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Sums Involving the Number of Distinct Prime Factors Function

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Cover Page Footnote

I wrote this when I was 17, and looking back now I'm still amazed. This was my first ever paper, and helped launch my research career -- and I couldn't have done it were it not for literally dozens of people around me. Many thanks go to my research supervisors at the National Institute of Standards and Technology: Guru Khalsa, Mark Stiles, Kyoung-Whan Kim, and Vivek Amin. I am also indebted to Howard Cohl, who took a frankly horrifying manuscript and turned it into a publishable paper. Without him this project wouldn't have gotten off the ground. Bruce Berndt, Krishna Alladi, and the anonymous referee also provided valuable feedback. Finally, many thanks go out to my old high school teachers -- William Wuu, Joseph Boettcher, Colleen Adams, Joshua Schuman, Jamie Andrews, and others -- who miraculously put up with me through the years.

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Sums Involving the Number of Distinct Prime Factors Function

Abstract. We find closed form expressions for finite and infinite sums that are weighted by $\omega(n)$, where $\omega(n)$ is the number of distinct prime factors of n. We then derive general convergence criteria for these series. The approach of this paper is to use the theory of symmetric functions to derive identities for the elementary symmetric functions, then apply these identities to arbitrary primes and values of multiplicative functions evaluated at primes. This allows us to reinterpret sums over symmetric polynomials as divisor sums and sums over the natural numbers.

1 Introduction

The number of distinct prime factors of a natural number, $\omega(n)$, is a fundamental arithmetic quantity, and has been the subject of extensive study from Hardy and Ramanujan [5] to Erdős and Kac [2]. We seek to expand the list of known series identities involving $\omega(n)$, with potential applications to the asymptotics of $\omega(n)$ and its higher moments $\omega^k(n)$. The prototype for our results is the Dirichlet series

$$\sum_{n\in\mathbb{N}} \frac{\omega(n)}{n^s} = \zeta(s)P(s),$$

where $\zeta(s)$ is the Riemann zeta function and $P(s)$ is the *prime zeta* function, $P(s) := \sum_p \frac{1}{p^s}$. Results such as this allow us to study the asymptotic behavior of the partial sums $\sum_{n\leq x}\omega(n)$. However, sometimes it is simpler to study smoothed asymptotics $\sum_{n\leq x}\omega(n)f(n)$, for some suitable arithmetic function f. Therefore, in this work we compute closed forms for Dirichlet series involving $\omega(n)f(n)$, for any arbitary multiplicative weights f, and explicitly calculate many special cases. In the last section, we also lay out an extension for higher powers $\omega(n)^k f(n)$, and explicitly calculate some examples for $k = 2$.

We also describe finite versions of the above identities, which involve divisor sums over $\omega(n)f(n)$ for arbitary multiplicative f. This again helps expand our knowledge of the ω function, and the many of the special cases available here do not seem to be present anywhere else in the literature, perhaps because the easiest arithmetic functions to study are *multiplicative*, while $\omega(n)$ is *additive*. Identities mixing both additive and multiplicative functions are not readily available using standard methods, so we use nonstandard symmetric function expansions to study sums involving both. The methods readily generalize, and can be applied to other additive functions.

The paper is organized as follows. After standardizing some notation, we derive identities for the elemtary symmetric functions in Section 3. In Section 4, we explore divisor sums of $\omega(n)$ weighted by other functions as corollaries of our results on symmetric polynomials. In Section 5, Dirichlet series of $\omega(n)$ weighted by other functions are explored. We also compile an extensive list of corollaries, which may prove handy when considering exotic divisor sums. Finally, in Section 6 we present an extension to higher orders, and present a short summary in Section 7.

2 Background and notation

We recall some definitions and background material which will be essential in later parts of this paper. Letting $n = \prod_{1\leq i\leq k} p_i^{\alpha_i}$ denote the prime factorization of n, we have $\omega(n) = k$. For example, $\omega(5) = 1$, $\omega(10) = 2$, and $\omega(100) = \omega(2^2 5^2) = 2$. Recall that a multiplicative function satisfies $f(nm) = f(n)f(m)$ for $(n, m) = 1$, where (n, m) gives the greatest common divisor of n and m. A completely multiplicative function satisfies $f(nm) = f(n)f(m)$ for any n and m. We assume that $f(1) = 1$. An additive function satisfies $f(nm) = f(n)+f(m)$

for $(n,m)=1$, while a completely additive function satisfies $f(nm)=f(n)+f(m)$ for any n and m.

Now, we address some notation. We let $\sum_p a_p$ and $\prod_p a_p$ denote sums and products over all primes p, beginning with $p=2$. We let $\sum_{p|n} a_p$ and $\prod_{p|n} a_p$ denote sums and products over the distinct primes that divide a positive integer n. Finally, we let $\sum_{d|n} a_d$ and $\prod_{d|n} a_d$ denote sums and products over the positive divisors of a positive integer n, including 1 and n. For example, if $n=12$, $\prod_{d|12} a_d = a_1a_2a_3a_4a_6a_{12}$ and $\prod_{p|12} a_p = a_2a_3$. For the duration of this paper, the symbol $\mathbb{N} := \{1,2,3\ldots\}$ will denote the set of positive integers and will be referred to as the natural numbers. We obey the convention that for $s\in\mathbb{C}$, $s=\sigma+it$ with $\sigma,t\in\mathbb{R}$. We finally define $\mathbb{C}_\alpha := \{z\in\mathbb{C} : z\neq\alpha\}$.

We also utilize the theory of Euler products. An Euler product, described in [8, (27.4.1-2)], is the product form of a Dirichlet series. For any multiplicative function f we have [1, (11.8)]

$$\sum_{n\in\mathbb{N}} \frac{f(n)}{n^s} = \prod_p \left(1+a_{f,s}(p)\right), \text{ where } a_{f,s}(p) := \sum_{m=1}^{\infty} \frac{f(p^m)}{p^{ms}}. \tag{1}$$

The left-hand side of the previous equation is a Dirichlet series, and the right-hand is an Euler product. Furthermore, letting $s=\sigma+it$, the abscissa of absolute convergence is the unique real number such that $\sum_{n\in\mathbb{N}} \frac{f(n)}{n^s}$ absolutely converges if and only if $\sigma>\sigma_a$.

Some multiplicative functions which will be used in this paper include denotes the Euler totient function $\phi(n)$, which counts the number of natural numbers less than or equal to n which are coprime to n [8, (27.2.7)]. We will frequently use the multiplicative representation [8, (27.3.3)]

$$\phi(n) = n\prod_{p|n}\left(1-\frac{1}{p}\right). \tag{2}$$

A standard generalization of ϕ is Jordan's totient function [8, (27.3.4)], given by

$$J_k(n) := n^k\prod_{p|n}\left(1-\frac{1}{p^k}\right). \tag{3}$$

Using $\omega(n)$, we can also define the Möbius function, which gives the parity of the number of prime factors in a squarefree number, and is formally given by [8, (27.2.12)]

$$\mu(n) := \begin{cases} 1, & \text{if } n{=}1, \\ 0, & \text{if } n \text{ is non-squarefree}, \\ (-1)^{\omega(n)}, & \text{if } n \text{ is squarefree}. \end{cases} \tag{4}$$

3 Factorization identities

The main results of this paper are based on the following standard symmetric function identity.

Proposition 3.1. *Let $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in \mathbb{C}_1$. Then*

$$\left(\prod_{i=1}^{n}(1-x_i)\right)\left(\sum_{i=1}^{n}\frac{x_i}{x_i-1}\right) = \sum_{k=1}^{n}(-1)^k k e_k, \tag{5}$$

where $e_k = e_k(x_1, \ldots, x_n)$ denotes the elementary symmetric polynomial defined as

$$e_k := \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} x_{i_1} x_{i_2} \cdots x_{i_k}. \tag{6}$$

Proof. Following Macdonald [6, (Chapter 1)], we have the generating product

$$E(t) = \prod_{i=1}^{n}(1-tx_i) = \sum_{k=0}^{n}(-1)^k t^k e_k, \tag{7}$$

with $x_i \in \mathbb{C}$. Taking a logarithmic derivative of $E(t)$ for indeterminate t yields

$$E'(t) = E(t)\sum_{k=1}^{n}\frac{x_i}{tx_i - 1}. \tag{8}$$

Taking $t = 1$ completes the proof. $\square$

Corollary 3.2. *Let $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in \mathbb{C}_{-1}$. Then*

$$\left(\prod_{i=1}^{n}(1+x_i)\right)\left(\sum_{i=1}^{n}\frac{x_i}{x_i+1}\right) = \sum_{k=1}^{n} k e_k. \tag{9}$$

Proof. We repeat the previous argument in the proof of Proposition 3.1 but map t to $-t$ before taking the logarithmic derivative. $\square$

Note that by comparing coefficients of t^n we recover the Newton-Girard identities [6, (Chapter 1)], but these are in fact identities for the *generating functions* for various symmetric functions.

4 Divisor sums

The following theorems are obtained by reinterpreting (5). We consider divisor sums of a multiplicative function f weighted by $\omega(n)$ and $\mu(n)$ or $|\mu(n)| = \mu^2(n)$. Throughout, $f(p)$ will refer to the value of f evaluated at any prime p. We also define $\sum_{p|n} g(p)$ as 0 if $n = 1$, where g is any function, not necessarily multiplicative, because 1 has no distinct prime factors. Under this convention, the theorems in this section also hold for $n = 1$.

Theorem 4.1. *Let $f(n)$ be a multiplicative function defined such that $f(p) \neq 1$ for any prime p that divides n. Then*

$$\sum_{d|n} \mu(d)\omega(d)f(d) = \left(\prod_{p|n}(1 - f(p))\right)\left(\sum_{p|n} \frac{f(p)}{f(p)-1}\right). \tag{10}$$

Proof. First take any squarefree natural number n, so that $n = \prod_{i=1}^{\omega(n)} p_i$ by the fundamental theorem of arithmetic. We then let $x_i = f(p_i)$ in (5), such that each x_i is an arithmetic function evaluated at each distinct prime that divides n. We can evaluate e_k, yielding

$$e_k = \sum_{1 \leq i_1 < i_2 \cdots < i_k \leq n} f(p_{i_1})f(p_{i_2}) \cdots f(p_{i_k}).$$

Since f is multiplicative and each p_i is coprime to the others by definition, we have

$$e_k = \sum_{1 \leq i_1 < i_2 \cdots < i_k \leq n} f(p_{i_1}p_{i_2} \cdots p_{i_k}).$$

Now we can regard e_k as the sum of $f(n)$ evaluated at the divisors of n with k prime factors, since each term in e_k trivially has k prime factors, and every possible product of k primes that divide n is included in e_k. This is equivalent to partitioning the divisors of n based on their number of distinct prime factors. Then (5) transforms into

$$\left(\prod_{i=1}^{\omega(n)}(1 - f(p_i))\right)\left(\sum_{i=1}^{\omega(n)} \frac{f(p_i)}{f(p_i)-1}\right) = \sum_{d|n} \omega(d)(-1)^{\omega(d)}f(d).$$

Each divisor d is squarefree since n is squarefree, so we can replace $(-1)^{\omega(d)}$ by $\mu(d)$, where $\mu(d)$ is the Möbius function which is defined by (4). Rewriting the product and sum over p_i, $1 \leq i \leq \omega(n)$, as a product and sum over p gives (10) for squarefree numbers. However, we can immediately see that if n is non-squarefree, $\mu(d)$ eliminates any non-squarefree divisors on the left-hand side. Meanwhile, the right-hand side is evaluated over the distinct primes that divide n so changing the multiplicities of these primes will not affect the sum in any way. Therefore (10) is valid for all $n \in \mathbb{N}$, which completes the proof. □

Corollary 4.2. *Let $f(n)$ be a multiplicative function, $n \in \mathbb{N}$, and $f(p) \neq 1$ for any prime p that divides n. Then*

$$\sum_{d|n} \mu(d)\omega(d)f(d) = \left(\sum_{d|n} \mu(d)f(d)\right)\left(\sum_{p|n} \frac{f(p)}{f(p)-1}\right). \tag{11}$$

Proof. A general theorem for any multiplicative function $f(n)$, found in [1, (2.18)], is that

$$\sum_{d|n} \mu(d)f(d) = \prod_{p|n}(1 - f(p)). \tag{12}$$

Substituting this relation into (10) completes the proof. □

Theorem 4.3. *Let $f(n)$ be a multiplicative function, $n \in \mathbb{N}$, and $f(p) \neq -1$ for any prime p that divides n. Then*

$$\sum_{d|n} |\mu(d)|\omega(d)f(d) = \left(\prod_{p|n}(1+f(p))\right)\left(\sum_{p|n}\frac{f(p)}{1+f(p)}\right). \tag{13}$$

Proof. We follow the proof of Theorem 4.1 but substitute $x_i = f(p_i)$ into (9) instead of (5). Hence, we only have to take the divisor sum over squarefree divisors without multiplying by $(-1)^{\omega(d)}$. We do this by multiplying the divisor sums by $|\mu(d)|$, the characteristic function of the squarefree numbers. This completes the proof. □

Corollary 4.4. *Let $f(n)$ be a multiplicative function with $f(1) = 1$, $f(p) \neq -1$, and $n \in \mathbb{N}$. Then*

$$\sum_{d|n} |\mu(d)|\omega(d)f(d) = \left(\sum_{d|n}|\mu(d)|f(d)\right)\left(\sum_{p|n}\frac{f(p)}{1+f(p)}\right). \tag{14}$$

Proof. We substitute $f(n) = \mu(n)g(n)$ into (12), where $g(n)$ is multiplicative, ensuring that $f(n)$ is also multiplicative. Noting that $\mu^2(n) = |\mu(n)|$ since the Möbius function only takes values of ± 1 and 0 yields

$$\sum_{d|n} |\mu(d)|g(d) = \prod_{p|n}(1+g(p)).$$

Mapping g to f to maintain consistent notation and substituting into (13) completes the proof. □

Specializing $f(n)$ yields an assortment of new formulae involving convolutions with $\omega(n)$. Below we use a variety of functions $f(n)$ along with (10), (11), and (13) to find new expressions for divisor sums involving $\omega(n)$.

Corollary 4.5. *Let $n \in \mathbb{N}$. Then*

$$\sum_{d|n} |\mu(d)|\omega(d) = \omega(n)2^{\omega(n)-1}. \tag{15}$$

Proof. Substituting $f(n) = 1$ into (13) gives $\sum_{d|n}|\mu(d)|\omega(d) = \left(\prod_{p|n} 2\right)\left(\sum_{p|n}\frac{1}{2}\right)$. Since the product and sum on the right-hand side are over the distinct primes that divide n, each is evaluated $\omega(n)$ times. This simplifies to $\sum_{d|n}|\mu(d)|\omega(d) = 2^{\omega(n)}\frac{\omega(n)}{2}$. □

Corollary 4.6. *Let $n \in \mathbb{N}$. Then*

$$\sum_{d|n} \mu(d)\omega(d)\left(\frac{n}{d}\right)^k = J_k(n)\sum_{p|n}\frac{1}{1-p^k}. \tag{16}$$

Proof. Substituting $f(n) = \frac{1}{n^k}$ into (10) gives

$$\sum_{d|n} \frac{\mu(d)\omega(d)}{d^k} = \left(\prod_{p|n}\left(1-\frac{1}{p^k}\right)\right)\left(\sum_{p|n}\frac{1}{1-p^k}\right). \tag{17}$$

Using (3) to see that $\prod_{p|n}(1-\frac{1}{p^k}) = \frac{J_k(n)}{n^k}$ and substituting this relation into (17) completes the proof. □

Corollary 4.7. *Let $n \in \mathbb{N}$. Then*

$$\sum_{d|n} |\mu(d)|\omega(d)\left(\frac{n}{d}\right)^k = \frac{J_{2k}(n)}{J_k(n)}\sum_{p|n}\frac{1}{1+p^k}. \tag{18}$$

Proof. Substituting $f(n) = \frac{1}{n^k}$ into (13) gives

$$\sum_{d|n} \frac{|\mu(d)|\omega(d)}{d^k} = \left(\prod_{p|n}\left(1+\frac{1}{p^k}\right)\right)\left(\sum_{p|n}\frac{1}{1+p^k}\right).$$

Now we complete the proof by using (3) to see that

$$\prod_{p|n}\left(1+\frac{1}{p^k}\right) = \frac{n^{2k}}{n^{2k}}\frac{\prod_{p|n}(1-\frac{1}{p^{2k}})}{\prod_{p|n}(1-\frac{1}{p^k})} = \frac{1}{n^k}\frac{J_{2k}(n)}{J_k(n)}.$$

□

Corollary 4.8. *Let $n \in \mathbb{N}$ with n squarefree. Then*

$$\sum_{d|n} \omega(d)d^k = \frac{J_{2k}(n)}{J_k(n)}\sum_{p|n}\frac{p^k}{1+p^k}. \tag{19}$$

Proof. Substituting $f(n) = n^k$ into (13) gives

$$\sum_{d|n} |\mu(d)|\omega(d)d^k = \prod_{p|n}(1+p^k)\sum_{p|n}\frac{p^k}{1+p^k}. \tag{20}$$

Now if we let n be squarefree, then $n = \prod_{p|n} p$. We also know that (3) transforms into

$$J_k(n) = n^k\prod_{p|n}\left(1-\frac{1}{p^k}\right) = n^k\frac{\prod_{p|n}(p^k-1)}{\prod_{p|n}p^k}.$$

However we now have $n^k = \prod_{p|n} p^k$, so $J_k(n) = \prod_{p|n}(p^k-1)$. Then

$$\frac{J_{2k}(n)}{J_k(n)} = \frac{\prod_{p|n}(p^{2k}-1)}{\prod_{p|n}(p^k-1)} = \prod_{p|n}\frac{(p^{2k}-1)}{(p^k-1)} = \prod_{p|n}(p^k+1). \tag{21}$$

Letting n be a squarefree natural number in (13), we can also eliminate $|\mu(d)|$ from the sum on the left-hand side since every divisor of n will already be squarefree. Substituting (21) into (20) completes the proof. □

5 Infinite sums

The propositions (5) and (9) hold for a finite number of elements x_i. However, we can take the limit $i \to \infty$ to extend this sum. This enables us to find closed form product expressions for Dirichlet series, which are described in [8, (27.4.4)], of the form $\sum_{n\in\mathbb{N}} \frac{\omega(n)f(n)}{n^s}$ for many commonly encountered multiplicative functions $f(n)$.

We introduce the prime zeta function [3, (0.1)], denoted by $P(s)$. We define it by

$$P(s) := \sum_p \frac{1}{p^s},$$

and note that it converges for $\Re(s) > 1$. It is an analog of the Riemann zeta function, described in [8, (25.2.1)], with the sum taken over prime numbers instead of all natural numbers. We also define the shifted prime zeta function $P(s,a)$ as

$$P(s,a) := \sum_p \frac{1}{p^s + a},$$

such that $P(s,0) = P(s)$.

Lemma 5.1. *Let $a \in \mathbb{C}$ and $|a| < 2$. Then P(s,a) converges absolutely if and only if $s \in \mathbb{C}$, $\Re(s) > 1$.*

Proof. Consider $a \in \mathbb{C}$. Now if $\Re(s) = 1$, we prove that $P(s,a)$ diverges if $|a| < 2$. This implies that $P(s,a)$ diverges for all s such that $\Re(s) < 1$ for these values of a.

To prove divergence, it suffices to let $a \in \mathbb{R}$ since $\mathbb{R} \subset \mathbb{C}$. We let $s = 1$, $a \in \mathbb{R}$, and $2 > a \geq 0$. Peeling off the $i = 1$ term, where $p_1 = 2$, we have $p_{i+1} - p_i \geq 2 > a \geq 0$ since the minimum prime gap for primes not equal to 2 and 3 is 2. This means

$$P(1,a) = \sum_{i=1}^{\infty} \frac{1}{p_i + a} = \frac{1}{2+a} + \sum_{i=2}^{\infty} \frac{1}{p_i + a} \geq \frac{1}{2+a} + \sum_{i=2}^{\infty} \frac{1}{p_{i+1}} = \frac{1}{2+a} + P(1) - \frac{5}{6}.$$

However, [3] gives us that $P(1)$ diverges. Now we let $s = 1$ and $-2 < a \leq 0$. Then by the direct comparison test, $\frac{1}{p+a} > \frac{1}{p}$. This means $P(s,a)$ is bounded below by $P(1)$ and the sum diverges. The restriction $a > -2$ is because there is a simple pole if $s = 1$ and $a = -2$.

Now we let $\Re(s) > 1$, $a \in \mathbb{C}$, and $0 \leq |a| < 2$. By the triangle inequality, $|p^s| = |p^s + a - a| \leq |p^s + a| + |-a| = |p^s + a| + |a|$ so then $|p^s + a| \geq |p^s| - |a| = p^{\Re(s)} - |a| > 0$, since $p \in \mathbb{N}$ and $p \geq 2$. Then we can bound $P(s,a)$ so that

$$|P(s,a)| \leq \sum_p \left| \frac{1}{p^s + a} \right| \leq \sum_p \frac{1}{p^{\Re(s)} - |a|}.$$

We have now reduced the consideration of a complex s to considering a real s. Letting $\Re(s) = \sigma$ for convenience, we have that $p_i^\sigma - p_{i-1}^\sigma \geq 2 > |a| \geq 0$. This allows us to form the bound

$$P(s,a) \leq \sum_{i=1}^{\infty} \frac{1}{p_i^{\sigma} - |a|} = \frac{1}{2^{\sigma} - |a|} + \sum_{i=2}^{\infty} \frac{1}{p_i^{\sigma} - |a|} \leq \frac{1}{2^{\sigma} - |a|} + \sum_{i=2}^{\infty} \frac{1}{p_{i-1}^{\sigma}} = \frac{1}{2^{\sigma} - |a|} + P(\sigma).$$

We know that $P(s)$ converges for $\Re(s) = \sigma > 1$, so $P(s,a)$ will also converge for $0 \leq |a| < 2$, completing the proof. □

Lemma 5.2. *Let $a, s, k \in \mathbb{C}$ with $|a| < 2$. Then $\sum_p \frac{p^k}{p^s + a}$ converges absolutely if and only if $\Re(s) > \max(1, 1 + \Re(k))$.*

Proof. First we assume $\Re(k) \leq 0$. Then

$$\left| \sum_p \frac{p^k}{p^s + a} \right| \leq \sum_p \left| \frac{p^k}{p^s + a} \right| \leq \sum_p \left| \frac{1}{p^s + a} \right|,$$

which by (5.1) converges absolutely if and only if $s > 1$. Now we assume $\Re(k) > 0$. Then

$$\left| \sum_p \frac{p^k}{p^s + a} \right| \leq \sum_p \left| \frac{1}{p^{s-k} + ap^{-k}} \right|.$$

Since $\left| \frac{a}{p^k} \right| < \frac{2}{2} < 2$, the conditions for (5.1) are satisfied and the series converges absolutely if and only if $\Re(s - k) > 1$. Combining the two requirements for $\Re(s)$ completes the proof. □

Our results on infinite series are based on a certain symmetric polynomial evaluation, which is a refinement of the Euler product/Dirichlet series duality

$$\sum_{n \in \mathbb{N}} \frac{f(n)}{n^s} = \prod_p \left(1 + a_{f,s}(p)\right), \text{ where } a_{f,s}(p) := \sum_{m=1}^{\infty} \frac{f(p^m)}{p^{ms}}.$$

In particular, given the evaluation of e_k from Lemma 5.3, we can formally consider the generating product

$$E(t) = \sum_{k \geq 0}^{\infty} e_k t^k = \prod_{i=1}^{\infty} (1 + x_i t),$$

set $t = 1$ and $x_i = a_{f,s}(p_i)$, and rearrange the resulting series as

$$\prod_{i=1}^{\infty} \left(1 + a_{f,s}(p_i)\right) = \sum_{k=0}^{\infty} \sum_{\substack{n \geq 1 \\ \omega(n) = k}} \frac{f(n)}{n^s} = \sum_{n \geq 1} \frac{f(n)}{n^s}.$$

This is precisely the definition of an Euler product, though we have ignored questions of convergence in this comparison. The following lemma precisely encodes the correspondence above, and is the key element in the proof of Theorem 5.4.

Lemma 5.3. *Let $1 \leq i < \infty$, $i \in \mathbb{N}$, p_i denote the i^{th} prime, f denote any multiplicative function, and $x_i = \sum_{m=1}^{\infty} \frac{f(p_i{}^m)}{p_i{}^{ms}}$. Furthermore, let $k \in \mathbb{N}$ and*

$$S_k := \{n \in \mathbb{N} : \omega(n) = k\}, \tag{22}$$

so that S_k is the set of natural numbers with k distinct prime factors. If $\sum_{n\in\mathbb{N}} \frac{f(n)}{n^s}$ has an abscissa of absolute convergence σ_a, then

$$e_k := \sum_{1\leq i_1<i_2<\cdots<i_k} x_{i_1}x_{i_2}\cdots x_{i_k} = \sum_{n\in S_k} \frac{f(n)}{n^s}, \tag{23}$$

which converges absolutely for $\sigma > \sigma_a$.

Proof. Let $1 \leq i < \infty$, $i \in \mathbb{N}$, and $1 \leq \epsilon_i < \infty$, $\epsilon_i \in \mathbb{N}$. We can directly evaluate e_k as

$$e_k = \sum_{1\leq i_1<i_2\cdots<i_k} \frac{f(p_{i_1}^{\epsilon_{i_1}})f(p_{i_2}^{\epsilon_{i_2}})\cdots f(p_{i_k}^{\epsilon_{i_k}})}{\left(p_{i_1}^{\epsilon_{i_1}}p_{i_2}^{\epsilon_{i_2}}\cdots p_{i_k}^{\epsilon_{i_k}}\right)^s}.$$

Here ϵ_i varies because it goes over every single power of p which is present in x_i. Since x_i is the sum of a subsequence of $\sum_{n\in\mathbb{N}} \frac{f(n)}{n^s}$, it will also converge absolutely for $\sigma > \sigma_a$ and any rearrangement of its terms will not change the value of the sum. Since f is multiplicative and each p_i is coprime to the others by definition, we have

$$e_k = \sum_{1\leq i_1<i_2\cdots<i_k} \frac{f\left(p_{i_1}^{\epsilon_{i_1}}p_{i_2}^{\epsilon_{i_2}}\cdots p_{i_k}^{\epsilon_{i_k}}\right)}{\left(p_{i_1}^{\epsilon_{i_1}}p_{i_2}^{\epsilon_{i_2}}\cdots p_{i_k}^{\epsilon_{i_k}}\right)^s}.$$

If we take an arbitrary natural number n with k distinct prime factors, it will be present in the sum with the k^{th} symmetric function, e_k. The k^{th} symmetric function contains every natural number with k prime factors, since k dictates the number of terms that are multiplied together to form every term in e_k. The multiplicity also doesn't matter, since that varies with ϵ_i which is independent of k.

We also know that by the fundamental theorem of arithmetic, there is a bijection between the natural numbers and the products of distinct primes with any multiplicity. This means that every product of distinct primes in the expression for e_k corresponds to a natural number n. Taking it all together, it follows that e_k runs over every natural number n with k distinct prime factors. Rewriting each product of primes as n then gives equation (23). Since $\sum_{n\in S_k} \frac{f(n)}{n^s}$ is the sum over a subsequence of $\sum_{n\in\mathbb{N}} \frac{f(n)}{n^s}$, it will also converge absolutely for $\sigma > \sigma_a$. □

On a high level, the proof of the following theorem follows by substituting the symmetric polynomial evaluation of Lemma 5.3 into the symmetric polynomial identity (9). We can then rewrite the sum over k in (9) as a sum over all natural numbers, while recognizing the linear k weight as an occurrence of $\omega(n)$. However, there are several technical restrictions we need in order to ensure convergence of the resulting Dirichlet series.

Theorem 5.4. *Let $f(n)$ be a multiplicative function, $s \in \mathbb{C}$, $a_{f,s}(p) := \sum_{m=1}^{\infty} \frac{f(p^m)}{p^{ms}}$, and $a_{f,s}(p) \neq -1$ for any prime p. If $\sum_{n\in\mathbb{N}} \frac{f(n)}{n^s}$ and $\sum_p \frac{a_{f,s}(p)}{1+a_{f,s}(p)}$ both converge absolutely for $\sigma > \sigma_a$, then*

$$\sum_{n\in\mathbb{N}} \frac{\omega(n)f(n)}{n^s} = \left(\sum_{n\in\mathbb{N}} \frac{f(n)}{n^s}\right)\left(\sum_p \frac{a_{f,s}(p)}{1+a_{f,s}(p)}\right), \ \sigma > \sigma_a. \tag{24}$$

Proof. We let p_i denote the i^{th} prime number. We must choose a suitable x_i to substitute into (9), so we let $x_i = a_{f,s}(p_i)$, $1 \leq i < \infty$. We still retain the condition $x_i = a_{f,s}(p_i) \neq -1$. Substituting this specialization of x_i into (9) gives

$$\sum_{k=1}^{\infty} ke_k = \prod_p (1 + a_{f,s}(p)) \sum_p \frac{a_{f,s}(p)}{1+a_{f,s}(p)}.$$

The product and sum on the right now run through every prime p since each x_i is in a one-to-one correspondence with a sum over the ith prime. We also note that the product over primes is the Euler product for the Dirichlet series $\sum_{n\in\mathbb{N}} \frac{f(n)}{n^s}$. We can then appeal to the result of Lemma 5.3, but we now interpret k as the weight $\omega(n)$.

We have the series rearrangement

$$\sum_{k=1}^{\infty} ke_k = \sum_{k=1}^{\infty} k \sum_{n\in S_k} \frac{f(n)}{n^s} = \sum_{k=0}^{\infty} \sum_{n\in S_k} \omega(n)\frac{f(n)}{n^s} = \sum_{n\in\mathbb{N}} \frac{\omega(n)f(n)}{n^s}.$$

The most important fact about this proof is that k is constant on each set S_k by definition; in fact, k is equal to the number of distinct prime factors of every integer in S_k. Therefore, we can move it inside the second sum and rewrite it as $\omega(n)$. The inner sum converges absolutely for $\sigma > \sigma_a$, but the sum over n does not converge on this half plane in general. This rearrangement is valid if $\sum_{n\in\mathbb{N}} \frac{\omega(n)f(n)}{n^s}$ converges absolutely. However, as a Dirichlet series it is guaranteed to have an abscissa of absolute convergence and therefore the rearrangement is valid for some σ.

Simplifying (9) finally shows that

$$\sum_{n\in\mathbb{N}} \frac{\omega(n)f(n)}{n^s} = \left(\sum_{n\in\mathbb{N}} \frac{f(n)}{n^s}\right)\left(\sum_p \frac{a_{f,s}(p)}{1+a_{f,s}(p)}\right).$$

The left-hand side has the same convergence criteria as the right-hand side. Therefore if $\sum_{n\in\mathbb{N}} \frac{f(n)}{n^s}$ has an abscissa of absolutely convergence σ_a and $\sum_p \frac{a_{f,s}(p)}{1+a_{f,s}(p)}$ converges absolutely for some $\sigma > \sigma_b$, the left-hand side will converge absolutely for $\sigma > \max(\sigma_a, \sigma_b)$. This shows that weighting the terms of the Dirichlet series of any multiplicative function $f(n)$ by $\omega(n)$ multiplies the original series by a sum of f over primes. $\square$

Theorem 5.5. *Let $f(n)$ be a completely multiplicative function, $s \in \mathbb{C}$, and $n \in \mathbb{N}$. If $\sum_{n\in\mathbb{N}} \frac{f(n)}{n^s}$ has an abscissa of absolute convergence σ_a, then*

$$\sum_{n\in\mathbb{N}} \frac{\omega(n)f(n)}{n^s} = \left(\sum_{n\in\mathbb{N}} \frac{f(n)}{n^s}\right)\left(\sum_{p} \frac{f(p)}{p^s}\right), \ \sigma > \sigma_a. \tag{25}$$

Proof. If $\sigma > \sigma_a$, then $\sum_{m=1}^{\infty} \frac{f(p_i^m)}{{p_i}^{ms}}$ converges absolutely and therefore $\left|\frac{f(p)}{p^s}\right| < 1$. Then

$$x_i = a_{p_i} = \sum_{m=1}^{\infty} \frac{f(p_i^m)}{{p_i}^{ms}} = \sum_{m=1}^{\infty} \frac{f(p_i)^m}{{p_i}^{ms}} = \frac{1}{1-\frac{f(p_i)}{{p_i}^s}} - 1.$$

Replacing $a_{f,s}(p)$ by $\left(1-\frac{f(p)}{p^s}\right)^{-1} - 1$ in (24) and simplifying completes the proof. We also note that $\sum_{n\in\mathbb{N}} \frac{f(p)}{p^s}$ is the sum over a subsequence of $\sum_{n\in\mathbb{N}} \frac{f(n)}{n^s}$, so it will also converge absolutely for $\sigma > \sigma_a$ and we can simplify our convergence criterion. □

Theorem 5.6. *Let $f(n)$ be a multiplicative function. Let $\sum_{n\in\mathbb{N}} \frac{|\mu(n)|f(n)}{n^s}$ and $\sum_p \frac{f(p)}{p^s+f(p)}$ both converge absolutely for $\sigma > \sigma_a$. Assume that for all prime p and $s \in \mathbb{C}$ such that $\sigma > \sigma_a$ we have $f(p) \neq -p^s$. Then*

$$\sum_{n\in\mathbb{N}} \frac{|\mu(n)|\omega(n)f(n)}{n^s} = \left(\sum_{n\in\mathbb{N}} \frac{|\mu(n)|f(n)}{n^s}\right)\left(\sum_{p} \frac{f(p)}{p^s+f(p)}\right), \tag{26}$$

which converges absolutely for $\sigma > \sigma_a$.

Proof. To begin, we note that a product of multiplicative functions is also multiplicative. Letting $f(n) = |\mu(n)|g(n)$ in (24), where g is any multiplicative function which guarantees that f is multiplicative, we can simplify $a_{f,s}(p)$. We have that $|\mu(p^m)|$ is 0 for $m \geq 2$ and 1 for $m = 1$, since $|\mu(n)|$ is the characteristic function of the squarefree integers. If $m = 1$, we also have that $f(p) = |\mu(p)|g(p) = g(p)$, which means that $a_{f,s}(p) = \sum_{m=1}^{\infty} \frac{f(p^m)}{p^{ms}} = \frac{g(p)}{p^s}$. We still retain the $a_{f,s}(p) = \frac{g(p)}{p^s} \neq -1$ condition. Assuming that the sum over primes converges, substituting into (24) gives

$$\sum_{n\in\mathbb{N}} \frac{|\mu(n)|\omega(n)g(n)}{n^s} = \left(\sum_{n\in\mathbb{N}} \frac{|\mu(n)|g(n)}{n^s}\right)\left(\sum_{p} \frac{\frac{g(p)}{p^s}}{1+\frac{g(p)}{p^s}}\right).$$

We utilize the same convergence criterion as Theorem 5.4. Simplifying the fraction and letting g be represented by f in order to maintain consistent notation completes the proof. □

Theorem 5.7. *Let $f(n)$ be a multiplicative function. Let $\sum_{n\in\mathbb{N}} \frac{\mu(n)f(n)}{n^s}$ and $\sum_p \frac{f(p)}{p^s-f(p)}$ both converge absolutely for $\sigma > \sigma_a$. Assume that for all prime p and $s \in \mathbb{C}$ such that $\sigma > \sigma_a$ we have $f(p) \neq p^s$. Then*

$$\sum_{n\in\mathbb{N}} \frac{\mu(n)\omega(n)f(n)}{n^s} = \left(\sum_{n\in\mathbb{N}} \frac{\mu(n)f(n)}{n^s}\right)\left(\sum_{p} \frac{f(p)}{f(p)-p^s}\right), \tag{27}$$

which converges absolutely for $\sigma > \sigma_a$.

Proof. We let $f(n) = \mu(n)g(n)$ in (24), where g is any multiplicative function which guarantees that f is multiplicative. We can then simplify $a_{f,s}(p)$, since $a_{f,s}(p) := \sum_{m=1}^{\infty} \frac{f(p^m)}{p^{ms}} = -\frac{g(p)}{p^s}$. We still retain the $a_{f,s}(p) = -\frac{g(p)}{p^s} \neq -1$ condition. Assuming that the sum over primes converges, substituting into (24) gives

$$\sum_{n\in\mathbb{N}} \frac{\mu(n)\omega(n)g(n)}{n^s} = \left(\sum_{n\in\mathbb{N}} \frac{\mu(n)g(n)}{n^s}\right)\left(\sum_{p} \frac{-\frac{g(p)}{p^s}}{1-\frac{g(p)}{p^s}}\right).$$

We utilize the same convergence criterion as Theorem 5.4. Simplifying the fraction and letting g be represented by f in order to maintain consistent notation completes the proof. □

We note that the following proposition, the simplest application of (24), can be found in [4, (D-17)].

Corollary 5.8. *Let $\zeta(s)$ be the Riemann zeta function. For $s \in \mathbb{C}$ such that $\sigma > 1$,*

$$\sum_{n\in\mathbb{N}} \frac{\omega(n)}{n^s} = \zeta(s)P(s). \tag{28}$$

Proof. We let $f(n) = 1$ in (25), since this is a completely multiplicative function. We then note that $\sum_p \frac{1}{p^s}$ is the prime zeta function. The zeta and prime zeta functions both converge absolutely for $\sigma > 1$, so the left-hand side will too. □

Corollary 5.9. *Let $\lambda(s)$ be Liouville's function. For $s \in \mathbb{C}$ such that $\sigma > 1$,*

$$\sum_{n\in\mathbb{N}} \frac{\omega(n)\lambda(n)}{n^s} = -\frac{\zeta(2s)}{\zeta(s)}P(s). \tag{29}$$

Proof. We let $f(n) = \lambda(n)$ in (25). Liouville's function, found in [8, (27.2.13)], is completely multiplicative. We note that $\lambda(p) = -1$ for every prime p, since they trivially only have a single prime divisor with multiplicity 1. This gives

$$\sum_{n\in\mathbb{N}} \frac{\omega(n)\lambda(n)}{n^s} = \left(\sum_{n\in\mathbb{N}} \frac{\lambda(n)}{n^s}\right)\left(\sum_{p} \frac{(-1)}{p^s}\right).$$

We then note that [8, (27.4.7)] gives $\sum_{n\in\mathbb{N}} \frac{\lambda(n)}{n^s} = \frac{\zeta(2s)}{\zeta(s)}$ and states that it converges for $\sigma > 1$, which completes the proof since both the zeta and prime zeta functions converge for $\sigma > 1$. □

Corollary 5.10. *Let $\chi(n)$ denote a Dirichlet character. For $s \in \mathbb{C}$ such that $\sigma > 1$,*

$$\sum_{n\in\mathbb{N}} \frac{\omega(n)\chi(n)}{n^s} = L(s,\chi)\sum_{p} \frac{\chi(p)}{p^s}. \tag{30}$$

Proof. We let $f(n) = \chi(n)$ in (25). Here $\chi(n)$ is a Dirichlet character, found in [8, (27.8.1)], which is a completely multiplicative function that is periodic with period k and vanishes for $(n, k) > 1$. Substituting it in (25) gives

$$\sum_{n\in\mathbb{N}} \frac{\omega(n)\chi(n)}{n^s} = \left(\sum_{n\in\mathbb{N}} \frac{\chi(n)}{n^s}\right)\left(\sum_{p} \frac{\chi(p)}{p^s}\right).$$

We then note that a Dirichlet L-series, an important number theoretic series, is defined in [8, (25.15.1)] as $L(s,\chi) = \sum_{n\in\mathbb{N}} \frac{\chi(n)}{n^s}$. Simplifying to write the sum over n as an L-series while noting that [8, (25.15.1)] states that an L-series converges absolutely for $\sigma > 1$ completes the proof, since both the L series and prime L series converge absolutely for $\sigma > 1$. □

Corollary 5.11. *For $s \in \mathbb{C}$ such that $\sigma > 1$,*

$$\sum_{n\in\mathbb{N}} \frac{|\mu(n)|\omega(n)}{n^s} = \frac{\zeta(s)}{\zeta(2s)} P(s, 1). \tag{31}$$

Proof. We let $f(n) = 1$ in (26) and note that [8, (27.4.8)] states that $\sum_{n\in\mathbb{N}} \frac{|\mu(n)|}{n^s} = \frac{\zeta(s)}{\zeta(2s)}$ and that this converges for $\sigma > 1$. We note that (5.1) states that $P(s, 1)$ will also converge absolutely for $\sigma > 1$, which completes the proof. □

Corollary 5.12. *For $s \in \mathbb{C}$ such that $\sigma > 1$,*

$$\sum_{n\in\mathbb{N}} \frac{\mu(n)\omega(n)}{n^s} = -\frac{1}{\zeta(s)} P(s, -1). \tag{32}$$

Proof. We let $f(n) = 1$ in (27) and note that [8, (27.4.5)] states $\sum_{n\in\mathbb{N}} \frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)}$ and that it converges for $\sigma > 1$. We note that (5.1) states that $P(s, -1)$ will also converge absolutely for $\sigma > 1$, which completes the proof. □

While the previous sums have involved completely multiplicative functions or convolutions with the Möbius function, we can sometimes directly evaluate $\sum_{m=1}^{\infty} \frac{f(p^m)}{p^{ms}}$. Taking (24) but converting $\sum_{n\in\mathbb{N}} \frac{f(n)}{n^s}$ back to its Euler product means that

$$\sum_{n\in\mathbb{N}} \frac{\omega(n)f(n)}{n^s} = \prod_{p} (1 + a_{f,s}(p)) \sum_{p} \left(\frac{a_{f,s}(p)}{1 + a_{f,s}(p)}\right),$$

where $a_{f,s}(p) := \sum_{m=1}^{\infty} \frac{f(p^m)}{p^{ms}}$. We can extract the coefficient $a_{f,s}(p)$ through a variety of methods.

Corollary 5.13. *For $s \in \mathbb{C}$ such that $\sigma > 1$,*

$$\sum_{n\in\mathbb{N}} \frac{\omega(n)2^{\omega(n)}}{n^s} = 2\frac{\zeta^2(s)}{\zeta(2s)} P(s, 1). \tag{33}$$

Proof. We know from [8, (27.4.9)] that $\sum_{n\in\mathbb{N}} \frac{2^{\omega(n)}}{n^s} = \frac{\zeta^2(s)}{\zeta(2s)}$ and that it converges for $\Re(s) > 1$, where $2^{\omega(n)}$ is the number of squarefree divisors of n. We now directly evaluate $a_{f,s}(p)$, summing it as a geometric series. We note that $2^{\omega(p^m)} = 2$, since p^m trivially has a single distinct prime factor. Substituting into the formula for $a_{f,s}(p)$ shows that

$$a_{f,s}(p) = \sum_{m=1}^{\infty} \frac{2^{\omega(p^m)}}{p^{ms}} = \sum_{m=1}^{\infty} 2\left(\frac{1}{p^s}\right)^m = \frac{2}{p^s - 1}.$$

Then $\frac{a_{f,s}(p)}{1+a_{f,s}(p)} = \frac{2}{p^s+1}$. Substituting into (24) shows that

$$\sum_{n\in\mathbb{N}} \frac{\omega(n)2^{\omega(n)}}{n^s} = \left(\sum_{n\in\mathbb{N}} \frac{2^{\omega(n)}}{n^s}\right)\left(\sum_p \frac{2}{p^s+1}\right).$$

Rewriting the right-hand side in terms of zeta and prime zeta functions while noting that they will both converge if $\sigma > 1$ completes the proof. □

Corollary 5.14. *Let $J_k(n)$ denote Jordan's totient function. For $s, k \in \mathbb{C}$ such that $\sigma > \max(1, 1 + \Re(k))$,*

$$\sum_{n\in\mathbb{N}} \frac{\omega(n)J_k(n)}{n^s} = \frac{\zeta(s-k)}{\zeta(s)} \sum_p \frac{p^k - 1}{p^s - 1}. \tag{34}$$

Proof. Taking $a_{f,s}(p) = \frac{p^k-1}{p^s-p^k}$, we have $(1 + a_{f,s}(p)) = \left(1 - \frac{1}{p^s}\right)\left(1 - \frac{1}{p^{s-k}}\right)^{-1}$ and $\frac{a_{f,s}(p)}{1+a_{f,s}(p)} = \frac{p^k-1}{p^s-1}$. Substituting into (24) shows that

$$\sum_{n\in\mathbb{N}} \frac{\omega(n)J_k(n)}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)\left(1 - \frac{1}{p^{s-k}}\right)^{-1} \left(\sum_p \frac{p^k - 1}{p^s - 1}\right).$$

The series $\sum_p \frac{p^k-1}{p^s-1}$ does not have a representation as a sum of prime zeta and shifted prime zeta functions in general, but in special cases such as $s = 2k$, $k > 1$, it does. Rewriting in terms of zeta functions and taking convergence criteria based on (5.2) completes the proof. □

We conclude with some additional nontrivial examples, which will perhaps be of use when considering exotic divisor sums. They are all special cases of Theorem 5.4 and can be proved similarly to Corollary 5.14. In each case we begin with a known Euler product for a function $f(n)$, then use that to extract $a_{f,s}(p)$.

Corollary 5.15. *For $s, k \in \mathbb{C}$ such that $\sigma > \max(1, 1 + \Re(k))$,*

$$\sum_{n\in\mathbb{N}} \frac{\omega(n)\sigma_k(n)}{n^s} = \zeta(s)\zeta(s-k)\left(P(s) + P(s-k) - P(2s-k)\right). \tag{35}$$

Proof. We know from [8, (27.4.11)] that $\sum_{n\in\mathbb{N}} \frac{\sigma_k(n)}{n^s} = \zeta(s)\zeta(s-k)$, where $\sigma_k(n)$ is the sum of the kth powers of the divisors of n. Expanding the Euler product gives us that $\zeta(s)\zeta(s-k) = \prod_p \left(1-\frac{1}{p^s}\right)^{-1}\left(1-\frac{1}{p^{s-k}}\right)^{-1} = \prod_p (1+a_{f,s}(p))$. We obtain $1+a_{f,s}(p) = \left(1-\frac{1}{p^s}\right)^{-1}\left(1-\frac{1}{p^{s-k}}\right)^{-1}$. Simplifying gives that $\frac{a_{f,s}(p)}{1+a_{f,s}(p)} = \frac{1}{p^s} + \frac{1}{p^{s-k}} - \frac{1}{p^{2s-k}}$. Substituting into (24) shows that

$$\sum_{n\in\mathbb{N}} \frac{\omega(n)\sigma_k(n)}{n^s} = \left(\sum_{n\in\mathbb{N}} \frac{\sigma_k(n)}{n^s}\right) \sum_p \left(\frac{1}{p^s} + \frac{1}{p^{s-k}} - \frac{1}{p^{2s-k}}\right).$$

Rewriting the right hand side in terms of zeta and prime zeta functions while noting that they will both converge if $\Re(s) > \max(1, 1+\Re(k))$ completes the proof. □

Corollary 5.16. *Let $s \in \mathbb{C}$. Then*

$$\sum_{n\in\mathbb{N}} \frac{\omega(n)d(n^2)}{n^s} = \frac{\zeta^3(s)}{\zeta(2s)} \left(4P(s,1) - P(s)\right), \tag{36}$$

which converges if $\Re(s) > 1$.

Proof. We know from [7, (3.41)] that

$$\sum_{n\in\mathbb{N}} \frac{\sigma_a(n^2)}{n^s} = \frac{\zeta(s)\zeta(s-a)\zeta(s-2a)}{\zeta(2s-2a)}.$$

As in the proof of (35), expanding the Euler product gives us that

$$\frac{\zeta(s)\zeta(s-a)\zeta(s-2a)}{\zeta(2s-2a)} = \prod_p \left(1-\frac{1}{p^s}\right)^{-1}\left(1-\frac{1}{p^{s-a}}\right)^{-1}\left(1-\frac{1}{p^{s-2a}}\right)^{-1}\left(1-\frac{1}{p^{2s-2a}}\right).$$

The term inside the product is equal to $(1+a_{f,s}(p))$ for every prime p, so on simplifying we get that

$$\frac{a_{f,s}(p)}{1+a_{f,s}(p)} = \left(\frac{1}{p^s} + \frac{1}{p^{s-a}} + \frac{1}{p^{s-2a}} - \frac{1}{p^{2s-a}} - \frac{2}{p^{2s-2a}} - \frac{1}{p^{2s-3a}} + \frac{1}{p^{3s-3a}}\right)\left(1-\frac{1}{p^{2s-2a}}\right)^{-1}.$$

However, there is no convenient decomposition into a sum of terms of the form $\frac{1}{p^\alpha}$ and $\frac{1}{p^\alpha \pm 1}$ so we restrict our attention to the case $a=0$. Then $\sigma_0(n^2) = d(n^2)$, where $d(n^2)$ counts the number of divisors of n^2. With $a=0$, we have

$$\frac{a_{f,s}(p)}{1+a_{f,s}(p)} = \left(\frac{3}{p^s} - \frac{4}{p^{2s}} + \frac{1}{p^{3s}}\right)\left(1-\frac{1}{p^{2s}}\right)^{-1}.$$

Using the identity $p^{2s} - 1 = (p^s - 1)(p^s + 1)$ allows us to simplify this to

$$\frac{a_{f,s}(p)}{1 + a_{f,s}(p)} = \frac{4}{p^s + 1} - \frac{1}{p^s}.$$

Substituting this into (24) gives us that

$$\sum_{n\in\mathbb{N}} \frac{\omega(n)d(n^2)}{n^s} = \left(\sum_{n\in\mathbb{N}} \frac{d(n^2)}{n^s}\right) \sum_p \left(\frac{4}{p^s+1} - \frac{1}{p^s}\right).$$

Rewriting in terms of zeta and prime zeta functions while finding convergence criteria to match both completes the proof. □

Corollary 5.17. *Let $s \in \mathbb{C}$. Then*

$$\sum_{n\in\mathbb{N}} \frac{\omega(n)d^2(n)}{n^s} = \frac{\zeta^4(s)}{\zeta(2s)} \left(8P(s,1) + P(2s) - 4P(s)\right), \tag{37}$$

which converges if $\Re(s) > 1$.

Proof. Mirroring the proof of the last result, [7, (3.64)] shows that

$$\sum_{n\in\mathbb{N}} \frac{\sigma_a(n)\sigma_b(n)}{n^s} = \frac{\zeta(s)\zeta(s-a)\zeta(s-b)\zeta(s-a-b)}{\zeta(2s-a-b)}.$$

As in the proof of (35), expanding the Euler product gives us that

$$\begin{aligned}\frac{\zeta(s)\zeta(s-a)\zeta(s-b)\zeta(s-a-b)}{\zeta(2s-a-b)} = \prod_p &\left(1 - \frac{1}{p^s}\right)^{-1} \left(1 - \frac{1}{p^{s-a}}\right)^{-1} \left(1 - \frac{1}{p^{s-b}}\right)^{-1} \\ &\times \left(1 - \frac{1}{p^{s-a-b}}\right)^{-1} \left(1 - \frac{1}{p^{2s-a-b}}\right).\end{aligned}$$

The term inside the product is equal to $(1 + a_{f,s}(p))$ for every prime p, so on simplifying we get that

$$\begin{aligned}\frac{a_{f,s}(p)}{1 + a_{f,s}(p)} = &\left(\frac{1}{p^s} + \frac{1}{p^{s-a}} + \frac{1}{p^{s-b}} + \frac{1}{p^{s-a-b}} - \frac{1}{p^{2s-a}} - \frac{1}{p^{2s-b}} - \frac{3}{p^{2s-a-b}} - \frac{1}{p^{2s-a-2b}} - \frac{1}{p^{2s-2a-b}}\right. \\ &\left. + \frac{1}{p^{3s-a-b}} + \frac{1}{p^{3s-2a-b}} + \frac{1}{p^{3s-a-2b}} + \frac{1}{p^{3s-2a-2b}} - \frac{1}{p^{4s-2a-2b}}\right) \left(1 - \frac{1}{p^{2s-a-b}}\right)^{-1}.\end{aligned}$$

However, there is no good decomposition into a sum of terms of the form $\frac{1}{p^\alpha}$ and $\frac{1}{p^\alpha \pm 1}$. Even the case $a = b$ proves intractable so we restrict our attention to the case $a = b = 0$. Then

$\sigma_0(n)\sigma_0(n) = d^2(n)$, where $d^2(n)$ is the square of the number of divisors of n. With $a = b = 0$, we have

$$\frac{a_{f,s}(p)}{1 + a_{f,s}(p)} = \left(\frac{4}{p^s} - \frac{7}{p^{2s}} + \frac{4}{p^{3s}} - \frac{1}{p^{4s}}\right)\left(1 - \frac{1}{p^{2s}}\right)^{-1}.$$

Using the identity $p^{2s} - 1 = (p^s - 1)(p^s + 1)$ allows us to simplify this to

$$\frac{a_{f,s}(p)}{1 + a_{f,s}(p)} = \frac{8}{p^s + 1} + \frac{1}{p^{2s}} - \frac{4}{p^s}.$$

Substituting this into (24) gives us that

$$\sum_{n \in \mathbb{N}} \frac{\omega(n) d^2(n)}{n^s} = \left(\sum_{n \in \mathbb{N}} \frac{d^2(n)}{n^s}\right) \sum_p \left(\frac{8}{p^s + 1} + \frac{1}{p^{2s}} - \frac{4}{p^s}\right).$$

Rewriting in terms of zeta and prime zeta functions while finding convergence criteria to match both completes the proof. □

6 Extensions

Lastly, we show how to generalize the methods of this paper to second and higher order derivatives. We only sketch the following proofs, which can be made rigorous by paying sufficient attention to convergence criteria.

Proposition 6.1. *We have the symmetric function identity*

$$\sum_{k=1}^{n} k^2 e_k t^k = \left(\prod_{i=1}^{n}(1 + x_i)\right)\left(\left(\sum_{i=1}^{n} \frac{x_i}{1 + x_i}\right)^2 + \sum_{i=1}^{n} \frac{x_i}{(1 + x_i)^2}\right). \tag{38}$$

Proof. Starting with the generating product for e_k, $\prod_{k=1}^{n}(1 + tx_i) = \sum_{k=0}^{n} e_k t^k$, we study the action of the differential operator $D := t\frac{d}{dt}$. Applying it once yields

$$D\left(\prod_{k=1}^{n}(1 + tx_i)\right) = \sum_{k=1}^{n} k e_k t^k,$$

from which we recover the familiar (5) after evaluating at $t = 1$. Applying D a second time results in

$$\sum_{k=1}^{n} k^2 e_k t^k = D\left(t \prod_{k=1}^{n}(1 + tx_i) \sum_{k=1}^{n} \frac{x_i}{1 + tx_i}\right) \tag{39}$$

$$= \left(\prod_{i=1}^{n}(1 + x_i)\right)\left(\left(\sum_{i=1}^{n} \frac{x_i}{1 + x_i}\right)^2 + \sum_{i=1}^{n} \frac{x_i}{1 + x_i} - \sum_{i=1}^{n} \frac{x_i^2}{(1 + x_i)^2}\right). \tag{40}$$

Some algebra completes the proof. □

In general, iterating the application of D will yield higher order identities for the elementary symmetric polynomials.

Proposition 6.2. *Let $a_{f,s}(p) := \sum_{m=1}^{\infty} \frac{f(p^m)}{p^{ms}}$ as usual. If f is multiplicative and s has sufficiently large real part to ensure convergence of all the following series, then*

$$\sum_{n\in\mathbb{N}} \frac{\omega(n)^2 f(n)}{n^s} = \left(\sum_{n\in\mathbb{N}} \frac{f(n)}{n^s}\right) \left(\left(\sum_p \frac{a_{f,s}(p)}{1+a_{f,s}(p)}\right)^2 + \sum_p \frac{a_{f,s}(p)}{(1+a_{f,s}(p))^2}\right). \tag{41}$$

If f is instead completely multiplicative, we have

$$\sum_{n\in\mathbb{N}} \frac{\omega(n)^2 f(n)}{n^s} = \left(\sum_{n\in\mathbb{N}} \frac{f(n)}{n^s}\right) \left(\left(\sum_p \frac{f(p)}{p^s}\right)^2 + \sum_p \frac{f(p)}{p^s} - \sum_p \left(\frac{f(p)}{p^s}\right)^2\right). \tag{42}$$

Proof. Letting $x_i = a_{f,s}(p_i)$ in (38) while letting $t \to 1$ and $n \to \infty$ gives the first result. Summing $a_{f,s}(p)$ as a geometric series gives the second. $\square$

As an example, letting $f(n) = 1$ gives

$$\sum_{n\in\mathbb{N}} \frac{\omega(n)^2}{n^s} = \zeta(s)\left(P^2(s) + P(s) - P(2s)\right), \tag{43}$$

which converges for $\Re(s) > 1$. This result, though the simplest application of Proposition 6.2, appears to be new. In general, applying D to $E(t)$ k times will result in a Dirichlet series of the form $\sum_{n\in\mathbb{N}} \frac{\omega(n)^k f(n)}{n^s}$. This may prove useful when considering asymptotics for the weighted k-th moments

$$\sum_{n\leq x} \omega^k(n) f(n),$$

for some multiplicative f.

7 Conclusion

These theorems apply to any multiplicative functions. Together, this allows for a large class of infinite and divisor sums weighted by $\omega(n)$ to be addressed for the first time. This leads to some surprising results such as that the Dirichlet series for products of $\omega(n)$ and other multiplicative functions often have a convenient closed form expression in terms of zeta and prime zeta functions. The methods of this paper also suggest an obvious generalization; taking the kth derivative of $E(t)$ will lead to finite and infinite sums involving $\omega(n)^k$. This also suggests deep connections between the theory of symmetric functions and Dirichlet series, since with the right choice of x_i we can interpret a Dirichlet series as a sum over symmetric polynomials. Different identities for symmetric polynomials with correspond to general expressions for Dirichlet series weighted by different functions. Due to this, a systematic study of identities for symmetric functions should correspond to surprising identities for Dirichlet series.

References

[1] Apostol, T. M. *Introduction to Analytic Number Theory.* Springer-Verlag, New York, NY, 1976.

[2] Erdös, P. and Kac, M.. The Gaussian law of errors in the theory of additive number theoretic functions. *American Journal of Mathematics*, 62:738–742, 1940.

[3] Fröberg, C.-E. On the prime zeta function. *Nordisk Tidskr. Informationsbehandling (BIT)*, 8:187–202, 1968.

[4] Gould, H. W. and Shonhiwa, T.. A catalog of interesting Dirichlet series. *Missouri J. Math. Sci.*, 1:2–18, 2008.

[5] Hardy, G. H. and Ramanujan, S. The normal number of prime factors of a number n [Quart. J. Math. **48** (1917), 76–92]. *Collected papers of Srinivasa Ramanujan.* AMS Chelsea Publ., Providence, RI, 2000.

[6] I. G. Macdonald, *Symmetric functions and Hall polynomials,* 2nd ed., Oxford University Press, New York, 1995.

[7] Mathar, R. J. Survey of Dirichlet series of multiplicative arithmetic functions. *Pre-print*, 2012. http://arxiv.org/abs/1106.4038.

[8] NIST Digital Library of Mathematical Functions. `http://dlmf.nist.gov`, Release 1.0.9 of 2014-08-29. Online companion to [9].

[9] Olver, F. W. J. and Lozier, D. W. and Boisvert, R. F. and Clark, C. W., editors. *NIST Handbook of Mathematical Functions*. Cambridge University Press, New York, NY, 2010. Print companion to [8].