteq \mathbb{R}^n$  be non-empty, convex and closed. The *support function*  $h(K, \cdot) = h_K$  is defined by

$$h(K, u) := \sup\{\langle x, u \rangle; x \in K\} \text{ for } u \in \mathbb{R}^n.$$

Furthermore we define for  $u \in \text{dom } h(K, \cdot) \setminus \{0\}$

$$\begin{aligned} H(K, u) &:= \{x \in \mathbb{R}^n; \langle x, u \rangle = h(K, u)\} \text{ a supporting plane of } K, \\ H^- &:= \{x \in \mathbb{R}^n; \langle x, u \rangle \leq h(K, u)\} \text{ a support half space of } K, \text{ and} \\ F(K, u) &:= H(K, u) \cap K \text{ a support domain of } K. \end{aligned}$$

**Remark 4.6.** The support function for  $u \in \mathbb{S}^{n-1} \cap \text{dom } h(K, \cdot)$  can be viewed as the distance of the supporting plane at  $K$  with outer unit normal to the origin.

**Remark 4.7.** For  $B^n$  and  $u \in \mathbb{S}^{n-1}$  the support function can be computed easily. For arbitrary  $x \in B^n$  and  $u \in \mathbb{S}^{n-1}$  Cauchy–Schwarz gives us

$$|\langle x, u \rangle| \leq \|u\| \|x\| \leq 1.$$

The support function is equal 1 on  $B^n$  since the maximum is also 1.

**Remark 4.8.** The support function has the following properties:

1.  $h_K \leq h_L$  if and only if  $K \subseteq L$ ,
2.  $h(\{z\}, u) = \langle z, u \rangle$  for  $z, u \in \mathbb{R}^n$ ,
3.  $h(K + t, u) = h(K, u) + \langle t, u \rangle$  for  $t, u \in \mathbb{R}^n$ ,

4.  $h(\lambda K, u) = \lambda h(K, u) = h(K, \lambda u)$  for  $\lambda \in \mathbb{R}_0^+$ ,  $u \in \mathbb{R}^n$ ,
5.  $h(-K, u) = h(K, -u)$  for  $u \in \mathbb{R}^n$ ,
6.  $h(K, u_1 + u_2) \leq h(K, u_1) + h(K, u_2)$  for  $u_1, u_2 \in \mathbb{R}^n$ ,
7.  $h(K, \cdot) =: h_K$  is convex.

**Lemma 4.9.** Let  $K, L \in \mathcal{K}^n$ , then  $h(K + L, u) = h(K, u) + h(L, u)$ .

*Proof.* Let  $u \in \mathbb{R}^n \setminus \{0\}$ .

First, we show  $h(K + L, u) \geq h(K, u) + h(L, u)$ . Since  $K$  and  $L$  are compact, the supremum of the support function is attained. Thus, there exists  $x \in K$  and  $y \in L$  such that  $h(L, u) = \langle y, u \rangle$  and  $h(K, u) = \langle x, u \rangle$ .

Then we have

$$h(K, u) + h(L, u) = \langle u, x \rangle + \langle u, y \rangle = \langle u, x + y \rangle \leq h(K + L, u).$$

Secondly, we want to show  $h(K + L, u) \leq h(K, u) + h(L, u)$ . Let  $z \in K + L$  so that  $h(K + L, u) = \langle u, z \rangle$ . Then there exists  $x \in K$  and  $y \in L$  such that  $z = x + y$ . It follows

$$h(K + L, u) = \langle z, u \rangle = \langle x + y, u \rangle = \langle x, u \rangle + \langle y, u \rangle \leq h(K, u) + h(L, u).$$

Both inequalities are shown and so the claim follows.  $\square$

### 4.3 The Hausdorff metric

To prove the continuity of the volume functional and of the support function with respect to convex bodies, we need a metric on the set of convex bodies. This metric will be introduced in the following subsection.

**Definition 4.10.** For  $K, L \in \mathcal{K}^n$ , define the *Hausdorff distance* or *Hausdorff metric* by

$$\delta(K, L) := \min\{\lambda \geq 0; K \subseteq L + \lambda B^n, L \subseteq K + \lambda B^n\}.$$

**Remark 4.11.** The Hausdorff distance is indeed a metric on the set of convex bodies. This follows directly from the properties of minima and from the convexity of  $K$ .

**Lemma 4.12.** Let  $K, L \in \mathcal{K}^n$ ,  $K, L \subseteq RB^n$ , where  $R > 0$ , and  $u, v \in \mathbb{R}^n$ . Then

$$|h(K, u) - h(L, v)| \leq R|u - v| + \max\{|u|, |v|\}\delta(K, L).$$

*Proof.* Can be found in [8] (Section 1.8, Lemma 1.8.12).  $\square$

**Remark 4.13.** Lemma 4.12 proves the local Lipschitz continuity of the support function in both arguments.

**Theorem 4.14.** *Let  $K, L \in \mathcal{K}^n$ , then*

$$\delta(K, L) = \sup_{u \in \mathbb{S}^{n-1}} |h(K, u) - h(L, u)| =: \left\| \bar{h}_K - \bar{h}_L \right\|,$$

where  $\bar{h}_K = h_K|_{\mathbb{S}^{n-1}}$ .

*Proof.* Let  $\delta(K, L) \leq \alpha$ . This implies  $K \subseteq L + \alpha B^n$ . For  $u \in \mathbb{S}^{n-1}$ , it follows

$$h(K, u) \leq h(L + \alpha B^n, u) = h(L, u) + \alpha h(B^n, u) = h(L, u) + \alpha.$$

After switching  $K$  and  $L$ , we can conclude for  $u \in \mathbb{S}^{n-1}$  that

$$|h(K, u) - h(L, u)| \leq \alpha.$$

This implies  $\left\| \bar{h}_K - \bar{h}_L \right\| \leq \delta(K, L)$ . Using a similar argument one can show the other estimation. □

**Lemma 4.15.** *Let  $K_1, K_2 \in \mathcal{K}^n$  and  $K_2 \subseteq \mathring{K}_1$ . Then there exists  $\eta > 0$  such that for all  $K \in \mathcal{K}^n$  with  $\delta(K, K_1) < \eta$ ,  $K_2 \subseteq K$  holds.*

*Proof.* Can be found in [8] (Section 1.8, Lemma 1.8.18) □

#### 4.4 Volume and surface measure

In this subsection, we will introduce the volume functional and show its continuity. The results of this subsection are important in the proof of Brunn–Minkowski.

**Definition 4.16.** Let  $M \subseteq \mathbb{R}^n$ ,  $p \geq 0$ . For  $\delta > 0$  define

$$\mathcal{H}_\delta^p(M) := \inf \left\{ \sum_{i=1}^{\infty} \alpha(p) \left( \frac{\text{diam } C_i}{2} \right)^p ; (C_i)_{i \in \mathbb{N}} \text{ open sets in } \mathbb{R}^n \right. \\ \left. \text{with } \text{diam } C_i \leq \delta \text{ and } M \subseteq \bigcup_{i \in \mathbb{N}} C_i \right\}$$

with  $\alpha(p) = \frac{\pi^{\frac{p}{2}}}{\Gamma(p)(\frac{p}{2}+1)}$  for  $p \neq 0$ , where  $\Gamma(p) := \int_0^\infty t^{p-1} e^{-t} dt$  and  $\alpha(0) = 0$ .

Define the  $p$ -dimensional Hausdorff measure of  $M$  by

$$\mathcal{H}^p(M) := \sup_{\delta > 0} \mathcal{H}_\delta^p(M) = \lim_{\delta \rightarrow 0} \mathcal{H}_\delta^p(M).$$

**Remark 4.17.** Let  $t \in \mathbb{R}^n$  and  $\lambda \in \mathbb{R}$ , then we have

$$\mathcal{H}^p(\lambda M) = \lambda^p \mathcal{H}^p(M) \text{ and } \mathcal{H}^p(M + t) = \mathcal{H}^p(M).$$

Furthermore, for  $A \subseteq B$  we have  $\mathcal{H}^p(A) \leq \mathcal{H}^p(B)$ .

**Definition 4.18.** The *volume functional*  $V_n$  on  $\mathcal{K}^n$  is defined by the restriction of the  $n$ -dimensional Hausdorff measure on  $\mathcal{K}^n$ .

**Theorem 4.19.** The volume functional  $V_n : \mathcal{K}^n \rightarrow \mathbb{R}$  is continuous.

*Proof.* Case 1  $V_n(K) = 0$ :

Let  $\bar{K} \in \mathcal{K}^n$  and, without loss of generality, assume  $\delta(K, \bar{K}) = \alpha \leq 1$ . Thus, it holds that  $\bar{K} \subseteq K + \alpha B^n$ . Since  $V_n(K) = 0$  and  $K$  is convex, we know that  $K$  has to be contained in a hyperplane. From the monotony of the Hausdorff measure it follows for  $u$  orthogonal on the hyperplane, that

$$\begin{aligned} V_n(\bar{K}) &\leq V_n(K + \alpha B^n) \\ &= \int_{-\alpha}^{\alpha} V_{n-1}([K + \alpha B^n] \cap H_{u,\zeta}) d\zeta \\ &\leq \int_{-\alpha}^{\alpha} V_{n-1}(K + (\alpha B^n \cap H_{u,0})) d\zeta \\ &= 2\alpha V_{n-1}(K + (\alpha B^n \cap H_{u,0})) \\ &\leq \underbrace{2V_{n-1}(K + (B^n \cap H_{u,0}))}_{:=c(K)} \cdot \alpha, \end{aligned}$$

In the second line we used that  $K + \alpha B^n$  varies in the fixed coordinate of  $K$  by  $2\alpha$ . Since  $c(K)$  is fixed and does not depend on  $\alpha$  the continuity follows.

Case 2  $V_n(K) > 0$ :

Without loss of generality, assume that  $0 \in \overset{\circ}{K}$  (if not, we can shift it, since the Hausdorff measure is invariant under shifts). Let  $\varepsilon > 0$  and choose  $\lambda > 1$  such that

$$(\lambda^n - 1)\lambda^n V_n(K) < \varepsilon$$

and  $\rho > 0$  so that  $\rho B^n \subseteq \overset{\circ}{K}$ . From Lemma 4.15, we know that there is an  $\alpha > 0$  with  $\alpha \leq (\lambda - 1)\rho$  so that  $\rho B^n \subseteq \overset{\circ}{K} \subseteq \bar{K}$  for all  $\bar{K}$  with  $\delta(K, \bar{K}) < \alpha$ . From  $\delta(K, \bar{K}) < \alpha$  it follows

$$K \subseteq \bar{K} + \alpha B^n \subseteq \bar{K} + (\lambda - 1)\rho B^n \subseteq \bar{K} + (\lambda - 1)\bar{K} = \lambda \bar{K}.$$

With the same argument, we can also show that  $\bar{K} \subseteq \lambda K$ . Hence, we have the inequalities  $V_n(K) \leq V_n(\lambda \bar{K}) = \lambda^n V_n(\bar{K})$  and  $V_n(\bar{K}) \leq V_n(\lambda K) = \lambda^n V_n(K)$ .

This implies

$$\begin{aligned} V_n(K) - V_n(\bar{K}) &\leq (\lambda^n - 1)V_n(\bar{K}) \leq (\lambda^n - 1)\lambda^n V_n(K) \text{ and} \\ V_n(\bar{K}) - V_n(K) &\leq (\lambda^n - 1)V_n(K) \stackrel{\lambda > 1}{\leq} (\lambda^n - 1)\lambda^n V_n(K). \end{aligned}$$

As a consequence we have the estimate

$$\left|V_n(K) - V_n(\bar{K})\right| \leq (\lambda^n - 1)\lambda^n V_n(K) < \varepsilon.$$

This implies the continuity of  $V_n$ . □

**Theorem 4.20.** (*Steiner's Formula*)

There are  $V_m : \mathcal{K}^n \rightarrow \mathbb{R}$  and coefficients  $\kappa_i \in \mathbb{R}$  for  $i = 0, \dots, n$  such that

$$V_n(K + \rho B^n) = \sum_{m=0}^n \rho^{n-m} \kappa_{n-m} V_m(K)$$

for all  $K \in \mathcal{K}^n$ ,  $\rho \geq 0$ .

A proof of this can be found in [8] (Theorem 4.2.1).

**Remark 4.21.** Steiner's formula implies that  $\rho \mapsto V_n(K + \rho B^n)$  is a polynomial of degree  $n$ .

**Definition 4.22.** (surface measure)

Let  $K \subseteq \mathcal{K}^n$ . Then we define the *surface measure* of  $K$  by

$$S(K) := \lim_{\varepsilon \rightarrow 0} \frac{V_n(K + \varepsilon B^n) - V_n(K)}{\varepsilon}.$$

**Remark 4.23.** This limit exists, because  $V_n(K + \varepsilon B^n)$  is a polynomial and, therefore, it is smooth.

**Remark 4.24.** Let  $B^n \subseteq \mathbb{R}^n$  be a ball. We then have

$$S(rB^n) = nr^{n-1}V_n(B^n).$$

*Proof.* Using the convexity of  $B^n$  and the properties of the volume functional, we can compute

$$\begin{aligned} S(B^n) &= \lim_{\rho \downarrow 0} \frac{V_n(B^n + \rho B^n) - V_n(B^n)}{\rho} \\ &= \lim_{\rho \downarrow 0} \frac{((r + \rho)^n - r^n)V_n(B^n)}{\rho} \\ &= V_n(B^n) \lim_{\rho \downarrow 0} \frac{(r + \rho)^n - r^n}{\rho} = V_n(B^n) \cdot n \cdot r^{n-1}. \end{aligned}$$

□

#### 4.5 The theorem of Brunn–Minkowski

First, we will show an important theorem of convex geometry, the Brunn–Minkowski Theorem, which we will use in the proof of the Isoperimetric Inequality in the next subsection.

**Definition 4.25.**  $K_0, K_1 \in \mathcal{K}^n$  are called *homothetic* if they satisfy that  $K_0 = \lambda K_1 + t$  or  $K_1 = \lambda K_0 + t$  with  $\lambda \geq 0$  and  $t \in \mathbb{R}^n$ .

**Theorem 4.26.** (*Brunn–Minkowski*)

Let  $K_0, K_1 \in \mathcal{K}^n$  be two convex bodies and  $\lambda \in [0, 1]$ , then

$$V_n((1 - \lambda)K_0 + \lambda K_1)^{\frac{1}{n}} \geq (1 - \lambda)V_n(K_0)^{\frac{1}{n}} + \lambda V_n(K_1)^{\frac{1}{n}}.$$

Equality holds for  $\lambda \in (0, 1)$  if and only if  $K_0$  and  $K_1$  are contained in parallel hyperplanes or are homothetic.

*Proof.* First, consider the case that  $K_0, K_1 \in \mathcal{K}^n$  are contained in parallel hyperplanes. Let  $K_0 \subseteq E$  and  $K_1 \subseteq F$  with  $E$  and  $F$  hyperplanes. Without loss of generality, assume that for all  $x \in E$  we have  $x_1 = t$  for some  $t \in \mathbb{R}$ , and for all  $y \in F$  we have  $y_1 = r$  for some  $r \in \mathbb{R}$ . It then follows for all  $z \in (1 - \lambda)K_0 + \lambda K_1$  that there are  $x \in K_0$  and  $y \in K_1$  with

$$z_1 = (1 - \lambda)x_1 + \lambda y_1 = (1 - \lambda)t + \lambda r.$$

Therefore,  $(1 - \lambda)K_0 + \lambda K_1$  is also contained in a parallel hyperplane. Since hyperplanes are  $n - 1$  dimensional, equality holds trivially.

Now, let  $K_0, K_1 \in \mathcal{K}^n$  be homothetic and  $K_0 = \mu K_1 + t$  for  $\mu \in \mathbb{R}$  and  $t \in \mathbb{R}^n$ . For  $\lambda \in [0, 1]$  we have

$$\begin{aligned} V_n((1 - \lambda)K_0 + \lambda K_1)^{\frac{1}{n}} &= V_n([(1 - \lambda)\mu + \lambda]K_1 + t)^{\frac{1}{n}} \\ &= V_n([(1 - \lambda)\mu + \lambda]K_1)^{\frac{1}{n}} \\ &= [(1 - \lambda)\mu + \lambda]V_n(K_1)^{\frac{1}{n}} \\ &= (1 - \lambda)V_n(K_0)^{\frac{1}{n}} + \lambda V_n(K_1)^{\frac{1}{n}}, \end{aligned}$$

and therefore equality holds.

In the following, we will show the inequality in Theorem 4.26.

*Case 1*  $\dim K_0 = \dim K_1 < n$ . Since  $V_n(K_0) = V_n(K_1) = 0$  this case is trivial. Equality implies  $\dim((1 - \lambda)K_0 + \lambda K_1) < n$ . This implies that all three convex bodies have to be contained in parallel hyperplanes.

*Case 2*  $\dim K_0 < n, \dim K_1 = n$ . For  $x \in K_0$  we have

$$(1 - \lambda)K_0 + \lambda K_1 \supseteq (1 - \lambda)x + \lambda K_1,$$



which implies that

$$V_n((1 - \lambda)K_0 + \lambda K_1)^{1/n} \geq \lambda V_n(K_1)^{1/n}.$$

This implies the Brunn–Minkowski Inequality. If equality holds then  $K_0 = \{x\}$ . Hence,  $K_0$  and  $K_1$  are homothetic.

*Case 3*  $\dim K_0 = \dim K_1 = n$ . This case will be proven by induction.

The case  $n = 1$  is trivial. Now consider the iteration from  $n - 1$  to  $n$ .

Without loss of generality, assume  $V_n(K_0) = V_n(K_1) = 1$ . This is allowed because the general inequality follows from this special case. Indeed, for  $K_0, K_1 \in \mathcal{K}^n$  being  $n$ -dimensional with arbitrary volume and  $\lambda \in [0, 1]$  consider

$$\bar{K}_i := \frac{K_i}{V_n(K_i)^{1/n}} \quad \text{und} \quad \bar{\lambda} := \frac{\lambda V_n(K_1)^{1/n}}{(1 - \lambda)V_n(K_0)^{1/n} + \lambda V_n(K_1)^{1/n}} \in [0, 1].$$

Then we get

$$\begin{aligned} V_n((1 - \bar{\lambda})\bar{K}_0 + \bar{\lambda}\bar{K}_1)^{\frac{1}{n}} &= V_n\left(\frac{(1 - \lambda)K_0 + \lambda V_n(K_1)^{1/n}\bar{K}_0 - \lambda V_n(K_1)^{1/n}\bar{K}_0 + \lambda K_1}{(1 - \lambda)V_n(K_0)^{1/n} + \lambda V_n(K_1)^{1/n}}\right)^{\frac{1}{n}} \\ &= \frac{1}{(1 - \lambda)V_n(K_0)^{1/n} + \lambda V_n(K_1)^{1/n}} V_n((1 - \lambda)K_0 + \lambda K_1)^{\frac{1}{n}}. \end{aligned}$$

In addition, we also have (because the inequality is already proven for  $V_n(\bar{K}_i) = 1$ )

$$V_n((1 - \bar{\lambda})\bar{K}_0 + \bar{\lambda}\bar{K}_1)^{\frac{1}{n}} \geq (1 - \bar{\lambda})V_n(\bar{K}_0)^{\frac{1}{n}} + \bar{\lambda}V_n(\bar{K}_1)^{\frac{1}{n}} = 1.$$

This implies

$$\frac{1}{(1 - \lambda)V_n(K_0)^{1/n} + \lambda V_n(K_1)^{1/n}} V_n((1 - \lambda)K_0 + \lambda K_1)^{\frac{1}{n}} \geq 1.$$

Thus, the inequality in Theorem 4.26 is proven.

Therefore let  $V_n(K_0) = V_n(K_1) = 1$ . Choose  $u \in \mathbb{S}^n$  and  $\lambda \in [0, 1]$ . We now introduce the following notations to shorten the computations. We set  $K_\lambda := (1 - \lambda)K_0 + \lambda K_1$  and  $\beta_\lambda := h(K_\lambda, u)$ , and  $\alpha_\lambda := -h(K_\lambda, -u)$ . Furthermore, for  $\zeta \in (\alpha_i, \beta_i)$   $i = 0, 1$  we set the half space  $H^-(\zeta) := \{x \in \mathbb{R}^n; \langle x, u \rangle \leq \zeta\}$  and the hyperplane  $H(\zeta) := \{x \in \mathbb{R}^n; \langle x, u \rangle = h(K, u)\}$ , as well as the volume  $v_i(\zeta) := V_{n-1}(K_i \cap H(\zeta))$ .

Let us define for every  $\zeta \in [\alpha_i, \beta_i]$

$$w_i(\zeta) := \int_{\alpha_i}^{\zeta} v_i(t) dt.$$

We know that the support functions  $h$  and  $V_n$  are continuous. Hence,  $V_n(K_i \cap H^-(\zeta))$  is also continuous in  $\zeta$  and we can apply the Fundamental Theorem of Calculus on  $w_i$ . It

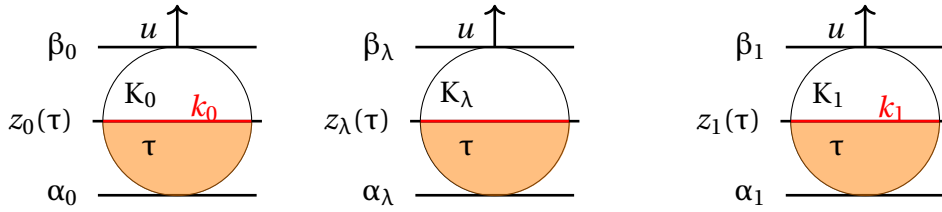


Figure 3: Proof of Brunn–Minkowski.

follows that  $w_i$  is differentiable and it holds that  $w'_i(\zeta) = v_i(\zeta) > 0$  (because  $\dim K_i = n$ ). This implies that  $w_i$  is strictly monotonically increasing and is therefore a bijection on its image  $(0, 1)$ . For  $i=0,1$  define  $z_i$  as the inverse function of  $w_i$ . Intuitively, this means that for a volume  $\tau \in (0, 1)$  with  $V_n(K_i \cap H^-(z_i(\tau))) = \tau$ , the function  $z_i$  maps  $\tau$  to the "height" of  $H$  (see Figure 3).

Consider  $z_\lambda(\tau) := (1 - \lambda)z_0 + \lambda z_1$  for  $\lambda \in (0, 1)$ ,  $i = 0, 1$ , and  $\tau \in (0, 1)$ . Let  $k_i(\tau) := K_i \cap H(z_i(\tau))$ , then we have

$$(1 - \lambda)k_0(\tau) + \lambda k_1(\tau) \subseteq K_\lambda \cap H(z_\lambda(\tau)) \quad \forall \lambda, \tau \in (0, 1). \tag{2}$$

Now, we can prove the inequality in Theorem 4.26 by induction. As mentioned before, the case  $n = 1$  holds trivially. For the iteration from  $n - 1$  to  $n$ , we can compute

$$\begin{aligned} V_n(K_\lambda) &\stackrel{\text{Fubini}}{=} \int_{\alpha_\lambda}^{\beta_\lambda} V_{n-1}(K_\lambda \cap H(\zeta)) \, d\zeta \\ &\stackrel{\zeta=z_\lambda(\tau)}{=} \int_0^1 V_{n-1}(K_\lambda \cap H(z_\lambda(\tau))) \cdot z'_\lambda(\tau) \, d\tau \\ &\stackrel{(2)}{\geq} \int_0^1 V_{n-1}((1 - \lambda)k_0(\tau) + \lambda k_1(\tau)) \cdot z'_\lambda(\tau) \, d\tau \\ &\geq \int_0^1 \left[ (1 - \lambda) \underbrace{V_{n-1}(k_0(\tau))^{\frac{1}{n-1}}}_{=v_0(z_0(\tau))^{\frac{1}{n-1}}} + \lambda \underbrace{V_{n-1}(k_1(\tau))^{\frac{1}{n-1}}}_{=v_1(z_1(\tau))^{\frac{1}{n-1}}} \right]^{n-1} \cdot \left( \frac{1 - \lambda}{v_0(z_0(\tau))} + \frac{\lambda}{v_1(z_1(\tau))} \right) \, d\tau \\ &= \int_0^1 \left( (1 - \lambda) v_0(z_0(\tau))^{\frac{1}{n-1}} + \lambda v_1(z_1(\tau))^{\frac{1}{n-1}} \right)^{n-1} \cdot \left( \frac{1 - \lambda}{v_0(z_0(\tau))} + \frac{\lambda}{v_1(z_1(\tau))} \right) \, d\tau, \end{aligned}$$

where we used in the third line the Inverse Function Theorem, which implies

$$z'_i(\tau) = \frac{1}{w'_i(z_i(\tau))} = \frac{1}{v_i(z_i(\tau))},$$

and the claim for  $n - 1$ .

To simplify the notation, define  $p := \frac{1}{n-1}$  and  $v_i = v_i(z_i(\tau))$ . The concavity of  $\log$  implies

$$\begin{aligned} \log\left([(1-\lambda)v_0^p + \lambda v_1^p]^{\frac{1}{p}} \cdot \left(\frac{1-\lambda}{v_0} + \frac{\lambda}{v_1}\right)\right) &= \frac{1}{p} \log[(1-\lambda)v_0^p + \lambda v_1^p] + \log\left(\frac{1-\lambda}{v_0} + \frac{\lambda}{v_1}\right) \\ &\geq \frac{1}{p} [p(1-\lambda) \log v_0 + p\lambda \log v_1] \\ &\quad - [(1-\lambda) \log v_0 + \lambda \log v_1] = 0. \end{aligned}$$

After applying exp, we see that

$$\left((1-\lambda)v_0(z_0(\tau))^{\frac{1}{n-1}} + \lambda v_1(z_1(\tau))^{\frac{1}{n-1}}\right)^{n-1} \cdot \left(\frac{1-\lambda}{v_0(z_0(\tau))} + \frac{\lambda}{v_1(z_1(\tau))}\right) \geq 1.$$

This implies

$$\int_0^1 \left((1-\lambda)v_0(z_0(\tau))^{\frac{1}{n-1}} + \lambda v_1(z_1(\tau))^{\frac{1}{n-1}}\right)^{n-1} \cdot \left(\frac{1-\lambda}{v_0(z_0(\tau))} + \frac{\lambda}{v_1(z_1(\tau))}\right) d\tau \geq 1$$

on  $(0, 1)$ . Hence, the inequality

$$V_n((1-\lambda)K_0 + \lambda K_1)^{\frac{1}{n}} = V_n(K_\lambda)^{\frac{1}{n}} \geq (1-\lambda)V_n(K_0)^{\frac{1}{n}} + \lambda V_n(K_1)^{\frac{1}{n}}$$

is proven. Now, it only remains to show that the equality implies that  $K_1$  and  $K_2$  are homothetic. Suppose for a  $\lambda \in (0, 1)$  that equality holds.

This implies  $K_\lambda \cap H(z_\lambda(\tau)) = (1-\lambda)k_0(\tau) + \lambda k_1(\tau)$ . Thus, we have  $k_1(\tau) = k_0(\tau)$  and therefore also  $v_1(z_1(\tau)) = v_0(z_0(\tau))$ . This implies  $z'_0(\tau) = z'_1(\tau)$ , and hence,  $z_1(\tau) - z_0(\tau)$  is constant.

Without loss of generality, assume that  $\int_{K_i} y dy = 0$ , for  $i = 0, 1$  (otherwise shift the sets).

Then we see that  $\int_{K_i} \langle y, u \rangle dy = \sum_{i=1}^n u_i \int_{K_i} y_i dy = 0$ . This implies

$$\begin{aligned} 0 &= \int_{K_i} \langle y, u \rangle dy = \int_{\alpha_i}^{\beta_i} \int_{K_i \cap H(\zeta)} \underbrace{\langle u, z \rangle}_{=\zeta} dz d\zeta \\ &= \int_{\alpha_i}^{\beta_i} V_{n-1}(K_i \cap H(\zeta)) \zeta d\zeta \\ &= \int_0^1 V_{n-1}(K_i \cap H(z_i(\tau))) z_i(\tau) \dot{z}_i(\tau) d\tau \\ &= \int_0^1 v_i(z_i(\tau)) z_i(\tau) \frac{1}{v_i(z_i(\tau))} d\tau = \int_0^1 z_i(\tau) d\tau, \end{aligned}$$

by using  $K_i = \bigcup_{\zeta} K_i \cap H(\zeta)$  and Fubini's Theorem.

Now, we know that  $\int_0^1 z_0(\tau) - z_1(\tau) d\tau = 0$ , and  $z_0 = z_1$  on  $(0, 1)$ . Therefore, we also know  $w_1 = w_0$  and  $v_1 = v_0$ . Furthermore, using

$$\int_{\alpha_1}^{\beta_1} v_1(t) dt = w_1 = w_0 = \int_{\alpha_0}^{\beta_0} v_0(t) dt,$$

we can conclude that  $\alpha_0 = \alpha_1$  and  $\beta_1 = \beta_0$ . This implies  $h(K_0, u) = h(K_1, u)$  and, with Remark 4.8, we get that  $K_0 = K_1$ . In particular  $K_0$  and  $K_1$  are homothetic.  $\square$

**Remark 4.27.** The Brunn–Minkowski Inequality can also be proven for domains with differentiable boundary. Using this, it is also possible to prove the Isoperimetric Inequality. In addition, there is also a Brunn–Minkowski Inequality for general domains. More information can be found in [5].

**Corollary 4.28.** *Let  $K_0, K_1 \in \mathcal{K}_n$  and  $I = [0, 1]$ . Then the function*

$$f : I \rightarrow \mathbb{R} \quad \lambda \mapsto V_n((1 - \lambda)K_0 + \lambda K_1)^{1/n}$$

*is concave. The function  $f$  is linear if and only if  $K_0$  and  $K_1$  are contained in parallel hyperplanes or are homothetic.*

The next lemma follows by using basic tools of analysis. Hence, we will not prove it here, but we will use it in the proof of the Isoperimetric Inequality.

**Lemma 4.29.** *Let  $f : I \rightarrow \mathbb{R}$  be a smooth, convex function such that  $f'(0) = f(1) - f(0)$  holds. Then  $f$  is linear.*

#### 4.6 The proof of the Isoperimetric Inequality II

In the last part of this paper, we will prove the Isoperimetric Inequality using methods of convex geometry. Therefore, we need to adapt the assumptions of our theorem.

**Theorem 4.30.** *(Isoperimetric Inequality)*

*Let  $K \in \mathcal{K}_0^n$ . Then*

$$S(K) \geq nV_n(B_n)^{1/n}V_n(K)^{1-1/n}.$$

*Equality holds if and only if  $K$  is a ball.*

*Proof.* Let  $K \in \mathcal{K}_0^n$ . We have  $V_n(K) \neq 0$ . Consider  $\varepsilon := \frac{t}{1-t}$ . We compute

$$\begin{aligned} S(K) &= \lim_{t \downarrow 0} \frac{V_n(K + \frac{t}{1-t}B^n) - V_n(K)}{\frac{t}{1-t}} \\ &= \lim_{t \downarrow 0} \frac{V_n((1-t)K + tB^n) - (1-t)^n V_n(K)}{(1-t)^{n-1} t} \\ &= \lim_{t \downarrow 0} \left[ \frac{V_n((1-t)K + tB^n) - V_n(K)}{t} + \frac{(1 - (1-t)^n)V_n(K)}{t} \right] \\ &= \lim_{t \downarrow 0} \left[ \frac{V_n((1-t)K + tB^n) - V_n(K)}{t} \right] + nV_n(K). \end{aligned}$$

In the third line, we used that the limits are the same. In the last line, we used the Theorem of l'Hôpital. This implies

$$S(K) - nV_n(K) = \lim_{t \downarrow 0} \frac{V_n((1-t)K + tB^n) - V_n(K)}{t}. \quad (*)$$

Now consider the function  $f(t) := V_n((1-t)K + tB^n)^{1/n}$  and observe

$$f'(t) = \frac{1}{n} V_n((1-t)K + tB^n)^{1/n-1} \cdot \frac{d}{dt} V_n((1-t)K + tB^n).$$

This implies  $f'(0) \stackrel{(*)}{=} \frac{1}{n} V_n(K)^{1/n-1} (S(K) - nV_n(K))$ . By using Corollary 4.28, it follows that  $f$  is concave on  $[0, 1]$ . Hence, we know that  $f'(0) \geq f(1) - f(0)$ . This implies the inequality

$$\frac{1}{n} V_n(K)^{1/n-1} S(K) - V_n(K)^{1/n} = \frac{1}{n} V_n(K)^{1/n-1} (S(K) - nV_n(K)) \geq V_n(B^n)^{1/n} - V_n(K)^{1/n}.$$

Therefore, the Isoperimetric Inequality

$$S(K) \geq nV_n(B^n)^{1/n} V_n(K)^{1-1/n}$$

follows. If  $K$  is a ball, equality follows by using Remark 4.24.

Equality implies  $f'(0) = f(1) - f(0)$ . By using the concavity of  $f$  and Lemma 4.29, it follows that  $f$  is linear. From Corollary 4.28, we know that  $K$  and  $B^n$  are homothetic. Hence,  $K$  is a ball.  $\square$

**Remark 4.31.** For dimension two the Isoperimetric Inequality in Theorem 4.31 can be written as

$$U(K) \geq 2V_2(B^2)^{1/2} V_2(K)^{1/2} = 2\sqrt{\pi} A[K]^{1/2},$$

where  $A$  is the area and  $U$  is the perimeter of  $K$ . By squaring this, we get the Isoperimetric Inequality of Theorem 3.15

$$4\pi \cdot A[G] \leq U[G]^2.$$

## 5 Comparison and outlook

In this paper, we showed two different kinds of proof of the Isoperimetric Inequality. In the first proof, we needed a domain which has a boundary with high regularity. This proof cannot be generalized to  $n$  dimensions because it uses Fourier series, which are only defined on  $\mathbb{C}$ . For the second proof, we needed a convex, compact set, but we did not need any regularity assumptions on the boundary. The convexity assumption is not restrictive at all (as we argued in the introduction) and works in  $n$  dimension. The  $n$ -dimensional case can also be proven by methods from differential geometry, for example, by using *mean curvature flow* (see [7]). The interested reader can also find a list of different proofs of the Isoperimetric Inequality in [3].

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