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Probabilities Involving Standard Trirectangular Tetrahedral Dice Rolls

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Cover Page Footnote

We would like to thank our mentor and friend, Paul R. Hurst. Also, thanks to the MAA for letting us present this research at Mathfest 2017.

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Probabilities Involving Standard Trirectangular Tetrahedral Dice Rolls

Abstract. The goal is to be able to calculate probabilities involving irregular shaped dice rolls. Here it is attempted to model the probabilities of rolling standard trirectangular tetrahedral dice on a hard surface, such as a table top. The vertices and edges of a tetrahedron were projected onto the surface of a sphere centered at the center of mass of the tetrahedron. By calculating the surface areas bounded by the resultant geodesics, baseline probabilities were achieved. Using a 3D printer, dice were constructed of uniform density and the results of rolling them were recorded. After calculating the corresponding con
dence intervals, the results were signi
cantly different from the original calculated probabilities. Possible reasons for the discrepancy are noted, but further research is needed to better understand what is going on.

Acknowledgements: We would like to thank our mentor and friend, Paul R. Hurst. Also, thanks to the MAA for letting us present this research at Mathfest 2017.

1 Probabilities of Irregular Shaped Dice

As the title suggests, only one die shape is covered in this paper. Probabilities for fair symmetric dice have been thoroughly analyzed[1]. The goal now is to explore dice that are not symmetric. Using spherical projections of a die, a theoretical probability baseline was established. Then statistical analysis of actual dice rolls under defined conditions and methodology was carried out to test the theoretical probabilities. The objective was to test the accuracy of using spherical projections as a way to estimate the outcome of rolling irregular shaped dice. If it is not accurate, do the results give insight into an improved general technique for finding probabilities?

1.1 Standard Trirectangular Tetrahedron

A standard trirectangular tetrahedron has never been used in a spherical projection study and was chosen because of its relatively simple geometry while retaining the necessary quality of being a non-symmetric solid shape. A standard trirectangular tetrahedron is a four sided polyhedron where exactly one face is an equilateral triangle. This face will be referred to as the base. The remaining three faces are congruent right triangles[2] and will be referred to as the sides. This is according to the definition in [1]. For more information, see [5].

2 Projecting the die onto a sphere

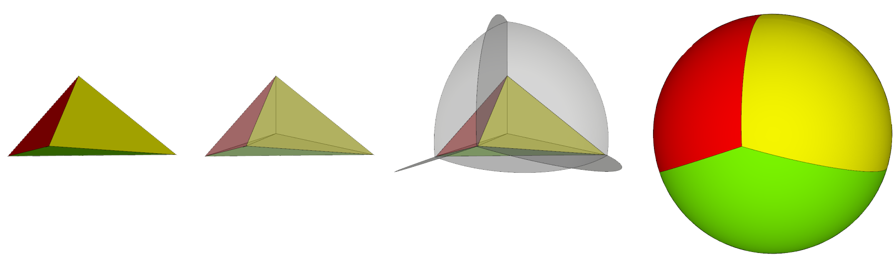


Fig. 1. *Left to Right.* Standard trirectangular tetrahedron, transparent to show center of mass, planes in gray from center of mass and each edge, and the sphere segmented by those planes.

The idea of "projecting the die onto a sphere" has been used by other mathematicians[3] although no one has previously used the method on the standard trirectangular tetrahedron. This can be visualized in figure 1. As a die tumbles through air around its center of mass it can take an infinite number of orientations. Let each point on the surface of the sphere correspond to the orientation

[1] See reference [2]

[2] Normal tetrahedra have equilateral triangles for all four sides.

[3] For examples, see references [3], [4], and [6]

of the die where that sphere surface point is lower than all other surface points. Projecting out from the center of mass through each face of the trirectangular tetrahedron gives us a region on the surface of the sphere that corresponds to that face.

2.1 Center of Mass

A standard trirectangular tetrahedron can be expressed by the inequalities (1). The mass of such a die with uniform density 1 can be found (2), and then used to find the center of mass (3), resulting in $\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$ as the center of mass.

$$x > 0,\ y > 0,\ z > 0, \text{ and } 0 < x + y + z < 1 \tag{1}$$

$$m = \int\int\int_D dV \tag{2}$$

$$\bar{x} = \frac{1}{m}\int\int\int_D xdV, \bar{y} = \frac{1}{m}\int\int\int_D ydV, \bar{z} = \frac{1}{m}\int\int\int_D zdV \tag{3}$$

2.2 Projected Surface Areas

The surface area of each corresponding region on the sphere can be calculated from the angles between the geodesics. For a region bounded by three geodesics and sphere of radius $r = 1$ see figure 2. By Girard's formula, the surface area of such a region is :

$$A = \alpha + \beta + \gamma - \pi \tag{4}$$

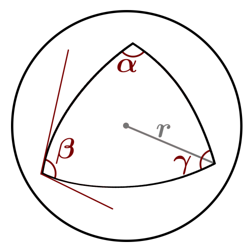


Fig. 2. Surface on a sphere bounded by three geodesics

2.3 Angles Between Planes

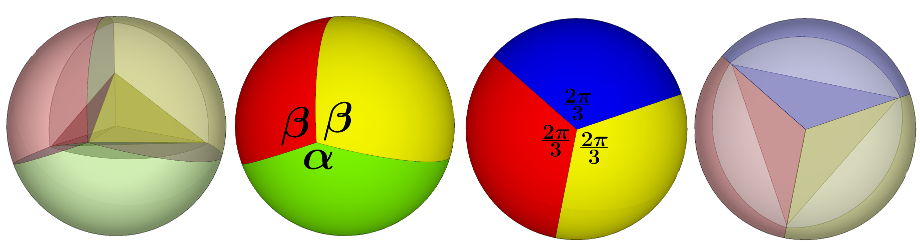


Fig. 3. Side and top views with their transparent counterparts to show the tetrahedron. Green corresponds to the base, and yellow, red, and blue are the sides.

Using (7), the acute angle is found with the normals, n_1 and n_2 in (5) and (6), to the planes through the center of mass and two non-rectangular vertices. The acute angle between planes is shown in (8) and the obtuse angle, α, in (9) and is the angle for all three of the base region angles. The two congruent angles, β, are found using (10). And finally the remaining side angle is simply $\frac{2\pi}{3}$.

$$n_1 = 4\left(\left[(0,0,1) - \left(\frac{1}{4},\frac{1}{4},\frac{1}{4}\right)\right] \times \left[(0,1,0) - \left(\frac{1}{4},\frac{1}{4},\frac{1}{4}\right)\right]\right) = 2i + j + k \quad (5)$$

$$n_2 = 4\left(\left[(0,0,1) - \left(\frac{1}{4},\frac{1}{4},\frac{1}{4}\right)\right] \times \left[(1,0,0) - \left(\frac{1}{4},\frac{1}{4},\frac{1}{4}\right)\right]\right) = i + 2j + k \quad (6)$$

$$\theta = \pi - \cos^{-1}\left(\frac{n_1 \cdot n_2}{\|n_1\|\|n_2\|}\right) \quad (7)$$

$$\cos^{-1}\left(\frac{n_1 \cdot n_2}{\|n_1\|\|n_2\|}\right) = \cos^{-1}\left(\frac{2+2+1}{\sqrt{4+1+1}\sqrt{4+1+1}}\right) = \cos^{-1}\left(\frac{5}{6}\right) \quad (8)$$

$$\alpha = \pi - \cos^{-1}\left(\frac{5}{6}\right) \quad (9)$$

$$\beta = \pi - \frac{\alpha}{2} \quad (10)$$

2.4 Theoretical Probability Calculation

The surface areas of both regions are found to be:

$$A_b = 3\alpha - \pi \quad (11)$$

$$A_s = 2\beta + \frac{2\pi}{3} - \pi \quad (12)$$

Where A_b is the surface area of the base region and A_s is the surface area of the side regions.The region proportion of the entire sphere is the theoretical probability of landing on the base or one of the 3 sides. Since the total surface area of a unit sphere is 4π, the theoretical probabilities are as follows;

$$P(\text{landing on the base}) = A_b/(4\pi) \approx 0.36018 \quad (13)$$

$$P(\text{landing on a side}) = 3A_s/(4\pi) = 1 - P(\text{base}) \approx 0.63982 \quad (14)$$

3 Statistical Analysis

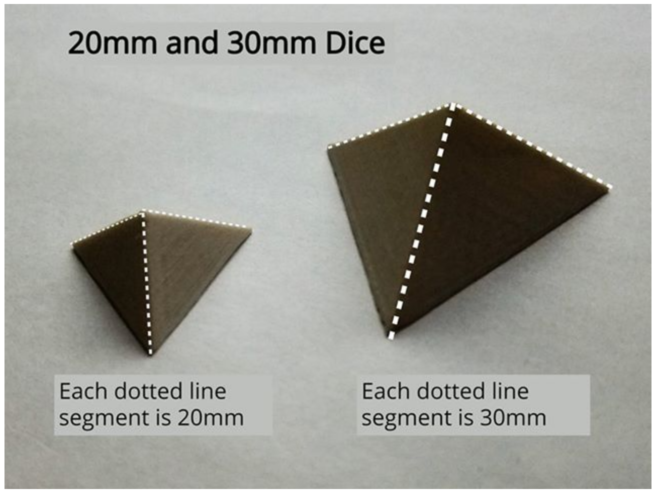


Fig. 4. 3D printed dice. Two sizes were used, 20mm and 30mm.

To test our theoretical probability, dice were made using a 3D printer to print out solid and uniform[4] density dice, see figure 4. Dice are typically used in games of chance by people sitting at a table. We decided to mimic this situation by throwing or dropping the dice onto a hard laminated table. This assured that the dice would bounce around before coming to a rest. [5] After an initial 3,000 rolls and realizing that how they are rolled likely affects the probability of these irregular dice, specific ways to roll the die were created. First, we decided we needed two different size dice. The length is given in millimeters and is the the length of the base edge. The same surface needed to be used, so all rolls were done on a table with a laminate hard surface. The dice were rolled 1,000 times for each trial. The eight trials can be reviewed below but in summary are as follows:

- Rolled onto the table top in pairs of two like it would be rolled for a board game
- Dropped onto the table top from two different heights
- Flicked in an upward direction and falling onto the table top.

3.1 Trials

The results for 8 different trial runs were recorded and given in the 3.2 Table of Trials. The column labeled 'Trial' corresponds to the y-axis in the graph, 3.3 95% Confidence Intervals. The column 'Proportion" is the total number of times

[4] Dice were printed in PLA using a 5^{th} generation Makerbot^{TM} Replicator+ with 100% infill and the floor set to the full height of the dice.

[5] A video of sample rolls can be found here: https://youtu.be/zAEmxOLIsqE

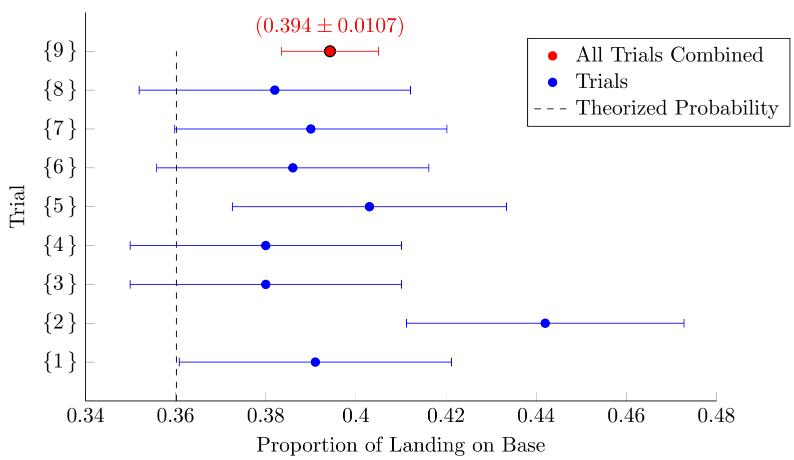
the dice landed on the base divided by the total number of rolls. Margins of error for 95% confidence are given. The column 'n' is the number of rolls per trial.

3.2 Table of Trials

Note trial 9 is the combination of all the trials 1 through 8.

Trial	Proportion	n	Margin of Error	Method
{9}	0.394	8000	0.0107	All Trials Combined
{8}	0.382	1000	0.0301	Two 20mm dice rolled onto table
{7}	0.390	1000	0.0302	Two 30mm dice rolled onto table
{6}	0.386	1000	0.0302	20mm die flicked 18in above table
{5}	0.403	1000	0.0304	30mm die flicked 18in above table
{4}	0.380	1000	0.0301	20mm dropped 36in onto table
{3}	0.380	1000	0.0301	20mm dropped 18in onto table
{2}	0.442	1000	0.0308	30mm dropped 36in onto table
{1}	0.391	1000	0.0302	30mm dropped 18in onto table

3.3 95% Confidence Intervals



There is statistically significant evidence (p-value < 0.0001) that the theorized probability is incorrect. The sphere projection suggested the probability of the die landing on its base would be approximately 0.36018, whereas we had a significantly higher proportion of bases when we rolled the dice.

4 Conclusion

The data suggested that the theoretical probability as calculated by the sphere projection method was not adequate at describing the situation. The dice landed on the base a significantly higher proportion of times than initially hypothesized. We suspect that if the dice had been thrown onto an absorbent surface, such as sand, then the results would have been significantly closer to the theoretical probability. The significant difference is likely due an energy barrier concept. As the die bounces on a hard surface, the amount of kinetic energy is constantly decreasing. If the die has the base side facing down, its center of mass is lower than if it has one of the other sides facing down. Thus, it requires more energy for the die to transition from having the base side down than is required to flip the other way. This seems to account for the difference between our theoretical and empirical results. Thus, the sphere projection with adjustments for other factors, such as height of center of mass of the different aspects and possibly properties of the shape of the dice, can be used to accurately estimate the probabilities.

4.1 Further Research

In the future more research could be done. There are several things that could improve the accuracy of this experiment and give insight into the mechanics of rolling irregular dice in general.

- More dice rolls to increase the sample size
- Different shapes, such as different sized tetrahedrons, cones, cylinders, triangular prisms, square pyramids, cuboids, and so on.
- Specifically using a tetrahedron, the height could be manipulated and several different-sized die can be created. Comparing their probabilities could produce interesting results.

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