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Partial Sum Trigonometric Identities and Chebyshev Polynomials

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Abstract. Using Eulers theorem, geometric sums and Chebyshev polynomials, we prove trigonometric identities involving sums and multiplications of cosine.

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Introduction

In [5] the authors proved a generalization of the identity $\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$, namely

$$\sum_{k=1}^{n} \cos\left(\frac{k\pi}{2n+1}\right)(-1)^{k+1} = \frac{1}{2}.$$

It's an obvious question whether there there exists an equation of a similar form that generalizes $\cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$. We determined that this was not the case so instead we looked towards $\cos^2\left(\frac{\pi}{4}\right) = \frac{1}{2}$. We found that

$$\cos^2\left(\frac{\pi}{6}\right) - \cos^2\left(\frac{2\pi}{6}\right) = \frac{1}{2},$$

$$\cos^2\left(\frac{\pi}{8}\right) - \cos^2\left(\frac{2\pi}{8}\right) + \cos^2\left(\frac{3\pi}{8}\right) = \frac{1}{2},$$

$$\cos^2\left(\frac{\pi}{10}\right) - \cos^2\left(\frac{2\pi}{10}\right) + \cos^2\left(\frac{3\pi}{10}\right) - \cos^2\left(\frac{4\pi}{10}\right) = \frac{1}{2},$$

and so forth. This led us to a generalization of $\cos^2(\frac{\pi}{4}) = \frac{1}{2}$ and, by extension, a series of other identities. In this paper we will discuss different methods for proving these identities, primarily focusing on the use of Chebyshev polynomials.

Elementary Proofs

Theorem 1. *Let n be a natural number. Then*

$$\sum_{k=1}^{n} \cos^2\left(\frac{k\pi}{2n+2}\right)(-1)^{k+1} = \frac{1}{2}.$$

Proof. Our proof will use the following fact:

(1) If $x + y = \pi$ then $\cos x = -\cos y$.

First, applying the power reducing formula for $\cos^2\theta$ gives us:

$$\sum_{k=1}^{n}\cos^2\left(\frac{k\pi}{2n+2}\right)(-1)^{k+1} = \frac{1}{2}\sum_{k=1}^{n}\left(1+\cos\left(\frac{k\pi}{n+1}\right)\right)(-1)^{k+1}$$

$$= \frac{1}{2}\sum_{k=1}^{n}(-1)^{k+1} + \frac{1}{2}\sum_{k=1}^{n}\cos\left(\frac{k\pi}{n+1}\right)(-1)^{k+1}.$$

At this point it is important to note two important details:

1. If n is even, $\sum_{k=1}^{n}(-1)^{k+1}=0$. Then we must show that $\sum_{k=1}^{n}\cos(\frac{k\pi}{n+1})(-1)^{k+1}=1$.
2. If n is odd, $\sum_{k=1}^{n}(-1)^{k+1}=1$. Then we must show that $\sum_{k=1}^{n}\cos(\frac{k\pi}{n+1})(-1)^{k+1}=0$.

Case 1. *Let n be even.*

We will write $n=2m$.
By splitting the sum $\sum_{k=1}^{2m}\cos\left(\frac{k\pi}{2m+1}\right)(-1)^{k+1}$ and letting the summation of the terms for $k\geq m+1$ run backwards we get:

$$\sum_{k=1}^{m}\cos\left(\frac{k\pi}{2m+1}\right)(-1)^{k+1} + \sum_{k=1}^{m}\cos\left(\frac{(2m+1-k)\pi}{2m+1}\right)(-1)^{2m+2-k}.$$

In [5] it was shown that

$$\sum_{k=1}^{m}\cos\left(\frac{k\pi}{2m+1}\right)(-1)^{k+1} = \frac{1}{2},$$

leaving us to show that

$$\sum_{k=1}^{m}\cos\left(\frac{(2m+1-k)\pi}{2m+1}\right)(-1)^{2m+2-k} = \frac{1}{2}.$$

By equation (1) we have $\cos(\frac{k\pi}{2m+1}) = -\cos\left(\frac{(2m+1-k)\pi}{2m+1}\right)$. But $(-1)^{k+1}$ and $(-1)^{2m+2-k}$ are also negatives of each other. Therefore

$$\sum_{k=1}^{m}\cos\left(\frac{(2m+1-k)\pi}{2m+1}\right)(-1)^{2m+2-k} = \sum_{k=1}^{m}\cos\left(\frac{k\pi}{2m+1}\right)(-1)^{k+1} = \frac{1}{2}$$

as desired.

Case 2 can be shown by the same argument and therefore has been omitted. ✓

A similar proof can be used to show that for $n \geq 2$,

$$\sum_{k=1}^{n} \sin^2\left(\frac{k\pi}{2n+2}\right)(-1)^{k+1} = \frac{(-1)^{n+1}}{2}.$$

Another way to look at this theorem is by observing that the difference between the sum of terms with odd values in the numerator and the the sum of terms with even values in the numerator is equal to $\frac{1}{2}$. Interestingly enough, the values of these two separate sums have their own distinct values. This led to the observation that by taking their sum instead of their difference we also get a distinct value. This led to the following theorem.

Theorem 2. *Let $n \geq 2$ be a natural number. Then*

$$\sum_{k=1}^{n} \cos^2\left(\frac{k\pi}{n}\right) = \frac{n}{2}.$$

Proof. First, applying the power reducing formula for $\cos^2\theta$ gives us:

$$\sum_{k=1}^{n} \cos^2\left(\frac{k\pi}{n}\right) = \frac{1}{2}\left[\sum_{k=1}^{n} 1 + \sum_{k=1}^{n} \cos\left(\frac{2k\pi}{n}\right)\right] = \frac{n}{2} + \frac{1}{2}\sum_{k=1}^{n} \cos\left(\frac{2k\pi}{n}\right)$$

So, we must show that $\sum_{k=1}^{n} \cos\left(\frac{2k\pi}{n}\right) = 0$. To do so, we will also show $\sum_{k=1}^{n} \sin\left(\frac{2k\pi}{n}\right) = 0$. Looking towards [1] we can use Euler's formula ($\cos x + i \sin x = e^{ix}$):

$$\sum_{k=1}^{n} \left(\cos\left(\frac{2k\pi}{n}\right) + i \sin\left(\frac{2k\pi}{n}\right)\right) = \sum_{k=1}^{n} (e^{\frac{i2\pi}{n}})^k$$

$$= \sum_{k=0}^{n-1} (e^{\frac{i2\pi}{n}})^k = \frac{e^{(\frac{i2\pi}{n})}\left(1 - \left(e^{\frac{i2\pi}{n}}\right)^n\right)}{1 - e^{\frac{i2\pi}{n}}} = \frac{e^{(\frac{i2\pi}{n})}\left(1 - 1\right)}{1 - e^{\frac{i2\pi}{n}}} = 0.$$

Therefore, $\sum_{k=1}^{n} \cos\left(\frac{2k\pi}{n}\right) = 0$ and $\sum_{k=1}^{n} \sin\left(\frac{2k\pi}{n}\right) = 0$.

✓

A similar proof can be used to show that for $n \geq 2$,

$$\sum_{k=1}^{n} \sin^2\left(\frac{k\pi}{n}\right) = \frac{n}{2},$$

or you can prove this using Theorem 2 and $\sin^2\theta + \cos^2\theta = 1$.

Background

The first author in [5] initially proved specific cases of $\cos(\frac{\pi}{3}) = \frac{1}{2}$ with the use of Chebyshev polynomials. This work was unpublished [4] but the proofs used in the remainder of this paper were inspired by his work. Before we go into the proofs involving Chebyshev polynomials, it is first important to develop an understanding of what Chebyshev polynomials are and how they work. The following definition can be found in [3].

Definition 1. *The Chebyshev polynomial, $T_n(x)$, of the first kind is a polynomial in x of degree n, defined by the relation*

$$T_n(x) = \cos(n\theta) \ \textit{ when } x = \cos(\theta).$$

For example:

$$T_0(x) = \cos(0\theta) = 1$$
$$T_1(x) = \cos(\theta) = x$$
$$T_2(x) = \cos(2\theta) = 2\cos^2(\theta) - 1 = 2x^2 - 1$$
$$T_3(x) = \cos(3\theta) = \cos(3\theta)+\cos(\theta)-\cos(\theta) = 2\cos(\theta)\cos(2\theta)-\cos(\theta) = 4\cos^3(\theta)-3\cos(\theta) = 4x^3-3x$$

It doesn't take long for these calculations to become too complicated to do by hand. Fortunately the following is a recursion formula that makes finding these polynomials much more simple.

Recursion Formula $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$ for $n \geq 1$ and where $T_0(x) = 1$ and $T_1(x) = x$.

Proof. Let $T_0(x) = 1$ and $T_1(x) = x$ and let $n \geq 1$. Then

$$T_{n+1}(x) = \cos((n+1)\theta).$$
$$= 2\Big(\frac{\cos((n+1)\theta) + \cos((n-1)\theta)}{2}\Big) - \cos((n-1)\theta$$
$$= 2\cos(\theta)\cos(n\theta) - \cos((n-1)\theta)$$
$$2xT_n(x) - T_{n-1}(x)$$

✓

It then becomes easier to continue computing polynomials:

$$T_4(x) = 8x^4 - 6x^2 - 2x^2 + 1 = 8x^4 - 8x^2 + 1,$$
$$T_5(x) = 16x^5 - 16x^3 + x - 4x^3 + 3x = 16x^5 - 20x^3 + 4x,$$

and so on. The first five examples can be seen in the graph below. Note that each polynomial with even degree pass through the points (-1,1) and (1,1) while each polynomial with odd degree pass through the points (-1,-1) and (1,1). These will be defined in Properties of Chebyshev Polynomials (a) and (b).

Properties of Chebyshev Polynomials

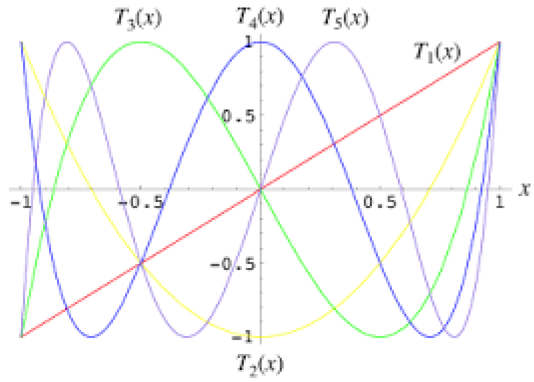

Figure 1: Chebyshev Polynomial of the First Kind

(a) Let n be a natural number. Then $T_n(-1) = (-1)^n$.

(b) Let n be a natural number. Then $T_n(1) = 1$.

(c) The leading term of $T_n(x)$ equals $2^{n-1}x^n$.

(d) If n is odd (even) then $T_n(x)$ involves only odd (even) powers of n.

(e) If n is odd then the constant term is equal to zero.

(f) If $n = 2m$ then the constant term of $T_n(x)$ is equal to $(-1)^m$.

(g) If $n = 2m + 1$ the last term of $T_n(x)$ is equal to $(-1)^m nx$.

(h) If n is even the second to last term of $T_n(x)$ is equal to $\frac{n^2}{2}x^2$.

(i) The extrema of $T_n(x)$ occur at

$$x = \cos\left(\frac{k\pi}{n}\right), (k = 0, 1, 2, \dots, n).$$

$T_n(x)$ is equal to 1 when k is even and -1 when k is odd.

Proof of (a)

Proof. Let $n = 0$. Then $T_0(-1) = 1$.
Let $n = 1$. Then $T_1(-1) = -1$.
Let $n = 2$. Then $T_2(-1) = -2(-1) - 1 = 1$.
Assume $T_n(-1) = -1$ for even n and $T_n(-1) = 1$ for odd n. Consider $n + 1$. Then

$$\begin{aligned} T_{n+1}(-1) &= 2(-1)T_n(-1) - T_{n-1}(-1) \\ &= 2(-1)(-1)^n - (-1)^{-1} \\ &= (-1)^{n-1}[2(-1)^2 - 1] \\ &= (-1)^{n-1} \\ &= (-1)^{n+1} \end{aligned}$$

✓

Properties (b)-(h) are proven similarly to (a) through the recursion formula.

Proof of (i)

Proof. The extrema of $T_n(x)$ correspond to the extrema of $\cos(n\theta)$. Since

$$\frac{d}{dx}T_n(x) = \frac{d}{dx}\cos(n\theta) = \frac{\frac{d}{d\theta}\cos(n\theta)}{\frac{dx}{d\theta}} = \frac{n\sin(n\theta)}{\sin(\theta)},$$

the extrema of $T_n(x)$ occur at the zeros of $\sin(n\theta)$, namely at

$$x = \cos\left(\frac{k\pi}{n}\right), (k = 0, 1, 2, \ldots, n).$$

Then, since $T_n(x) = \cos(n\theta)$, when we plug in $\theta = \frac{k\pi}{n}$ we are left with $\cos(k\pi)$. Hence we get the value of 1 for even k and -1 for odd k. ✓

The remaining facts will be needed in order to set up our proofs:

1. If a zero x_0 of a polynomial $f(x)$ is also a local extremum, then the zero is a repeated root of even multiplicity.
2. If $x + y = \pi$ then $\cos x = -\cos y$.
3. If $x + y = 2\pi$ then $\cos x = \cos y$.

Chebyshev Polynomial Proofs

The theorem below was first proven in [2], therefore the Chebyshev polynomial proof of this identity is an alternative method.

Theorem 3. *Let m be a non-negative integer. Then*

$$\prod_{k=0}^{m}\cos\left(\frac{k\pi}{2m+1}\right) = \frac{1}{2^m}$$

Proof. Let $n = 2m + 1$ where m is a non-negative integer. By property (i), $T_n(x) = -1$ at $x = \cos\left(\frac{k\pi}{n}\right)$ for odd k. Note that there are n zeros of $T_n(x) + 1$. By property (a), $x = -1$ is a root of $T_n(x) + 1$, so $T_n(x) + 1$ is divisible by $x + 1$. By property (c), $\frac{T_n(x)+1}{x+1}$ has a leading term of $2^{n-1}x^n$ where there are $n - 1$ zeros remaining. By fact 1, each root must have a multiplicity of two for $\frac{T_n(x)+1}{x+1}$. Using this information we can factor $T_n(x) + 1$ as follows:

$$T_n(x) + 1 = 2^{n-1}(x + 1)\prod_{k=1}^{\frac{n-1}{2}}\left(x - \cos\left(\frac{(2k-1)\pi}{n}\right)\right)^2$$

Note that by property (c) the last term of $T_n(x) + 1$ is the constant 1. Now, setting this equal to the last term derived from the previous equation:

$$1 = 2^{n-1} \prod_{k=1}^{\frac{n-1}{2}} \cos^2\left(\frac{(2k-1)\pi}{n}\right)$$

Since n is odd we will re-write it as $n = 2m + 1$ where $m \geq 1$ giving us

$$1 = 2^{2m} \prod_{k=1}^{m} \cos^2\left(\frac{(2k-1)\pi}{2m+1}\right)$$

Note that the numerators of the equation on the right run through all odd values less than or equal to $2m+1$. Using fact 2 when the numerator is greater than m the equation can be written as

$$\frac{1}{2^{2m}} = \prod_{k=1}^{m} \cos^2\left(\frac{k\pi}{2m+1}\right).$$

Then, taking the square root:

$$\frac{1}{2^m} = \prod_{k=1}^{m} \cos\left(\frac{k\pi}{2m+1}\right) = \prod_{k=0}^{m} \cos\left(\frac{k\pi}{2m+1}\right)$$

for $m \geq 0$.

✓

This equation of course implies that

$$\prod_{k=0}^{n} \cos^2\left(\frac{k\pi}{2n+1}\right) = \frac{1}{2^{2n}}$$

which implies

$$\prod_{k=0}^{\frac{m}{2}} \cos^2\left(\frac{2k\pi}{2m+2}\right) = \frac{1}{2^m}$$

where $m = 2n$. This observation led to the following theorem.

Theorem 4. *Let m be a non-negative integer. Then*

$$\prod_{k=0}^{m} \cos^2\left(\frac{k\pi}{2m+2}\right) = \frac{2m+2}{2^{2m+1}}$$

Proof. **Case 1.**
Part 1.
From our previous observation of $\prod_{k=0}^{n} \cos^2\left(\frac{k\pi}{2n+1}\right) = \frac{1}{2^{2n}}$ we can then say

$$\prod_{k=0}^{\frac{m}{2}} \cos^2\left(\frac{2k\pi}{2m+2}\right) = \frac{1}{2^m}$$

where $m = 2n$. Note that this covers the even terms in the numerator leaving us to show that when m is even:

$$\prod_{k=1}^{\frac{m}{2}} \cos^2\left(\frac{(2k-1)\pi}{2m+2}\right) = \frac{2m+2}{2^{m+1}}.$$

<u>Part 2.</u> Let $n = 2m$ where m is odd.
Using methods similar to the ones in the previous proof we can use properties (c) and (i) to derive the following:

$$T_n(x) + 1 = 2^{n-1} \prod_{k=1}^{\frac{n}{2}} \left(x - \cos\left(\frac{(2k-1)\pi}{n}\right)\right)^2$$

$$T_n(x) + 1 = 2^{2m-1} \prod_{k=1}^{m} \left(x - \cos\left(\frac{(2k-1)\pi}{2m}\right)\right)^2.$$

Since m is odd, the last term of $T_n(x) + 1$ is equal to zero. Now observe that an x^2 can be factored out of the equation on the left and on the right since we get a $x = 0$ when $k = \frac{m+1}{2}$. Then:

$$T_n(x) + 1 = 2^{2m-1} x^2 \prod_{k=1}^{\frac{m-1}{2}} \left(x - \cos\left(\frac{(2k-1)\pi}{2m}\right)\right)^2 \left(x - \cos\left(\frac{(2m-2k+1)\pi}{2m}\right)\right)^2$$

Dividing both sides by x^2 and by taking the last term (i.e. the second to last term of $T_n(x)$):

$$\frac{n^2}{2} = \frac{4m^2}{2} = 2^{2m-1} \prod_{k=1}^{\frac{m-1}{2}} \left(\cos^2\left(\frac{(2k-1)\pi}{2m}\right) \cos^2\left(\frac{(2m-2k+1)\pi}{2m}\right)\right)$$

$$\frac{4m^2}{2^m} = \prod_{k=1}^{\frac{m-1}{2}} \left(\cos^2\left(\frac{(2k-1)\pi}{2m}\right) \cos^2\left(\frac{(2m-2k+1)\pi}{2m}\right)\right).$$

Then by using fact 2 and by noting that the terms are squared:

$$\frac{4m^2}{2^{2m}} = \prod_{k=1}^{\frac{m-1}{2}} \left(\cos^4\left(\frac{(2k-1)\pi}{2m}\right)\right).$$

Taking the square root:

$$\frac{2m}{2^m} = \prod_{k=1}^{\frac{m-1}{2}} \left(\cos^2\left(\frac{(2k-1)\pi}{2m}\right)\right)$$

Then by replacing m with $m+1$:

$$\frac{2m+2}{2^{m+1}} = \prod_{k=1}^{\frac{m}{2}} \cos^2\left(\frac{(2k-1)\pi}{2m+2}\right)$$

when m is even, as desired. Then, by taking the results from part 1 and multiplying it by the results in part two we get:

$$\prod_{k=0}^{n} \cos^2\left(\frac{k\pi}{2n+2}\right) = \frac{2n+2}{2^{2n+1}}$$

For even m.

Case 2.

Part 1. Let $n = 2m$, where m is even. Using properties (a)-(c) and (i) we can derive the following:

$$T_n(x) - 1 = 2^{n-1}\Big(x-1\Big)\Big(x+1\Big)\prod_{k=1}^{\frac{n-2}{2}} \left(x - \cos\left(\frac{2k\pi}{n}\right)\right)^2$$

Replacing n with $2m$:

$$= 2^{2m-1}\Big(x-1\Big)\Big(x+1\Big)\prod_{k=1}^{m-1} \left(x - \cos\left(\frac{2k\pi}{2m}\right)\right)^2$$

Since m is even, the last term of $T_n(x) - 1$ is equal to zero. Now observe that an x^2 can be factored out of the equation on the left and on the right when $k = \frac{m}{2}$. Hence the equation can be re-written as follows:

$$T_n(x) - 1 = 2^{2m-1}x^2\Big(x^2-1\Big)\prod_{k=1}^{\frac{m-2}{2}} \left[\left(x - \cos\left(\frac{2k\pi}{2m}\right)\right)^2 \left(x - \cos\left(\frac{(2m-2k)\pi}{2m}\right)\right)^2\right]$$

Dividing out a x^2 from both sides and taking the last term:

$$\frac{n^2}{2} = \frac{4m^2}{2} = 2^{2m-1}\prod_{k=1}^{\frac{m-2}{2}} \cos^2\left(\frac{2k\pi}{2m}\right)\cos^2\left(\frac{(2m-2k)\pi}{2m}\right)$$

We then divide 2^{m-1} and take the square root of both sides:

$$\frac{2m}{2^m} = \prod_{k=1}^{\frac{m-2}{2}} \cos\left(\frac{2k\pi}{2m}\right)\cos\left(\frac{(2m-2k)\pi}{2m}\right)$$

When then use fact 2 to get:

$$\frac{2m}{2^m} = \prod_{k=1}^{m-2} \cos\left(\frac{2k\pi}{2m}\right)$$

Replacong m with $m+1$:

$$\frac{2m+2}{2^{m+1}} = \prod_{k=0}^{m-1} \cos\left(\frac{2k\pi}{2m+2}\right)$$

When m is odd.

Part 2.
Let n be an even number such that when $n = 2m$, m is even.

$$T_n(x) + 1 = 2^{n-1} \prod_{k=1}^{\frac{n}{2}} \left(x - \cos\left(\frac{(2k-1)\pi}{n}\right)\right)^2$$

Since m is even the last term of $T_n(x)$ is 2. Then, taking the the last term:

$$2 = 2^{n-1} \prod_{k=1}^{\frac{n}{2}} \cos^2\left(\frac{(2k-1)\pi}{n}\right)$$

Re-writing it in terms of m:

$$\frac{1}{2^{2m-2}} = \prod_{k=1}^{m} \cos^2\left(\frac{(2k-1)\pi}{2m}\right)$$

We can replace m with $m+1$ to get:

$$\frac{1}{2^{2m}} = \prod_{k=1}^{m+1} \cos^2\left(\frac{(2k-1)\pi}{2m+2}\right).$$

Taking the quare root:

$$\frac{1}{2^{m}} = \prod_{k=1}^{m+1} \cos\left(\frac{(2k-1)\pi}{2m+2}\right)$$

Then, using fact 2:

$$\frac{1}{2^{m}} = \prod_{k=1}^{m+1} \cos^2\left(\frac{(2k-1)\pi}{2m+2}\right)$$

when m is odd.

Then by taking the results from part 1 and multiplying them by the results of part 2 we get:

$$\prod_{k=0}^{m} \cos^2\left(\frac{k\pi}{2m+2}\right) = \frac{2m+2}{2^{2m+1}}$$

for odd m.
Thus, by case 1 and case 2, we get

$$\prod_{k=0}^{n} \cos^2\left(\frac{k\pi}{2n+2}\right) = \frac{2n+2}{2^{2n+1}}$$

for all n. ✓

Optional Reading

Although Theorem 1 and Theorem 2 have already been proven, it's important to highlight that these theorems can also be proven with Chebyshev polynomials and in more than one way. Since this section adds no new information than what has previously been given, this section is left to readers who wish to further familiarize themselves with Chebyshev polynomials and their relations to trigonometric identities.

Theorem 1. *Let n be a natural number. Then*

$$\sum_{k=1}^{n} \cos^2\left(\frac{k\pi}{2n+2}\right)(-1)^{k+1} = \frac{1}{2}.$$

Proof. **Case 1.** Let n be an odd integer where $n \geq 1$. Looking towards the proof given for Theorem 3, we can derive the following equation:

$$\frac{T_n(x)+1}{x+1} = 2^{n-1} \prod_{k=1}^{\frac{n-1}{2}} \left(x - \cos\left(\frac{(2k-1)\pi}{n}\right)\right)^2$$

Then separating the square and applying fact 2:

$$= 2^{n-1} \prod_{k=1}^{\frac{n-1}{2}} \left(x - \cos\left(\frac{(2k-1)\pi}{n}\right)\right)\left(x + \cos\left(\frac{(n-(2k-1))\pi}{n}\right)\right)$$

$$= 2^{n-1} \prod_{k=1}^{\frac{n-1}{2}} \left(x - 2\cos^2\left(\frac{(2k-1)\pi}{2n}\right) + 1\right)\left(x + 2\cos^2\left(\frac{(n-(2k-1))\pi}{2n}\right) - 1\right).$$

Then, the x term with the second highest power can be concluded to have the following coefficient:

$$-2^{n-1}x^{n-2} \sum_{k=1}^{\frac{n-1}{2}} \left(2\cos^2\left(\frac{(2k+1)\pi}{2n}\right) - 2\cos^2\left(\frac{(n-(2k+1))\pi}{2n}\right)\right).$$

Then, using fact 2 once more:

$$= -2^n x^{n-2} \sum_{k=1}^{n-1} \cos^2\left(\frac{k\pi}{2n}\right)(-1)^{k+1}.$$

It can then be shown by long division that the coefficient for the second term of $\frac{T_n(x)+1}{x+1}$ is -2^{n-1}.
Setting the two sides equal to each other, you get:

$$2^n x^{n-2} \sum_{k=1}^{n-1} \cos^2\left(\frac{k\pi}{2n}\right)(-1)^{k+1} = 2^{n-1}x^{n-2}$$

$$\sum_{k=1}^{n-1} \cos^2\left(\frac{k\pi}{2n}\right)(-1)^{k+1} = \frac{1}{2}.$$

Then replace n with $n+1$ to get:

$$\sum_{k=1}^{n} \cos^2\left(\frac{k\pi}{2n+2}\right)(-1)^{k+1} = \frac{1}{2},$$

For even n.

Case 2. Let n be even. By property (i), $T_n(x) = -1$ at $x = \cos\left(\frac{k\pi}{n}\right)$ for odd k and $T_n(x) = 1$ at $x = \cos\left(\frac{k\pi}{n}\right)$ for even k.
Part 1: Using methods found in the proof of Theorem 3 we can derive the following equation:

$$T_n(x) + 1 = 2^{n-1} \prod_{k=1}^{\frac{n}{2}} \left(x - \cos\left(\frac{(2k-1)\pi}{n}\right)\right)^2$$

$$= 2^{n-1} \prod_{k=1}^{\frac{n}{2}} \left(x - 2\cos^2\left(\frac{(2k-1)\pi}{2n}\right) + 1\right)^2.$$

By property (d), there are no odd powers, so the coefficient for x^{n-1} is equal to zero. Then, using the above equation, we can conclude the coefficient of x^{n-1} to be following:

$$0 = 2^n \sum_{k=1}^{\frac{n}{2}} \Big(- 2\cos^2\left(\frac{(2k-1)\pi}{2n}\right) + 1\Big)$$

$$0 = 2\sum_{k=1}^{\frac{n}{2}} \cos^2\left(\frac{(2k-1)\pi}{2n}\right) - \frac{n}{2}$$

$$\frac{n}{4} = \sum_{k=1}^{\frac{n}{2}} \cos^2\left(\frac{(2k-1)\pi}{2n}\right).$$

Then, by replacing n with $n+1$:

$$\frac{n+1}{4} = \sum_{k=1}^{\frac{n+1}{2}} \cos^2\left(\frac{(2k-1)\pi}{2n+2}\right)$$

For odd $n \geq 1$.

Part 2: Consider $T_n(x) - 1$. The zeros of $T_n(x) - 1$ occur at $x = \cos\left(\frac{k\pi}{n}\right)$ for even k. Note that this then means that there are $\frac{n}{2}$ zeros of $T_n(x) - 1$ and recall that $x = 1$ and $x = -1$

are endpoints. Then, all other zeros of $T_n(x)-1$ have a multiplicity of 2 giving us the following:

$$T_n(x)-1 = 2^{n-1}\Big(x-\cos\Big(\frac{0\pi}{n}\Big)\Big)\Big(x-\cos\Big(\frac{n\pi}{n}\Big)\Big)\prod_{k=1}^{\frac{n-2}{2}}\Big(x-\cos\Big(\frac{2k\pi}{n}\Big)\Big)^2$$

By separating the square and by using fact 2,

$$= 2^{n-1}\Big(x-1\Big)\Big(x+1\Big)\prod_{k=1}^{\frac{n-2}{2}}\Big(x-\cos\Big(\frac{2k\pi}{n}\Big)\Big)\Big(x-\cos\Big(\frac{2(n-k)\pi}{n}\Big)\Big)$$

$$= 2^{n-1}\Big(x-1\Big)\Big(x+1\Big)\prod_{k=1}^{\frac{n-2}{2}}\Big(x-2\cos^2\Big(\frac{k\pi}{n}\Big)+1\Big)\Big(x-2\cos^2\Big(\frac{(n-k)\pi}{n}\Big)+1\Big).$$

Finding coefficient for second term:

$$0x^{n-1} = 2^n x^{n-1}\sum_{k=1}^{\frac{n-2}{2}}\Big(-2\cos^2\Big(\frac{2k\pi}{2n}\Big)-2\cos^2\Big(\frac{2(n-k)\pi}{2n}\Big)+2\Big)$$

We can divide out $2^n x^{n-1}$. Note that $\cos(\frac{\pi}{2})=0$ so the previous term can be re-written as:

$$0 = \sum_{k=1}^{n-1}\Big(-2\cos^2\Big(\frac{2k\pi}{2n}\Big)+1\Big)-1$$

$$1 = \sum_{k=1}^{n-1}\Big(-2\cos^2\Big(\frac{2k\pi}{2n}\Big)+1\Big)$$

Then by replacing n with $n+1$:

$$1 = \sum_{k=1}^{n}\Big(-2\cos^2\Big(\frac{2k\pi}{2n+2}\Big)+1\Big)$$

$$1 = -2\sum_{k=1}^{n}\cos^2\Big(\frac{2k\pi}{2n+2}\Big)+n$$

$$\frac{n-1}{2} = \sum_{k=1}^{n}\cos^2\Big(\frac{2k\pi}{2n+2}\Big)$$

Note that $\cos(\frac{\pi}{2})=0$ (i.e. at $k=\frac{n+1}{2}$) allowing us to take out the middle term. Then, using fact 2 for k terms greater than $\frac{n+1}{2}$ we get:

$$\frac{n-1}{2} = 2\sum_{k=1}^{\frac{n-1}{2}}\cos^2\Big(\frac{2k\pi}{2n+2}\Big)$$

$$\frac{n-1}{4} = \sum_{k=1}^{\frac{n-1}{2}} \cos^2\left(\frac{2k\pi}{2n+2}\right)$$

For odd $n > 1$.

We then can subtract the results of part 2 from the results of part 1 to get:

$$\frac{n+1}{4} - \frac{n-1}{4} = \sum_{k=1}^{n}\left(\cos^2\left(\frac{(2k-1)\pi}{2n+2}\right) - \cos^2\left(\frac{2k\pi}{2n+2}\right)\right)$$

Which can be simplified to:

$$\frac{1}{2} = \sum_{k=1}^{n} \cos^2\left(\frac{k\pi}{2n+2}\right)(-1)^{k+1}$$

for odd n.

Therefore, by case 1 and case 2, $\sum_{k=1}^{n} \cos^2\left(\frac{k\pi}{2n+2}\right)(-1)^{k+1} = \frac{1}{2}$ for all natural numbers n. ✓

Note that if you instead add together the two results from part one and two in Case 2 of our proof, you instead get

$$\sum_{k=1}^{n} \cos^2\left(\frac{k\pi}{n}\right) = \frac{n}{2}$$

for odd $n \geq 3$. This leads us into our second theorem.

Theorem 2. *Let $n \geq 2$ be a natural number. Then*

$$\sum_{k=1}^{n} \cos^2\left(\frac{k\pi}{n}\right) = \frac{n}{2}.$$

Proof. We already know by our observation from the previous proof that out theorem holds true for $n \geq 3$ where n is odd. Then only have to show it holds true for even n. So, let $n \geq 2$ be an even integer. Using methods similar to the ones in the previous proof we can derive the following:

$$T_n(x) - 1 = 2^{n-1}\left(x - \cos\left(\frac{0\pi}{n}\right)\right)\left(x - \cos\left(\frac{n\pi}{n}\right)\right)\prod_{k=1}^{\frac{n-2}{2}}\left(x - \cos\left(\frac{2k\pi}{n}\right)\right)^2$$

By separating the square and using fact 2:

$$= 2^{n-1}(x-1)(x+1)\prod_{k=1}^{\frac{n-2}{2}}\left[\left(x - \cos\left(\frac{2k\pi}{n}\right)\right)\left(x - \cos\left(\frac{2(n-k)\pi}{n}\right)\right)\right]$$

$$= 2^{n-1}(x-1)(x+1)\prod_{k=1}^{\frac{n-2}{2}}\Big[\Big(x - 2\cos^2\Big(\frac{k\pi}{n}\Big)+1\Big)\Big(x - 2\cos^2\Big(\frac{(n-k)\pi}{n}\Big)+1\Big)\Big]$$

Taking the second term:

$$0x^{n-1} = 2^{n-1}x^{n-1}\sum_{k=1}^{\frac{n-2}{2}}\Big(-2\cos^2\Big(\frac{k\pi}{n}\Big) - 2\cos^2\Big(\frac{(n-k)\pi}{n}\Big)+2\Big)$$

Note that $\cos\frac{\pi}{2} = 0$ so the above equation can be written as:

$$0x^{n-1} = 2^{n-1}x^{n-1}\sum_{k=1}^{n-1}\Big(-2\cos^2\Big(\frac{k\pi}{n}\Big)+1\Big) - 1$$

$$0 = \sum_{k=1}^{n}\Big(-2\cos^2\Big(\frac{k\pi}{n}\Big)+1\Big)$$

$$0 = -2\sum_{k=1}^{n}\cos^2\Big(\frac{k\pi}{n}\Big)+n$$

$$\frac{n}{2} = \sum_{k=1}^{n}\cos^2\Big(\frac{k\pi}{n}\Big)$$

For even n.

✓

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